E3S Web of Conferences **38**, 02013 (2018) https://doi.org/10.1051/e3sconf/20183802013
*ICEMEE 2018*

# Research on the dynamic buckling of functionally graded material plates under conditions of one edge fixed and three edges simply supported

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**Abstract:** Based on the strain assumption and linear mixing rate of Vogit, the physical property parameter expression of functionally graded material plates is obtained. According to the theory of small deformation and Hamilton principle, the dynamic buckling governing equation of functionally graded material plates under longitudinal load is obtained. Using the method of trial function, the analytical expression of critical load and the buckling solution of the functionally graded material plate under conditions of one edge fixed and three edges simply supported is obtained. The analytical expression of critical load is numerically calculated by METLAB. The influence of geometric size, gradient index, modal order and material composition on critical load is discussed. The results show that the critical buckling load decreases exponentially with the increase of critical length, decreases with the increase of gradient index k, increases with the increase of modal order, and the elastic modulus of constituent materials has significant effect on the critical load. The higher-order buckling modes of functionally graded material plates are prone to occur under the condition of high longitudinal load.

## 1 Introduction

Functionally graded materials are non-homogenous new composite materials that show smooth and continuous characteristics among different materials. Functionally graded materials can effectively avoid or reduce stress concentration without sudden changes in physical properties. Due to its good mechanical properties, functionally graded materials have become research hot spot[1,2]. It has been widely used in aerospace, mechanical engineering, biomedical engineering and other fields. Researchers on the dynamic buckling of functionally graded materials plate are explored Constantly[3,4]. Feldman and Aboudi[5] studied the linear dynamic buckling of functionally graded plates under in-plane loading. Shen [6] used the quadratic perturbation method to solve the nonlinear bending problem of simply-supported functionally graded material plates under the coupling condition of thermo and mechanical. The results show that the changes of volume fraction index and temperature have significant influence on the nonlinear bending behavior of the plate. Najafizadeh and Eslami[7,8] discussed the thermal buckling of circular plate in the in-plane load under simply supported and clamped supported conditions.

In this paper, Vogit's strain assumption, linear mixing rate and Kirchhoff sheet theory are used to derive the governing equations with Hamilton's principle. The expression of critical load for the functionally graded material plate is obtained. The expressions are numerically calculated by METLAB software programming. The influences of critical length, gradient index, material properties and boundary conditions for critical load are discussed.

## 2 The governing equation of functionally graded material plates

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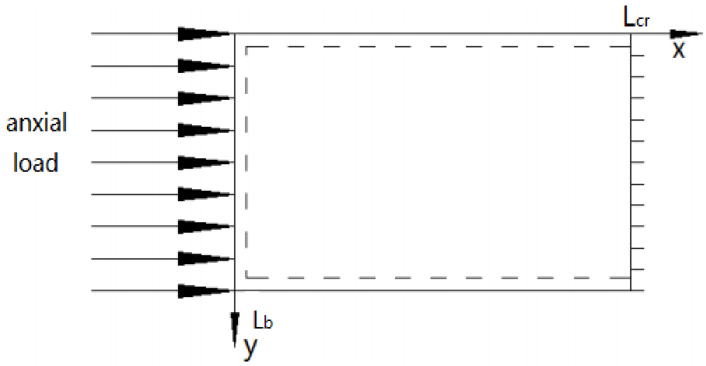



**Figure 1** The plate with the length Lcr, width Lb

The boundary conditions of one edge fixed and three edges simply supported can be written as

$$\begin{cases} w(0)=w_{xx}(0)=w(L_{cr})=w_x(L_{cr})=0 \\ w(0)=w_{yy}(0)=w(L_b)=w_{yy}(L_b)=0 \end{cases} \tag{1}$$

The material properties of functionally graded material along the z-direction can be written as

$$\begin{cases} E_{(z)}=(E_c-E_m)\left(\dfrac{2z+h}{2h}\right)^k+E_m \\ \rho_{(z)}=(\rho_c-\rho_m)\left(\dfrac{2z+h}{2h}\right)^k+\rho_m \\ \mu_{(z)}=(\mu_c-\mu_m)\left(\dfrac{2z+h}{2h}\right)^k+\mu_m \end{cases} \tag{2}$$

The constitutive equation of functionally graded material plates is

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ k_x \\ k_y \\ k_{xy} \end{bmatrix} \tag{3}$$

Kinetic energy with moment of inertia is

$$\begin{aligned} T &= \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{L_{cr}}\int_0^{L_b}\rho_{(z)}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]dxdydz \\ &= \frac{1}{2}\int_0^{L_{cr}}\int_0^{L_b}\left[I_0\left(\frac{\partial u_0}{\partial t}\right)^2+I_0\left(\frac{\partial v_0}{\partial t}\right)^2+I_0\left(\frac{\partial w_0}{\partial t}\right)^2-2I_1\frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial x\partial t}-2I_1\frac{\partial v_0}{\partial t}\frac{\partial^2 w_0}{\partial y\partial t}\right. \\ &\quad \left.+I_2\left(\frac{\partial^2 w_0}{\partial x\partial t}\right)^2+I_2\left(\frac{\partial^2 w_0}{\partial y\partial t}\right)^2\right]dxdy \end{aligned} \tag{4}$$

The deformation energy is

$$U=\frac{1}{2}\int_0^{L_{cr}}\int_0^{L_b}\left(N_x\varepsilon_x^0+N_y\varepsilon_y^0+N_{xy}\gamma_{xy}^0+M_xk_x+M_yk_y+M_{xy}k_{xy}\right)dxdy \tag{5}$$

The energy of external power is

$$W=\frac{1}{2}\int_0^{L_{cr}}\int_0^{L_b}N_{(t)}\left(\frac{\partial w_0}{\partial x}\right)^2dxdy \tag{6}$$

The coefficients are

$$A_{11}=A_{22}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E_{(z)}}{1-\mu_{(z)}^2}dz \quad A_{12}=A_{21}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\mu_{(z)}E_{(z)}}{1-\mu_{(z)}^2}dz \quad A_{66}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E_{(z)}}{2(1+\mu_{(z)})}dz$$

$$B_{11}=B_{22}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{zE_{(z)}}{1-\mu_{(z)}^2}dz \quad B_{12}=B_{21}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{z\mu_{(z)}E_{(z)}}{1-\mu_{(z)}^2}dz \quad B_{66}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{zE_{(z)}}{2(1+\mu_{(z)})}dz$$

$$D_{11}=D_{22}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{z^2E_{(z)}}{1-\mu_{(z)}^2}dz \quad D_{12}=D_{21}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{z^2\mu_{(z)}E_{(z)}}{1-\mu_{(z)}^2} \quad D_{66}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{z^2E_{(z)}}{2(1+\mu_{(z)})}dz$$

Functionally graded material plates have the isotropic mechanical properties

$$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$$

Generalized inertia is defined in Eq. 4:

$$\left(I_0, I_1, I_2\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_{(z)}\left(1, z, z^2\right)dz \tag{7}$$

Substituting Eq.3, Eq.4, Eq.5, Eq.6 and Eq.7 into Hamilton's principle $\delta\int_{t_2}^{t_1}\left(T - U + W\right)dt = 0$ ,the governing equation of functionally graded material plates can be written in the following form through calculation and simplification

$$D_{11}w_{xxxx} + \left(2D_{22} + 4D_{66}\right)w_{xxyy} + D_{11}w_{yyyy} + N_{(t)}w_{xx} - I_0\ddot{w} + I_2\ddot{w}_{xx} + I_2\ddot{w}_{yy} = 0 \tag{8}$$

## 3 Dynamic buckling analysis of functionally graded material plates

In y direction, set the buckling mode of functionally graded material plates is $\sin\frac{m\pi y}{l_b}$ [9], set

$$w = X(x)T(t)\sin\frac{m\pi y}{L_b} \tag{9}$$

Substituting Eq. 9 into Eq. 8, the following equations can be obtained:

$$D_{11}X_{xxxx}T - \left(2D_{12} + 4D_{66}\right)X_{xx}T\frac{m^2\pi^2}{L_b{}^2} + D_{11}XT\frac{m^4\pi^4}{L_b{}^4} + N_{(t)}X_{xx}T - I_0XT_{tt} + I_2X_{xx}T_{tt} - I_2XT_{tt}\frac{m^2\pi^2}{L_b{}^2} = 0 \tag{10}$$

Separating the variables, Eq.10 can be transformed into:

$$\begin{cases} X_{xxxx} + \alpha X_{xx} + \beta X + \lambda\xi X - \lambda\eta X_{xx} = 0 \\ T_{tt} + \lambda T = 0 \end{cases} \tag{11}$$

The coefficients of Eq.11 are

$$\alpha = \frac{N_{(t)}}{D_{11}} - \frac{\left(2D_{22} + 4D_{66}\right)m^2\pi^2}{D_{11}L_b{}^2} \qquad \beta = \frac{m^4\pi^4}{L_b{}^4}$$

$$\xi = \frac{I_2}{D_{11}} \qquad \eta = \frac{I_0}{D_{11}} + \frac{I_2m^2\pi^2}{D_{11}L_b{}^2} \tag{12}$$

The solution of Eq. 11 is divergent when λ < 0 ,the system is not stable and the dynamic buckling will occurs[10]. So the Eq. 11 has the dynamic buckling solution, the parameter λ of Eq. 11 must satisfy the following conditions

$$\lambda < 0, \left(\alpha - \lambda\xi\right)^2 - 4\left(\beta - \lambda\zeta\right) > 0 \tag{13}$$

The dynamic buckling solution to Eq. 13 is

$$X(x) = C_1\sin k_1x + C_2\cos k_1x + C_3\sin k_2x + C_4\cos k_2x \tag{14}$$

The coefficients of Eq.14 are

$$\begin{cases} k_1 = \sqrt{\dfrac{\alpha - \lambda\xi + \sqrt{\left(\alpha - \lambda\xi\right)^2 - 4\left(\beta + \lambda\eta\right)}}{2}} \\ k_2 = \sqrt{\dfrac{\alpha - \lambda\xi - \sqrt{\left(\alpha - \lambda\xi\right)^2 - 4\left(\beta + \lambda\eta\right)}}{2}} \end{cases}$$

Substituting Eq. 14 into Eq. 1, the linear equation group can be obtained

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -k_1^2 & 0 & -k_2^2 \\ \sin k_1L_{cr} & \cos k_1L_{cr} & \sin k_2L_{cr} & \cos k_2L_{cr} \\ k_1\cos k_1L_{cr} & -k_1\sin k_1L_{cr} & k_2\cos k_2L_{cr} & -k_2\sin k_2L_{cr} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \tag{15}$$

The conditions of Eq. 15 with non-zero solution are $k_1l = n_1\pi$ , $k_2l = n_2\pi$ [10],where $n_1 = 1,2,3\left(1,2,3\cdots\right), n_2 = n_1 + 1$

The following forms can be obtained

$$\begin{cases} k_1^2 + k_2^2 = \dfrac{\pi^2\left(n_1^2 + n_2^2\right)}{L_{cr}{}^2} = \partial - \lambda\xi \\ k_1^2k_2^2 = \dfrac{n_1^2n_2^2\pi^4}{L_{cr}{}^4} = \beta + \lambda\eta \end{cases} \tag{16}$$

Simplifying Eq.16, the expression of critical dynamic buckling load $N_{cr}$ is

$$N_{cr} = \frac{D_{11}\pi^2\left(n_1^2 + n_2^2\right)}{L_{cr}{}^2} + \frac{D_{11}\pi^4n_1^2n_2^2I_2}{L_{cr}{}^4\left(I_0L_b{}^2 + I_2m^2\pi^2\right)} - \frac{D_{22}m^4\pi^4I_2}{L_b{}^2\left(I_0L_b{}^2 + I_2m^2\pi^2\right)} + \left(2D_{12} + 4D_{66}\right)\frac{m^2\pi^2}{L_b{}^2} \tag{17}$$

Using the boundary conditions and the constraints conditions Eq.1 and Eq.14 the buckling mode can be obtained

$$w = T_{(t)}C_1\left[\left(\frac{n_2\sin\frac{n_1\pi}{L_{cr}} - n_1\sin\frac{n_2\pi}{L_{cr}}}{n_2\cos\frac{n_2\pi}{L_{cr}} - n_1\cos\frac{n_1\pi}{L_{cr}}}\right)\left(\cos\frac{n_1\pi x}{L_{cr}} - \cos\frac{n_2\pi x}{L_{cr}}\right) + \sin\frac{n_1\pi x}{L_{cr}} - \frac{n_1}{n_2}\sin\frac{n_2\pi x}{L_{cr}}\right]\sin\frac{m\pi y}{L_b} \tag{18}$$

## 4 Numerical results and discussion

**Table 1** Material parameters of functionally gradient materials plates

| materials | Elasticity Modulus $E(GPa)$ | Density $\rho(g/cm^3)$ | Poisson's Ratio $\mu$ |
|---|---|---|---|
| Ceramics | 385 | 3.96 | 0.23 |
| Titanium | 108.5 | 4.54 | 0.41 |
| Iron | 155 | 7.86 | 0.291 |
| Copper | 119 | 8.96 | 0.326 |

The relationship between the critical load *Lcr* and critical length *Ncr* in different situations can be acquired and different orders of the buckling modes can be obtained by MATLAB.

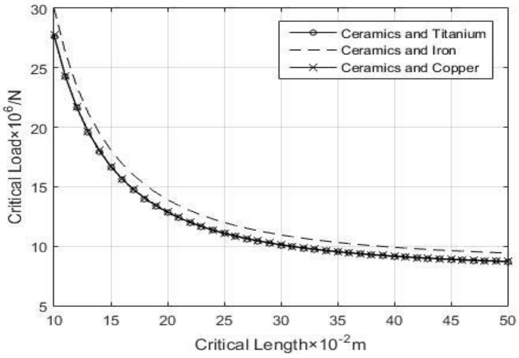



**Figure 1.** Relationship between the critical load and critical length when material is different

The critical load *Ncr* decreases exponentially with the increase of the critical length *Lcr*. Fig. 1 shows with Table 1 that the change of the elastic modulus of the material has obvious influence on the critical load of dynamic buckling of the functionally graded material plate, and the influence of Poisson's ratio is relatively small.

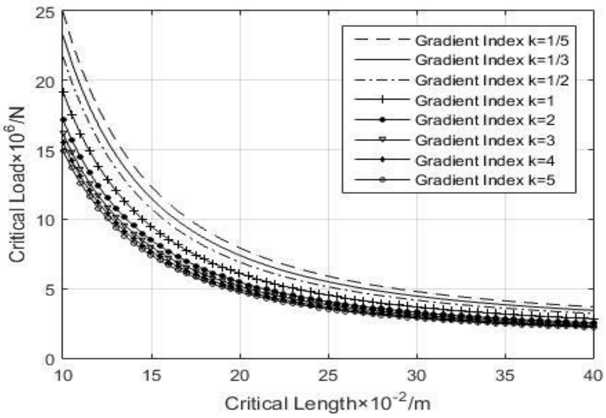



**Figure 2.** Relationship between the critical load and length when gradient index is different

The critical load *Ncr* decreases with the increase of gradient index *k*. The relationship between *Ncr* and *Lcr* changed obviously when the value of *k* is small and the change of *k* value in the range of (0-1) has great impact on *Ncr*. The critical load curve corresponding to different *k* values tends to level with the increase of *Lcr*.

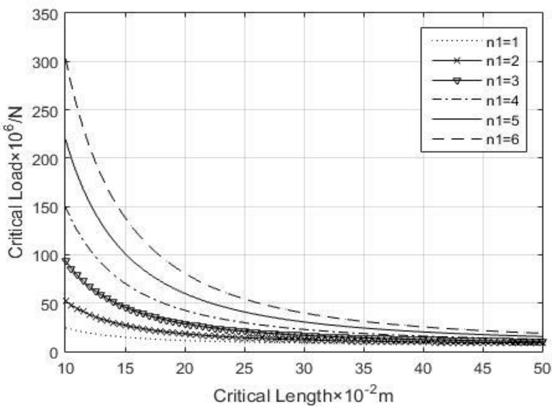
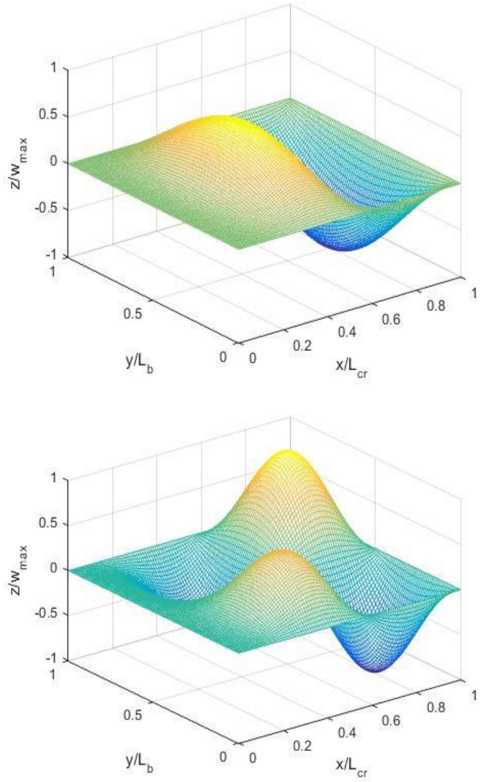



**Figure 3.** Relationship between critical load and length when the modal number of X direction is different

The critical load *Ncr* increases with the increase of modal number *n1*. The change of *Ncr* with *Lcr is* more obvious in the condition of high value of *n1*




$(a) n_1 = 2, m = 1$

$(b) n_1 = 2, m = 2$

**Figure 4.** Different orders of the buckling modes

The amplitude modes w of functionally graded material plates increasing positively in X direction.

## 5 Conclusion

1. The analytical expression of dynamic buckling critical load (17) and buckling modal expression (18) of functionally graded material plates are obtained. The buckling modes are shown in Fig.3.

2. Calculate Eqs. (17) and (18) by MATLAB, and discuss the influence of critical length, gradient index, material property and order of buckling modes on the critical buckling load and buckling mode. The results show that the critical load of dynamic buckling decreases exponentially with the increase of length, and the critical load of buckling decreases with the increase of gradient index *k*. The gradient index *k* have great influence on buckling critical load in conditions that the gradient index *k* changes in the range of (0-1). The buckling critical load is greatly affected by the elastic modulus and the boundary conditions have great influence on the buckling mode of the functionally graded material plates.

## Acknowledgements

The project is supported by Science Foundation for https://doi.org/10.1051/e3sconf/20183802013

Young Scholars of Shanxi Province of China（NO.201601D021127）and the Graduate Educational Reformation of Shanxi Province of China（NO.2015JG41）

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