

# The study of chaotic and regular regimes of the fractal oscillators FitzHugh-Nagumo

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**Abstract.** In this paper we study the conditions for the existence of chaotic and regular oscillatory regimes of the hereditary oscillator FitzHugh-Nagumo (FHN), a mathematical model for the propagation of a nerve impulse in a membrane. To achieve this goal, using the non-local explicit finite-difference scheme and Wolf's algorithm with the Gram-Schmidt orthogonalization procedure and the spectra of the maximum Lyapunov exponents were also constructed depending on the values of the control parameters of the model of the FHN. The results of the calculations showed that there are spectra of maximum Lyapunov exponents both with positive values and with negative values. The results of the calculations were also confirmed with the help of oscillograms and phase trajectories, which indicates the possibility of the existence of both chaotic attractors and limit cycles.

## 1 Introduction

The classical model of the FitzHugh-Nagumo oscillator (FN) was proposed Gin-Ichi Nagumo, Suguru Arimoto and Shuji Yoshizawa in 1962 [1], and a year earlier - by Richard FitzHugh [2] for a mathematical description in biophysics of the propagation of the nerve impulse in the membrane. The mathematical model of the FHN oscillator is a kind of Van der Pol-Duffing oscillator, and therefore this system has stable oscillations (limit cycles) and chaotic dynamics. Since the FN oscillator describes limit cycles, it can be used in modeling cyclic processes. For example, the temporal dynamics of seismic activity. In work [3] the diffusion model of FN migration of seismic activity along the fault is considered.

In the case of the heredity (heredity) effect, the classical FHN model has a generalization, according to which the current state of the system depends from a finite number of previous states. Such an effect is common in fractal media [4] and is studied in the framework of hereditary mechanics [5]. The hereditary oscillator FN (FHN), by analogy with the work of Volterra [6] can be described by means of integro-differential equations with difference kernels, which are called memory functions. Due to the wide distribution of power laws in nature [7], it makes sense to choose memory functions

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as power functions. Such an assumption makes it possible to proceed to equations with derivatives of fractional orders, which are studied in the framework of fractional calculus [8], and the hereditary oscillators in this case are called fractal ones.

In the works of the author [9, 10] the FHN oscillator (Cauchy problem) was considered and a nonlocal explicit finite difference scheme is obtained and investigated that allows us to find a numerical solution to the corresponding Cauchy problem. Oscillograms and phase trajectories were constructed and investigated using a numerical algorithm. This work is a logical continuation of the above articles and aims to establish the conditions for the existence of chaotic and regular regimes of the FHN.

## 2 Statement of the problem

Let us consider the generalized model of the nonlinear fractal oscillator:

$$\begin{cases} \partial_{0r}^{\beta} x(\tau) = c \cdot (y(t) - x(t) - \frac{x^3(t)}{3} + z), \\ \partial_{0r}^{\gamma} x(\tau) = -\frac{(x(t) - a + by(t))}{c}, \end{cases} \quad (1)$$

where the differential operators:

$$\partial_{0r}^{\beta} x(\tau) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\ddot{x}(\tau) d\tau}{(t-\tau)^{1-\beta}}, \quad \partial_{0r}^{\gamma} x(\tau) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{x}(\tau) d\tau}{(t-\tau)^{\gamma}}, \quad (2)$$

are defined in the sense of Gerasimov-Caputo [11, 12] with fractional orders  $1 < \beta < 2$ ,  $0 < \gamma < 1$ ,  $a$ ,  $b$ ,  $c$  are constants that satisfy conditions  $1 - 2b/3 < a < 1$ ,  $0 < b < 1$ ,  $b < c^2$ ,  $x(t)$  is the membrane potential,  $z$  is the intensity of the stimulus, the constant in the first approximation, which can also have the form of a rectangular pulse or delta function,  $t \in [0, T]$  is the process time,  $T > 0$  is the simulation time,  $x_0$  and  $y_0$  are the initial conditions.

The model of the FHN (1) describes a fractal nonlinear oscillator and in the case  $\beta = 2$  and  $\gamma = 1$  becomes a classical nonlinear oscillator of the FN. We write (1) as a system in the case when  $\{\beta\} \leq \{\gamma\}$  [13]:

$$\begin{cases} \partial_{0r}^{\alpha_1} x_1(\tau) = x_2(t), \alpha_1 = \gamma, \\ \partial_{0r}^{\alpha_2} x_2(\tau) = a + bz - qx(t) - gx^3(t) - c(x^2 + p)x_2(t), \alpha_2 = \beta - \gamma, \end{cases} \quad (3)$$

where  $p = b/c^2 - 1$ ,  $q = 1 - b$ ,  $g = b/3$  were taken from [1].

The case when  $\{\beta\} > \{\gamma\}$  leads to an increase in the system (3) by one equation and will be considered in other works.

In what follows we shall consider system (3) together with the equations in Variations for constructing the spectrum of Lyapunov exponents using Wolf's algorithm [14]. We form the equations in variations:

$$\begin{cases} \partial_{0r}^{\alpha_1} \Delta x_1(\tau) = \Delta x_2(t), \\ \partial_{0r}^{\alpha_2} \Delta x_2(\tau) = (-2cx(t)y(t) - q - 3gx^2(t))\Delta x_1 - c(x^2(t) + p) \cdot \Delta x_2(t), \end{cases} \quad (4)$$

The right-hand side of system (4) is the product of the elements of the Jacobi matrix system (3) by the corresponding elements of the vectors (variations)  $\{\Delta x_1, \Delta x_2\}$ .

Systems (3) and two sets of systems of the type (4) with orthogonal initial conditions are solved jointly with the help of numerical methods, for example, finite difference schemes. Next, vectors are constructed that are subjected to a procedure Gramma-Schmidt orthogonalization and the maximum values are calculated Lyapunov [14]. The Lyapunov exponents, constructed from different values

control parameters of the initial system (1) in aggregate form a spectrum. The positive values of the maximum Lyapunov exponent correspond to the regions the existence of a chaotic regime, and the negative — the realm of existence regular regime.

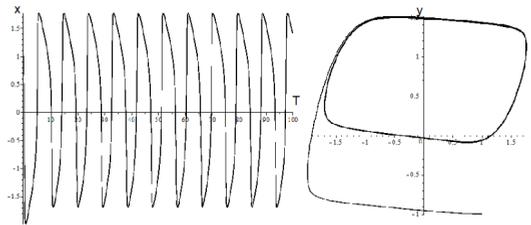
### 3 Results of the study

As a numerical method for solving systems (3) and (4), we chose a nonlocal explicit finite-difference scheme with a regular grid with an amount nodes  $N = 200$  and a sampling rate  $\tau = 0.05$ . The properties of such a scheme were are considered in the author's papers [9] and [10], and in [15] the condition stability and convergence of schemes of this type.

Note that since the parameters of model (1) are limited, the effects associated with with the rigidity of the system, we do not observe.

Note that the areas of changes in the values of the control parameters for the covering of the spectra of the Lyapunov maximal exponents can be extended if the conditions of the task require this. However, at the same time, computational costs will increase.

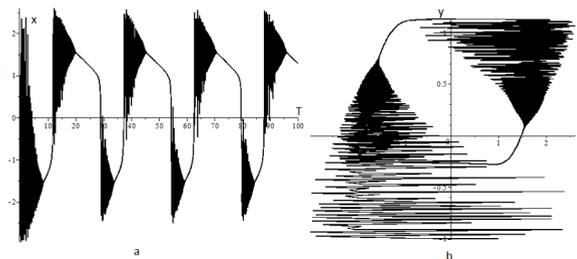
**Example 1.** Consider the values of the control parameters:  $T = 100, c = 3, a = 0.7, b = 0.8, z = -0.68, \beta = 1.6, \gamma = 1$ .



**Figure 1.** Oscillogram *a*) and phase trajectory *b*)

On the oscillogram (Fig. 1a) we can see that at the selected values control parameters of the amplitude of the oscillations reaches a constant level, and the phase trajectory (Fig. 1b) has a closed orbit and reaches the limiting cycle (regular attractor).

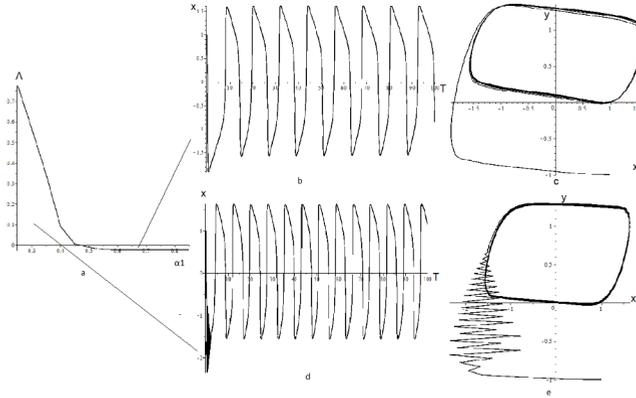
**Example 2.** Consider the following values of the control parameters:  $T = 100, c = 7, a = 0.7, b = 0.8, z = -0.68, \beta = 1.6, \gamma = 1$  and construct the graphs.



**Figure 2.** Oscillogram *a*) and phase trajectory *b*)

In Fig. 2 shows an example of a chaotic attractor, when the chaotic regime occurs within a certain limited region in the phase plane (Fig. 2b).

**Example 3.** Let us choose the values of the control parameters for the construction of the spectrum Lyapunov exponents:  $N = 2000, T = 100, c = 3, a = 0.7, b = 0.8, z = -0.4, \alpha_2 = 1, x_0 = 1, y_0 = -1$ .

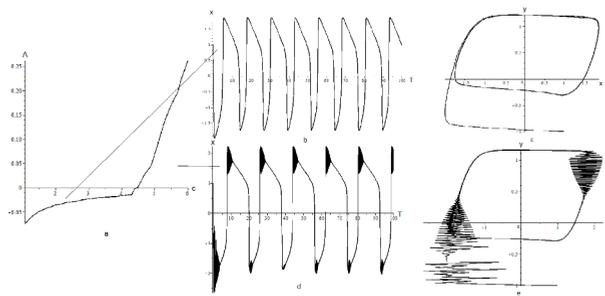


**Figure 3.** a) The spectrum of Lyapunov's maximum exponents from the values  $\alpha_1 \in [0.1, 0.85]$  with step  $h = 0.05$ ; b) the oscillogram for  $\alpha_1 = 0.6$ ; c) the phase trajectory for  $\alpha_1 = 0.6$ ; d) the oscillogram for  $\alpha_1 = 0.4$ ; e) the phase trajectory for  $\alpha_1 = 0.4$ .

In Fig. 3a shows the spectrum of the maximum Lyapunov exponents, constructed from the values of parameter  $\alpha_1$ , on which you can see the areas as positive  $\alpha_1 \in [0.1, 0.457]$ , and negative values of  $\alpha_1 \in (0.457, 0.85]$ . Next, oscillograms and phase trajectories are given as an example, constructed according to the values of the parameter  $\alpha_1$  from these regions (in Figure 3 marked with arrows). We see that the positive values of the indicator Lyapunov correspond to chaotic regimes, and negative ones correspond to regular modes.

We consider the spectrum of the maximal Lyapunov exponent constructed in depending on the values of the parameter  $c$ .

**Example 4.** Parameter values:  $T = 100, a = 0.7, b = 0.8, z = -0.4, \alpha_1 = 0.7, \alpha_2 = 1, x_0 = 1, y_0 = -1$ .



**Figure 4.** a) The spectrum of Lyapunov's maximum exponents from the values  $c \in [1, 6]$  with step  $h = 0.1$ ; b) the oscillogram for  $c = 4$ ; c) the phase trajectory for  $c = 4$ ; d) the oscillogram for  $c = 7$ ; e) the phase trajectory for  $c = 7$ .

According to the spectrum of Fig. 4a, we see that the values of the parameter  $c$  from interval  $[0, 4.4]$  there correspond regular regimes, otherwise we will receive chaotic regimes.

We note that the boundaries of the spectra of the maximal Lyapunov exponents can be To clarify, taking a smaller step of discretization, but this will lead to large computing costs.

## 4 Acknowledgment

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## 5 Conclusion

Using the spectra of Lyapunov maximum exponents constructed in accordance with the values of the control parameters  $\alpha_1$  and  $c$  of the system of FHN (1), it was shown that for such a system both chaotic regimes and regular regimes exist. A more detailed analysis of the system of FHN can be obtained by constructing an atlas of dynamic regimes. However, such a task will require considerable computing resources.

Another continuation of the work may be related to the generalization of the model (1) to the case of variable fractional orders  $\beta(t)$  and  $\gamma(t)$  by analogy with the papers [16] and [17].

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