

Contact problem for foundations with multilayer nonuniform coatings of variable thickness

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Abstract. The article is devoted to study of the contact problem for a punch with a complex shape and a base with a coating consisting of nonuniform layers of variable thickness. Such a foundations are often found in practice. Coating layers in them can play a role, for example, heat or electric insulators. Such layers can be used as protection against mechanical stress on the main structure. Mathematical model of the problem is constructed. It is a mixed integral equation containing functions that describe the properties and thicknesses of the layers, as well as the shapes of the contacting bodies, and additional integral conditions. Analytical solution of the problem is presented for one of the formulation options. In the resulting solution, the functions associated with the properties and forms of bodies are distinguished by separate terms. This allows one to perform high-quality calculations and analysis of the behavior of the punch on the layer, even if these functions are rapidly changing.

1 Introduction

A number of structures and mechanisms are covered with different layers. It can be some insulators, they can be used as protection against mechanical stress on the main structure. The presence of layers can be associated with the manufacturing process of the structure, when each subsequent layer differs in properties from the previous one due to differences in properties in different batches of material. Layers with different properties can be obtained, for example, by spraying, deposition, immersion in a melt, enameling, using additive technologies, etc.

The study of plane contact problems for bodies with complex coatings began with solving problems for foundations with coatings having a complex shape, consistent with the shape of the acting punch [1] and for bodies with nonuniform coatings [2]. The mathematical model of such problems contained one complex function. The case when the shapes of the contacting bodies are nonconformal, or the coating is inhomogeneous, and the punch profile is described by a complex function was more complicated. Solutions for described problems were constructed in [3–5]. Further studies made it possible to obtain similar solutions for bodies with multilayer nonuniform coatings and functionally gradient coatings [6] (but in this paper layers have constant thickness and punch base shape describes by “smooth” function).

This paper describes a solution for the most general case when: 1) we have multilayer coating, 2) coating

layers are nonuniform and have variable thickness, 3) punch base form describes by complex function.

2 Plane contact problem for layered foundation

The viscoelastic aging layer lies without friction on the underlying non-deformable foundation. It is covered by several thin elastic layers. There is smooth contact between all layers. The constant thickness of the lower layer is equal to h_{lower} . The thicknesses of the coating layers $h_k(x)$ are variable ($k=1, \dots, N$, where N is number of layers). The viscoelastic properties of the lower layer do not depend on the coordinate, but Young's modulus depend on time and Poisson's ratio is constant, i.e. $E_{\text{lower}} = E_{\text{lower}}(t)$, $\nu_{\text{lower}} = \text{const}$. The properties of the layers are constant in time but depend on the coordinate x , i.e. $E_k = E_k(x)$, $\nu_k = \nu_k(x)$. It is assumed that the rigidity of the lower layer exceeds the coating layers rigidities.

Starting from time τ_0 , a rigid punch begins to be pressed into the above coating. Its lower base is described by the function $f(x)$. There is no friction between the punch and the top layer of foundation. It is assumed that applied force has such value that contact occurs along the entire width of the punch $2a$. As a result of this interaction, the punch is immersed in layered foundation and rotates.

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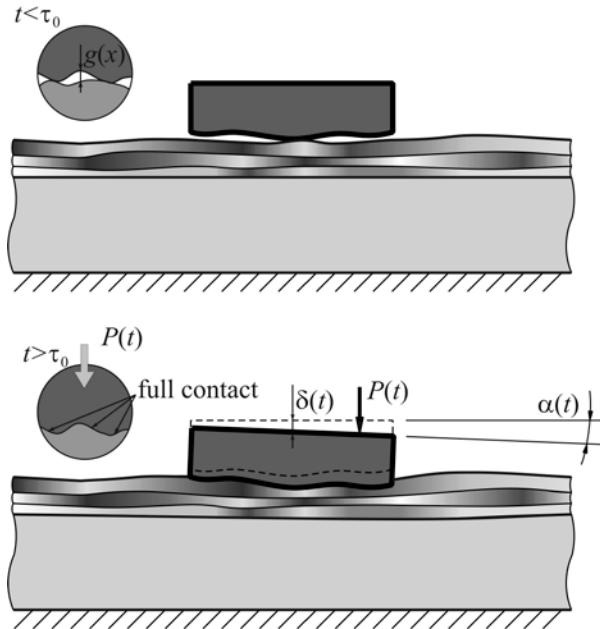


Fig. 1. Plane contact problem for foundation with coating.

Consider the case where the total thickness of the coating is much smaller than the width of the contact area, i.e.

$$\sum_{k=1}^N h_k(x) \ll 2a. \quad (1)$$

Assuming that $q(x,t)$ is contact pressure under the punch (or distributed load acting on region $x \in [-a,a]$) we can write out expressions for displacements of upper layer of the foundation due to this load (see, for example, [7–9]):

$$u_q(x,t) = -\sum_{k=1}^N \frac{[1-\nu_k^2(x)]h_k(x)}{E_k(x)} q(x,t) - \frac{2(1-\nu_{\text{lower}}^2)}{\pi} \left[\frac{1}{E_{\text{lower}}(t-\tau_{\text{lower}})} \int_{-a}^a k_{\text{pl}} \left(\frac{x-\xi}{h_{\text{lower}}} \right) q(\xi,t) d\xi + \int_{\tau_0}^t K(t-\tau_{\text{lower}}, \tau-\tau_{\text{lower}}) \frac{1}{E_{\text{lower}}(\tau-\tau_{\text{lower}})} \times \int_{-a}^a k_{\text{pl}} \left(\frac{x-\xi}{h_{\text{lower}}} \right) q(\xi,t) d\xi \right]. \quad (2)$$

In this equation ν_{lower} is constant Poisson's ratio of lower base, τ_{lower} is moment of lower layer production ($\tau_{\text{lower}} \leq \tau_0$), $k_{\text{pl}}(s)$ is known kernel of the plane contact problem

$$k_{\text{pl}}(s) = \int_0^{\infty} \frac{L(u)}{u} \cos(su) du, \quad L(u) = \frac{\cosh(2u) - 1}{\sinh(2u) + 2u}, \quad (3)$$

$K(t-\tau_{\text{lower}}, \tau-\tau_{\text{lower}})$ is creep kernel which has a form

$$K(t,\tau) = E_{\text{lower}}(\tau) \frac{\partial}{\partial \tau} \left[\frac{1}{E_{\text{lower}}(\tau)} + C_{\text{lower}}(t,\tau) \right], \quad (4)$$

$C_{\text{lower}}(t,\tau)$ is tensile creep function (see, for example, [8]). On the other hand, the movement of the upper face of the base is equal to the settlement of the punch base as a rigid body. Then we can obtain main integral equation for our

problem

$$\sum_{k=1}^N \frac{[1-\nu_k^2(x)]h_k(x)}{E_k(x)} q(x,t) + \frac{2(1-\nu_{\text{lower}}^2)}{\pi} \left[\frac{1}{E_{\text{lower}}(t-\tau_{\text{lower}})} \int_{-a}^a k_{\text{pl}} \left(\frac{x-\xi}{h_{\text{lower}}} \right) q(\xi,t) d\xi + \int_{\tau_0}^t K(t-\tau_{\text{lower}}, \tau-\tau_{\text{lower}}) \frac{1}{E_{\text{lower}}(\tau-\tau_{\text{lower}})} \times \int_{-a}^a k_{\text{pl}} \left(\frac{x-\xi}{h_{\text{lower}}} \right) q(\xi,t) d\xi \right] = \delta(t) + \alpha(t)x - g(x), \quad (5)$$

where $\delta(t)$ is punch settlement, $\alpha(t)$ is its tilt angle, and $g(x)$ is gap between contact surfaces in nondeformable state and called backlash function:

$$g(x) = f(x) - \sum_{k=1}^N h_k(x) - \min_{\xi \in [-a,a]} \left[f(\xi) - \sum_{k=1}^N h_k(\xi) \right]. \quad (6)$$

In equation (5) assumed that tilt angle $\alpha(t)$ is small, i.e. $\sin \alpha(t) \sim \tan \alpha(t) \sim \alpha(t)$.

The resulting integral equation should be supplemented by the equilibrium conditions of the punch, which is affected by concentrated force $P(t)$ and distributed load $q(x,t)$:

$$\int_{-a}^a q(\xi,t) d\xi = P(t), \quad \int_{-a}^a q(\xi,t) \xi d\xi = P(t)e(t), \quad (7)$$

where $e(t)$ is eccentricity of the application of the concentrated force of $P(t)$.

Note that properties $E_k = E_k(x)$, $\nu_k = \nu_k(x)$ and widths $h_k(x)$ of coating layers and punch base form $f(x)$ can be described by complex functions.

3 Versions of statement of contact problem

There are exist four different versions of statement for the problem (5), (7):

- 1) the settlement and tilt angle of the punch are given (i.e., the right-hand side of the integral equation is given); the force and the moment of its application must be found;
- 2) the punch settlement and the moment of the load application are given; tilt angle of the punch and force must be found
- 3) the tilt angle of the punch and the force are given; punch settlement and moment must be found;
- 4) the force and the moment of its application are given; the settlement and tilt angle of the punch must be found.

In versions 2–4 right-hand side of integral equation contain unknown parameters (settlement and/or tilt angle).

For all cases we should find contact pressure also.

4 Dimensionless form of main integral equation and additional conditions

We transform the obtained equations to dimensionless form. For this we introduce the following variables and functions

$$\begin{aligned}
 x^* &= \frac{x}{a}, \quad \xi^* = \frac{\xi}{a}, \quad t^* = \frac{t}{\tau_0}, \quad \tau^* = \frac{\tau}{\tau_0}, \quad \tau_{\text{lower}}^* = \frac{\tau_{\text{lower}}}{\tau_0}, \\
 \delta^*(t^*) &= \frac{\delta(t)}{a}, \quad \alpha^*(t^*) = \frac{\alpha(t)}{a}, \quad g^*(x^*) = \frac{g(x)}{a}, \\
 m^*(x^*) &= \frac{E_0}{2(1-\nu_{\text{lower}}^2)a} \sum_{k=1}^N \frac{[1-\nu_k^2(x)]h_k(x)}{E_k(x)}, \\
 c^*(t^*) &= \frac{E_{\text{lower}}(t-\tau_{\text{lower}})}{E_0}, \\
 q^*(x^*, t^*) &= \frac{2(1-\nu_{\text{lower}}^2)q(x, t)}{E_{\text{lower}}(t-\tau_{\text{lower}})}, \\
 P^*(t^*) &= \frac{2P(t)(1-\nu_{\text{lower}}^2)}{E_{\text{lower}}(t-\tau_{\text{lower}})a}, \\
 M^*(t^*) &= \frac{2P(t)e(t)(1-\nu_{\text{lower}}^2)}{E_{\text{lower}}(t-\tau_{\text{lower}})a^2}, \\
 K^*(t^*, \tau^*) &= K(t-\tau_{\text{lower}}, \tau-\tau_{\text{lower}})\tau_0, \\
 k^*(x^*, \xi^*) &= \frac{1}{\pi} k_{\text{pl}} \left(\frac{x-\xi}{h_{\text{lower}}} \right),
 \end{aligned} \tag{8}$$

Hence integral equation (5) take the form

$$\begin{aligned}
 &c^*(t^*)m^*(x^*)q^*(x^*, t^*) \\
 &+ \int_{-1}^1 k^*(x^*, \xi^*)q^*(\xi^*, t^*)d\xi^* \\
 &+ \int_{-1}^t K^*(t^*, \tau^*) \int_{-1}^1 k^*(x^*, \xi^*)q^*(\xi^*, \tau^*)d\xi^* d\tau^* \\
 &= \delta^*(t^*) + \alpha^*(t^*)x^* - g^*(x^*),
 \end{aligned} \tag{9}$$

or in operator form

$$\begin{aligned}
 &c^*(t^*)m^*(x^*)q^*(x^*, t^*) + (\mathbf{I} - \mathbf{V}^*)\mathbf{F}^*q^*(x^*, t^*) \\
 &= \delta^*(t^*) + \alpha^*(t^*)x^* - g^*(x^*),
 \end{aligned} \tag{10}$$

where \mathbf{I} is identity operator and

$$\begin{aligned}
 \mathbf{V}^*l(t^*) &= \int_{-1}^t K^*(t^*, \tau^*)l(\tau^*)d\tau^*, \\
 \mathbf{F}^*l(x^*) &= \int_{-1}^1 k^*(x^*, \xi^*)l(\xi^*)d\xi^*.
 \end{aligned} \tag{11}$$

Additional conditions transform also

$$\begin{aligned}
 &\int_{-1}^1 q^*(x^*, t^*)dx^* = P^*(t^*), \\
 &\int_{-1}^1 q^*(x^*, t^*)x^*dx^* = M^*(t^*).
 \end{aligned} \tag{12}$$

We should find function $q^*(z^*, t^*)$ from equations (10) and (12) and some unknown parameters: if we know $P^*(t^*)$ and $M^*(t^*)$ then we should find $\delta^*(t^*)$ and $\alpha^*(t^*)$, if we know $\delta^*(t^*)$ and $\alpha^*(t^*)$ then we should find $P^*(t^*)$ and $M^*(t^*)$, etc.

5 Analytical solution of the problem for one version of statement

Obtained mathematical model consist of mixed integral equation with functions $m^*(x^*)$ and $g^*(x^*)$ connected with coating layers properties and shapes of contacting

surfaces. Note that these functions can be rapidly changing or discontinuous. Moreover if we know only applied force and its eccentricity we have integral equation with unknown terms form left- ($q^*(x^*, t^*)$) and right-hand side ($\delta^*(t^*)$ and $\alpha^*(t^*)$).

Integral equation (10) with additional conditions (12) are similar to integral equation with additional conditions (1.2) from work [5]. Therefore, the solution obtained in the work [5] can be fully used for our problem. Finding solution in this paper based on special representation of the solution, special basis, and generalized projection method, described in [10]. It allow one to find solution of mixed integral equation with partially unknown right-hand side (10) and system of additional conditions (12).

Let the acting force and the moment of its application be given, i.e. functions $P^*(t^*)$ and $M^*(t^*)$ (which are connected with force and moment) are given and it is necessary to determine $q^*(x^*, t^*)$, $\alpha^*(t^*)$, and $\delta^*(t^*)$ (i.e. contact stresses under the punch, and tilt angle and settlement of the punch). According [5] final formulas for this functions has a form

$$\begin{aligned}
 q^*(x^*, t^*) &= \frac{1}{m^*(x^*)} \left[z_0(t^*)p_0(x^*) + z_1(t^*)p_1(x^*) \right. \\
 &\quad \left. + \sum_{k=2}^{\infty} z_k(t^*)\varphi_k(x^*) - \frac{g^*(x^*)}{c^*(t^*)} \right], \\
 \alpha^*(t^*) &= \sqrt{\frac{J_0}{J_0J_2 - J_1^2}} \left\{ c^*(t^*)z_1(t^*) - (\mathbf{I} - \mathbf{V}) \left[-\frac{g_1}{c^*(t^*)} \right. \right. \\
 &\quad \left. \left. + R_{10}z_0(t^*) + R_{11}z_1(t^*) + \sum_{k=2}^{\infty} K_k^{(1)}z_k(t^*) \right] \right\}, \\
 \delta^*(t^*) &= \frac{1}{\sqrt{J_0}} \left\{ c^*(t^*)z_0(t^*) - (\mathbf{I} - \mathbf{V}) \left[-\frac{g_0}{c^*(t^*)} \right. \right. \\
 &\quad \left. \left. + R_{00}z_0(t^*) + R_{01}z_1(t^*) + \sum_{k=2}^{\infty} K_k^{(0)}z_k(t^*) \right] \right\} \alpha^*(t^*) \frac{J_1}{J_0},
 \end{aligned} \tag{13}$$

where following variables and functions are introduced

$$z_0(t^*) = \frac{\tilde{P}(t^*)}{\sqrt{J_0}}, \quad z_1(t^*) = \frac{J_0\tilde{M}(t^*) - J_1\tilde{P}(t^*)}{\sqrt{J_0(J_0J_2 - J_1^2)}},$$

$$\begin{aligned}
 z_k(t^*) &= (\mathbf{I} + \mathbf{W}_k) \left[c^*(t^*) + \gamma_k \right]^{-1} \\
 &\quad \times (\mathbf{I} - \mathbf{V}) \left[\frac{g_k}{c^*(t^*)} - z_0(t^*)K_k^{(0)} - z_1(t^*)K_k^{(1)} \right],
 \end{aligned}$$

$$\varphi_k(x^*) = \sum_{i=2}^{\infty} \psi_i^{(k)} p_i(x^*),$$

$$K_k^{(0)} = \sum_{i=2}^{\infty} R_{0i} \psi_i^{(k)}, \quad K_k^{(1)} = \sum_{i=2}^{\infty} R_{1i} \psi_i^{(k)},$$

$$g_0 = \sum_{l=0}^{\infty} R_{0l} \int_{-1}^1 \frac{p_l(\xi^*)g^*(\xi^*)d\xi^*}{m^*(\xi^*)},$$

$$g_1 = \sum_{l=0}^{\infty} R_{1l} \int_{-1}^1 \frac{p_l(\xi^*)g^*(\xi^*)d\xi^*}{m^*(\xi^*)},$$

$$g_k = \sum_{i=2}^{\infty} \psi_i^{(k)} \sum_{l=0}^{\infty} R_{il} \int_{-1}^1 \frac{p_l(\xi^*)g^*(\xi^*)d\xi^*}{m^*(\xi^*)},$$

$$\mathbf{W}_k f(t^*) = \int_1^t R_k(t^*, \tau^*) f(\tau^*) d\tau^*, \quad (14)$$

$$J_i = \int_{-1}^1 \frac{(\xi^*)^i}{\sqrt{m^*(\xi^*)}} d\xi^*, \quad p_0(x^*) = \frac{1}{\sqrt{J_0}},$$

$$p_j(x^*) = \frac{1}{\sqrt{d_{j-1}d_j}} \begin{vmatrix} J_0 & J_1 & \dots & J_j \\ J_1 & J_2 & \dots & J_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x^* & \dots & (x^*)^j \end{vmatrix},$$

$$d_i = \begin{vmatrix} J_0 & J_1 & \dots & J_i \\ J_1 & J_2 & \dots & J_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ J_i & J_{i+1} & \dots & J_{2i} \end{vmatrix},$$

$$\tilde{P}(t^*) = P^*(t^*) + \frac{1}{c^*(t^*)} \int_{-1}^1 \frac{g^*(\xi^*) d\xi^*}{m^*(\xi^*)},$$

$$\tilde{M}(t^*) = M^*(t^*) + \frac{1}{c^*(t^*)} \int_{-1}^1 \frac{\xi^* g^*(\xi^*) d\xi^*}{m^*(\xi^*)},$$

coefficients $\psi_i^{(k)}$ and γ_k can be find from the solution of spectral problem

$$\sum_{i=0}^{\infty} R_{il} \psi_i^{(k)} = \gamma_k \psi_i^{(k)}, \quad i, k = 0, 1, 2, \dots, \quad (15)$$

R_{il} can be calculated from the formula

$$R_{il} = \int_{-1}^1 \int_{-1}^1 \frac{k^*(x^*, \xi^*) p_i(x^*) p_l(\xi^*)}{m^*(x^*) m^*(\xi^*)} dx^* d\xi^*, \quad (16)$$

and $R_k(t^*, \tau^*)$ are resolvents of the kernels

$$K_k(t^*, \tau^*) = \frac{\gamma_k K^*(t^*, \tau^*)}{c(t) + \gamma_k}. \quad (17)$$

The expression for the contact pressure under the punch in the dimensional form is ($x \in [-a, a]$, $t \geq \tau_0$)

$$q(x, t) = \frac{E_{\text{lower}}(t - \tau_{\text{lower}})a}{E_0} \left\{ \sum_{k=1}^N \frac{[1 - v_k^2(x)] h_k(x)}{E_k(x)} \right\}^{-1} \times \left[z_0 \left(\frac{t}{\tau_0} \right) p_0 \left(\frac{x}{a} \right) + z_1 \left(\frac{t}{\tau_0} \right) p_1 \left(\frac{x}{a} \right) + \sum_{k=2}^{\infty} z_k \left(\frac{t}{\tau_0} \right) \varphi_k \left(\frac{x}{a} \right) \right]. \quad (18)$$

So formulas (13)–(18) allow us to calculate contact stresses $q(x, t)$ under the punch, punch settlement $\delta(t)$, and its tilt angle $\alpha(t)$. Note that solutions for other cases has similar structure. Most complicated thing in these formulas is to calculate coefficients R_{il} . According to (3) (8), and (16) we must calculate following integrals:

$$R_{il} = \frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 \frac{L(u) p_i(x^*) p_l(\xi^*)}{u m^*(x^*) m^*(\xi^*)} \cos \frac{a(x^* - \xi^*) u}{h_{\text{lower}}} du dx^* d\xi^*. \quad (19)$$

To do this, the authors use special methods.

Note that in the expression for contact pressure, function associated with the thicknesses of the layers and their properties is a separate factor, and a function related to the gap is highlighted as a separate term. This allows

the use of a small number of members of the series to obtain a qualitative picture of the interaction.

6 Conclusions

The problem of the interaction of a rigid punch and a viscoelastic aging base with a multilayer nonuniform coating is posed and solved. The case when the thicknesses of the layers are variable and the shape of the punch is described by a complex function is considered. The solution to the problem was obtained under the assumption that the package of layers is thin compared to the width of the punch, its rigidity is less than the rigidity of the lower layer, and punch tilt angle is small.

It is shown that the mathematical model of such a problem is mixed integral equation with integral additional conditions. Its solution was built in previous works and written out in this article. Obtained solution allows for efficient numerical calculations even with a small number of terms in the resulting series.

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