Lamb waves in multilayered media: 3D and 6D formalisms

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Abstract. A mathematical model for analyzing Lamb waves propagating in stratified media with arbitrary elastic anisotropy is worked out. The model incorporates a combined Fundamental Matrix (FM) and Modified Transfer Matrix (MTM) methods. Multilayered unbounded plates with different types of boundary conditions imposed on the outer surfaces are considered. Closed form fundamental matrices and secular equations for dispersion relations are derived.

1. Introduction

The genuine Lamb waves [1] are surface acoustic waves propagating in an unbounded isotropic homogeneous plate subjected to traction-free boundary conditions at the outer planes:

The genuine Lamb waves [1] are surface acoustic waves propagating in an unbounded isotropic homogeneous plate subjected to traction-free boundary conditions at the outer planes:

$$\mathbf{t}_{\boldsymbol{\nu}}(\mathbf{x},t) \equiv \boldsymbol{\nu} \cdot \mathbf{C} \cdot\cdot \nabla_{\mathbf{x}} \mathbf{u}(\mathbf{x},t)\big|_{x'=-h}^{x'=0} = 0. \qquad (1.1)$$

In (1.1) $\mathbf{t}_{\boldsymbol{\nu}}$ are surface tractions, $\boldsymbol{\nu}$ is the unit normal to one of the boundary planes, $\mathbf{C}$ is the fourth order elasticity tensor (in the original Lamb's analysis tensor $\mathbf{C}$ was isotropic), $\mathbf{u}$ is the displacement field, t is time,

$$x' = \mathbf{x} \cdot \boldsymbol{\nu} \qquad (1.2)$$

is the coordinate normal to the plane $\mathbf{x} \cdot \boldsymbol{\nu} = 0$, and h is the depth of a plate; see, Fig.1, where $\mathbf{n}$ is the unit vector normal to the wave front.

We shall also consider clamped boundary planes:

$$\mathbf{u}(\mathbf{x},t)\big|_{x'=-h}^{x'=0} = 0 \qquad (1.3)$$

and mixed boundary conditions for the traction-free upper plane and clamped bottom plane:

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$$\mathbf{t_v}(\mathbf{x},t)\big|^{x'=0} = 0$$

$$\mathbf{u}(\mathbf{x},t)\big|_{x'=-h} = 0 \tag{1.4}$$

For a multilayered plate, the interface boundary conditions are conditions of the ideal mechanical contact:

$$\mathbf{t_v}^{(n)}(\mathbf{x},t)\Big|_{x'=-h_n} = \mathbf{t_v}^{(n+1)}(\mathbf{x},t)\Big|^{x'=0}$$

$$\mathbf{u}^{(n)}(\mathbf{x},t)\Big|_{x'=-h_n} = \mathbf{u}^{(n+1)}(\mathbf{x},t)\Big|^{x'=0} , \tag{1.5}$$

where index n is referred to the corresponding layer. Equations (1.5) are written in the local coordinate systems with origins placed at the upper boundaries of the corresponding layers.

Regarding polarization (that is the direction of particle motion at the wave front), the genuine Lamb wave resemble Rayleigh wave [2], which propagate in a halfspace with the traction-free boundary condition at the boundary plane, but Lamb wave differ from Rayleigh wave in absence of necessity to impose Sommerfield's emission condition:

$$\mathbf{u}(\mathbf{x},t) = O(|x'|^{-1}), \quad |x'| \to \infty . \tag{1.6}$$

Sommerfield's condition (1.6) is used to remove solutions for Rayleigh waves that do not attenuate with depth, as being physically unreasonable.

Generally, Lamb wave in a particular layer is sought in the form [3 − 6]:

$$\mathbf{u}(\mathbf{x},t) = \left(\sum_{k=1}^{6} C_k \mathbf{m}_k \, f_k(x'') \right) e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} , \tag{1.7}$$

where C_k are the unknown complex coefficients defined up to a multiplier from the boundary or interface conditions (1.1) − (1.5); $\mathbf{m}_k$ are the unknown polarization vectors; f_k are also unknown scalar complex-valued functions; the exponential multiplier $e^{ir(\mathbf{n}\cdot\mathbf{v}-ct)}$ corresponds to propagation of the plane wave front along direction $\mathbf{n}$ with the phase speed c; r is the wave number; dimensionless coordinate x'' in (1.7) is defined by

$$x'' = irx' . \tag{1.8}$$

In most of the works on surface acoustic waves a problem of constructing the displacement field (1.7) is not considered. Instead, by satisfying all the boundary and interface conditions, the disperse relations between the phase speed c and wave number r are derived either analytically for single-layered plates [7, 8], or numerically for multilayered plates [9 − 13]. Quite often relations $c - r$ are replaced by relations $c - \omega$, where ω is the frequency:

$$\omega = rc . \tag{1.9}$$

However, even for a simpler problem of finding disperse relations, polarization vectors $\mathbf{m}_k$ and displacement distribution functions f_k need generally to be constructed. Vectors $\mathbf{m}_k$ are either genuine or generalized eigenvectors of the Christoffel equation, introduced later. Analogously, functions f_k are eigenfunctions of an ordinary differential equation, to which the Christoffel equation is the characteristic equation.

Herein, an alternative approach will be introduced, which removes necessity to create and analyze vectors $\mathbf{m}_k$ and functions f_k. This approach is based on constructing the fundamental matrix for a system of the first-order differential equations and finding eigensolutions for the Sturm-Liouville problem; see also [14, 15], where a method based on constructing the fundamental matrix and utilizing Stroh's six-dimensional formalism [16],

was used for studying Flouqet's problem for periodically layered composite plates. In the present paper a different six-dimensional formalism developed in [17, 18] will be used. Applying this formalism will allow us to obtain closed form fundamental matrices and secular equations for Lamb waves propagating in multilayered plates with arbitrary elastic anisotropy.

2. Governing equations

All the regarded layers are assumed to be homogeneous, anisotropic, and linearly hyperelastic. Equation of motion for homogeneous anisotropic elastic medium can be written in the form:

$$\mathbf{L}(\partial_\mathbf{x}, \partial_t)\mathbf{u} \equiv \mathrm{div}_\mathbf{x}\,\mathbf{C} \cdot\cdot \nabla_\mathbf{x}\mathbf{u} - \rho\ddot{\mathbf{u}} = 0, \qquad (2.1)$$

where ρ is the material density, and $\mathbf{C}$ is the elasticity tensor assumed to be *positive definite*:

$$\underset{\mathbf{S}\in sym(R^3\otimes R^3),\ \mathbf{A}\neq 0}{\forall\mathbf{S}}\quad (\mathbf{S}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{S}) \equiv \sum_{i,j,\,m,n} S_{ij}C^{ijmn}S_{mn} > 0. \qquad (2.2)$$

Condition (2.2) ensures strong ellipticity of tensor $\mathbf{C}$:

$$\underset{\mathbf{a},\mathbf{b}\in R^3,\ \mathbf{a},\mathbf{b}\neq 0}{\forall\mathbf{a},\mathbf{b}}\quad \mathbf{a}\otimes\mathbf{b}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{b}\otimes\mathbf{a} > 0. \qquad (2.3)$$

The latter condition is equivalent to positive definiteness of the acoustic tensor

$$\mathbf{A}(\mathbf{b}) \equiv \mathbf{b}\cdot\mathbf{C}\cdot\mathbf{b} \qquad (2.4)$$

at any $\mathbf{b}\in R^3$, $\mathbf{b}\neq 0$.

2.1. 3D-formalism

Now, we outline a formal procedure for finding the unknown polarization vectors $\mathbf{m}_k$ and functions f_k. Substituting representation (1.7) into Eq. (2.1) yields a system of the second-order matrix equations with respect to functions $\mathbf{m}_k f_k$:

$$\left(\mathbf{A}_1\partial^2_{x''} + (\mathbf{A}_2 + \mathbf{A}_2{}^t)\partial_{x''} + \mathbf{A}_3\right)\cdot\mathbf{m}_k f_k = 0, \qquad (2.5)$$

where

$$\mathbf{A}_1 = \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{\nu}, \quad \mathbf{A}_2 = \mathbf{\nu}\cdot\mathbf{C}\cdot\mathbf{n}, \quad \mathbf{A}_3 = \mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2\mathbf{I}. \qquad (2.6)$$

Assuming functions f_k to have Euler's exponential representation

$$f_k(x'') = e^{ir\gamma_k x'} \qquad (2.7)$$

with the unknown parameters γ_k (these are sometimes called the Christoffel parameters) and substituting this representation into Eq. (2.5), we arrive at the Christoffel equation:

$$\left(\mathbf{A}_1\gamma_k^2 + (\mathbf{A}_2 + \mathbf{A}_2{}^t)\gamma_k + \mathbf{A}_3\right)\cdot\mathbf{m}_k = 0. \qquad (2.8)$$

The latter equation reveals that exponential parameters γ_k are roots of a polynomial of the sixth degree with respect to γ_k:

$$\det\left(\mathbf{A}_1\gamma_k^2 + (\mathbf{A}_2 + \mathbf{A}_2{}^t)\gamma_k + \mathbf{A}_3\right) = 0. \qquad (2.9)$$

On obtaining the Christoffel parameters from Eq. (2.9), we can find polarization vectors $\mathbf{m}_k$ as the eigenvectors of a matrix in the left-hand side of Eq. (2.8).

Remark 2.1. If a matrix in the left-hand side of Eq. (2.8) has Jordan blocks (this case is known as the non-semisimple degeneracy), then representation (2.7) becomes insufficient. Within framework of both 3D and 6D-formalisms this case was studied in [19]. □

2.2. 6D-formalism

Now it will be demonstrated that the preceding stage of finding the Christoffel parameters γ_k and polarization vectors $\mathbf{m}_k$ can be avoided.

Let

$$\mathbf{f}_k(x'') \equiv \mathbf{m}_k\, f_k(x'')$$
$$\mathbf{v}_k(x'') \equiv \partial_{x''}\mathbf{f}_k(x'') \tag{2.10}$$

be two vector-valued functions. Representation (2.10) ensures that these functions are dimensionless and have the same orientation in R^3. With these functions Eq. (2.5) can be represented in a form of a six-dimensional system of ordinary differential equations of the first order:

$$\frac{d}{dx''}\hat{\mathbf{z}}_k(x'') = \hat{\mathbf{G}}\cdot\hat{\mathbf{z}}_k(x''), \tag{2.11}$$

where $\hat{\mathbf{z}}_k(x'')$ is a six-dimensional vector:

$$\hat{\mathbf{z}}_k(x'') = \begin{pmatrix} \mathbf{f}_k(x'') \\ \mathbf{v}_k(x'') \end{pmatrix} \tag{2.12}$$

and $\hat{\mathbf{G}}$ is a 6×6 matrix:

$$\hat{\mathbf{G}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{A}_1^{-1}\cdot\mathbf{A}_3 & -\mathbf{A}_1^{-1}\cdot(\mathbf{A}_2 + \mathbf{A}_2{}') \end{pmatrix}. \tag{2.13}$$

In (2.13) $\mathbf{0}$ is the zero and $\mathbf{I}$ is the identity 3×3 matrix.

PROPOSITION 2.1. Matrix $\hat{\mathbf{G}}$ is not singular, provided

$$c^2 \notin \frac{1}{\rho}\mathrm{Sp}(\mathbf{A}_3), \tag{2.14}$$

where Sp denotes spectrum of the corresponding matrix.

Proof. The proof flows out from considering the determinant of matrix $\hat{\mathbf{G}}$:

$$\det(\hat{\mathbf{G}}) = \det\left(\mathbf{A}_1^{-1}\cdot\mathbf{A}_3\right) = \det(\mathbf{A}_1^{-1})\det(\mathbf{A}_3) \tag{2.15}$$

and since matrix $\mathbf{A}_1$ is not singular, due to assuming positive definiteness of the elasticity tensor, condition (2.14) follows from expression (2.6) for $\mathbf{A}_3$.□

As is known from the theory of ordinary differential equations [20, 21], the solution of Eq. (2.11) satisfying Cauchy's initial conditions at some value x_0'', can be represented in a form:

$$\hat{\mathbf{z}}(x'') = \hat{\mathbf{R}}(x'' - x_0'')\cdot\hat{\mathbf{z}}_0, \tag{2.16}$$

where $\hat{\mathbf{R}}$ is the fundamental matrix (fundamental system); see [21, Ch. IV, §2, n.4]:

$$\hat{\mathbf{R}}(x'' - x_0'') = e^{\hat{\mathbf{G}}(x'' - x_0'')} \tag{2.17}$$

and

$$\hat{\mathbf{z}}_0 \equiv \begin{pmatrix} \mathbf{f}_0 \\ \mathbf{v}_0 \end{pmatrix} \qquad (2.18)$$

is a 6-dimensional vector specifying the initial conditions at x_0''.

Remark 2.2. The solution in a form (2.16) is valid for any non-singular matrix $\hat{\mathbf{G}}$, regardless of its (semi)simplicity or non-semisimple degeneracy; see [20, Ch. IV, §5]. □

3. Solution for a homogeneous layer

3.1. Traction-free plate

Boundary conditions (1.1) can be rewritten in terms of convolution of a special 3×6 matrix composed of matrices (2.6) and vector $\hat{\mathbf{z}}(x'')$:

$$\left(\mathbf{A}_2 ; \mathbf{A}_1 \right) \cdot \hat{\mathbf{z}}(x'') \Big|_{x''=-irh}^{x''=0} \equiv \left(\mathbf{A}_1 \cdot \mathbf{v}(x'') + \mathbf{A}_2 \cdot \mathbf{f}(x'') \right) \Big|_{x''=-irh}^{x''=0} = 0 . \qquad (3.1)$$

Substituting expression (2.16) for the vector $\hat{\mathbf{z}}(x'')$ into conditions (3.1) and taking $x_0'' = 0$, we arrive at the following equation:

$$\begin{pmatrix} \mathbf{A}_2 & \mathbf{A}_1 \\ \mathbf{A}_2 \cdot \mathbf{R}_{11} + \mathbf{A}_1 \cdot \mathbf{R}_{21} & \mathbf{A}_2 \cdot \mathbf{R}_{12} + \mathbf{A}_1 \cdot \mathbf{R}_{22} \end{pmatrix} \cdot \hat{\mathbf{z}}_0 = 0 , (3.2)$$

where $\mathbf{R}_{ij}$ are 3×3-matrix components of the fundamental matrix $\hat{\mathbf{R}}$ at $x'' = -irh$. Existence of a non-trivial 6-dimensional vector $\hat{\mathbf{z}}_0$ satisfying Eq. (3.2) is equivalent to vanishing the following determinant:

$$\det \begin{pmatrix} \mathbf{A}_2 & \mathbf{A}_1 \\ \mathbf{A}_2 \cdot \mathbf{R}_{11} + \mathbf{A}_1 \cdot \mathbf{R}_{21} & \mathbf{A}_2 \cdot \mathbf{R}_{12} + \mathbf{A}_1 \cdot \mathbf{R}_{22} \end{pmatrix} = 0 . (3.3)$$

The last equation can be rewritten in a form of vanishing the determinant of the third-order:

$$\det \left(\mathbf{B}_{\text{left}} \cdot \hat{\mathbf{R}} \cdot \mathbf{B}^{\text{right}} \right) = 0 , (3.4)$$

where

$$\mathbf{B}_{\text{left}} = \left(\mathbf{A}_2 , \mathbf{A}_1 \right) , \qquad \mathbf{B}^{\text{right}} = \begin{pmatrix} \mathbf{I} \\ -\mathbf{A}_1^{-1} \cdot \mathbf{A}_2 \end{pmatrix} . \qquad (3.5)$$

Equation (3.4) is the secular equation for a traction-free plate.

3.2. Clamped plate

Boundary conditions (1.3) can be rewritten in terms of convolution of a special 3×6 matrix and vector $\hat{\mathbf{z}}(x'')$:

$$\left(\mathbf{I} ; \mathbf{0} \right) \cdot \hat{\mathbf{z}}(x'') \Big|_{x''=-irh}^{x''=0} \equiv \mathbf{f}(x'') \Big|_{x''=-irh}^{x''=0} = 0 . \qquad (3.6)$$

Substituting expression (2.16) for the vector $\hat{\mathbf{z}}(x'')$ into conditions (3.6) and taking $x_0'' = 0$, we arrive at:

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R}_{11} & \mathbf{R}_{12} \end{pmatrix} \cdot \hat{\mathbf{z}}_0 = 0 . \qquad (3.7)$$

As before, components $\mathbf{R}_{ij}$ are 3×3-matrix components of the fundamental matrix $\hat{\mathbf{R}}$ at $x'' = -irh$. Existence of a non-trivial 6-dimensional vector $\hat{\mathbf{z}}_0$ satisfying Eq. (3.7) is equivalent to vanishing the determinant:

$$\det \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R}_{11} & \mathbf{R}_{12} \end{pmatrix} = 0. \quad (3.8)$$

The latter equation admits representation in a form of Eq. (3.4) with

$$\mathbf{B}_{\text{left}} = (\mathbf{I}, \mathbf{0}), \qquad \mathbf{B}^{\text{right}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix}. \quad (3.9)$$

3.3. Plate with mixed boundary conditions

By analogy with Eqs. (3.2) and (3.7), boundary conditions (1.4) yield:

$$\begin{pmatrix} \mathbf{A}_2 & \mathbf{A}_1 \\ \mathbf{R}_{11} & \mathbf{R}_{12} \end{pmatrix} \cdot \hat{\mathbf{z}}_0 = 0. \quad (3.10)$$

It can be shown that Eq. (3.10) can be represented in a form (3.4) with

$$\mathbf{B}_{\text{left}} = (\mathbf{I}, \mathbf{0}), \qquad \mathbf{B}^{\text{right}} = \begin{pmatrix} \mathbf{I} \\ -\mathbf{A}_1^{-1} \cdot \mathbf{A}_2 \end{pmatrix}. \quad (3.11)$$

Remark 3.1. a) Thus, dispersion relations for Lamb waves propagating in a single layered plate with the considered three types of boundary conditions, satisfy the unique secular equation (3.4) with different matrices $\mathbf{B}_{\text{left}}$, $\mathbf{B}^{\text{right}}$.

b) Problems and algorithms for numerical evaluation of a matrix exponential are discussed in [22 - 25]. □

4. Solution for a layered plate

Let a multilayered plate consists of N-layers, Fig. 2. The lower index n will be referred to the corresponding layer.

Assuming ideal mechanical contact at the interfaces and taking into account non-degeneracy of matrix $\mathbf{A}_1$, Eqs. (1.5) can be replaced by equations of continuity of the solution of Eq. (2.16) across the interfaces:

$$\hat{\mathbf{z}}_{n+1}(0) = \hat{\mathbf{z}}_n(-irh_n), \quad n = 1, ..., N-1. \quad (4.1)$$

Equations (4.1) are written in local coordinate systems with origin at the upper boundary of each layer. Conditions (4.1) allow us to represent $\hat{\mathbf{z}}_N(-irh_N)$ in terms of vector $\hat{\mathbf{z}}_0$, which specifies initial conditions at the upper boundary, and convolution of the fundamental matrices:

$$\hat{\mathbf{z}}_N(-irh_N) = \hat{\mathbf{R}} \cdot \hat{\mathbf{z}}_0, \quad (4.2)$$

where

$$\hat{\mathbf{R}} = e^{-ir\hat{\mathbf{G}}_N h_N} \cdot ... \cdot e^{-ir\hat{\mathbf{G}}_1 h_1}. \quad (4.3)$$

Remark 4.1. a) Matrix $\hat{\mathbf{R}}$ plays role of the *transfer matrix*, as it transfers initial values $\hat{\mathbf{z}}_0$ at $x'' = 0$ to $\hat{\mathbf{z}}_N(-irh_N)$. Within framework of the 3-dimensional formalism applied to surface wave analysis, the transfer matrices were introduced in [26, 27]; see also [28 − 30], where different modifications of the transfer matrix method are suggested to improve numerical stability.

b) It should be noted that generally

$$e^{-ir\hat{\mathbf{G}}_N h_N} \cdot \ldots \cdot e^{-ir\hat{\mathbf{G}}_1 h_1} \neq e^{-ir\left(\sum \hat{\mathbf{G}}_n h_n\right)}, \quad (4.4)$$

because of the non-commutative property of matrices $\hat{\mathbf{G}}_n$.

c) However, matrices $\hat{\mathbf{G}}_n$ and $\exp(\hat{\mathbf{G}}_n)$ commute, whatever matrix $\hat{\mathbf{G}}_n$ may be.□
Now, by applying boundary conditions to Eq. (4.2), the secular equation for dispersion curves in a multilayered plate takes form (3.4) with matrix $\hat{\mathbf{R}}$ defined by (4.3), and the auxiliary matrices $\mathbf{B}_{\text{left}}$, $\mathbf{B}^{\text{right}}$ defined by (3.5) for a traction-free plate, by (3.9) for a clamped plate, and by (3.11) for a plate with mixed boundary conditions.

5. Conclusions

The Cauchy six-dimensional formalism is developed for analysing propagation of Lamb waves in anisotropic multilayered plates. Comparison with the formalism, known as the 3D formalism, is done, revealing the anticipated benefits of the Cauchy formalism in respect of its ability to construct the closed form dispersion equation and the corresponding solution for dispersion curves.

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