

Flat area characteristics of polygons circumscribing parabolic figures

Elvira Egereva^{1*}, *Dmitry Surovtsev*¹, and *Zakhar Nepovinnyy*¹

¹ National Research Moscow State University of Civil Engineering, Yaroslavskoye Shosse, 26, Moscow, 129337, Russia

Abstract. The work elucidates the extremum areas of the polygons circumscribing parabolic figures. It is shown that the ratio of the areas of the polygons circumscribed near parabolic figures to the areas of the corresponding figures always remains a constant value, independent of the coefficients characterizing the quadratic function. The only point at which the function under study reaches its minimum value is found. The question of the necessary conditions for the existence of a circle circumscribed about a parabolic quadrilateral found in the Ptolemy theorem is being considered.

1 Introduction

The article explores the relationship between the values of the areas of parabolic figures and the minimum values of the areas of polygons described near parabolic figures. The study of the area of figures of various configurations has always attracted the attention of scientists [1]. This knowledge is applied in many branches of science and technology. So, the authors of [2] found an algorithm for finding a convex polygon with a maximum area described near a given polygon.

Among the results obtained by Lao and Mayer, considered in [3], limit theorems for the maximum perimeter and maximum area of random triangles inscribed in a circle are of great interest. In [3], the problem for the minimum perimeter and the minimum area bounding m - polygons ($m \geq 3$), which has not yet been studied, is considered.

Several interesting issues related to circling a convex polygon with a polygon of a smaller number of sides with minimal addition of area were considered in [4], and in [5] the author considered a regular polygon and applied the results to cyclotomic model sets. Article [6] solves a similar problem for an affine regular polygon inscribed in a hyperbola passing through a given point, and for an equilateral triangle [7].

It was shown in [8] that the area function defines a pseudo-Hermitian structure in space that preserves the orientation of the similarity classes of plane closed curves, where the explicit geodesic characteristics of these figures are calculated.

Uwe Schwerdtfeger in an article [9] calculated the area rules for discrete excursions with an arbitrary distribution of the area of a convex column of polygons. We study the minimal decompositions of rectilinear polygons into rectangles associated with the task of

*Corresponding author: egerevaen@mail.ru

generating and propagating electromagnetic interference in multilayer printed circuit boards in [10]. Convex polygons of the minimum and maximum area are also considered in [11–13].

The problem of partitioning a convex polygon into polygons of the same area and perimeter was solved in [14]. The authors of [15] studied the algorithm for approximating the calculation of an inscribed rectangle having the largest area on a convex polygon.

The novelty of the studies presented in this article is the exact determination of the ratio of the ratio of the area values of parabolic triangles and quadrangles to the area values of the corresponding described triangles and quadrangles.

2 Materials and methods

The quadratic $f(x)$ and linear $l(x)$ functions defined on the plane are considered. Moreover, the parabola $y = f(x)$ and the line $y = l(x)$ have two common points. The part of the XOY plane bounded by the graphs of the functions $f(x)$ and $l(x)$ will be called a parabolic triangle whose area is F_{Δ} (Fig. 1)

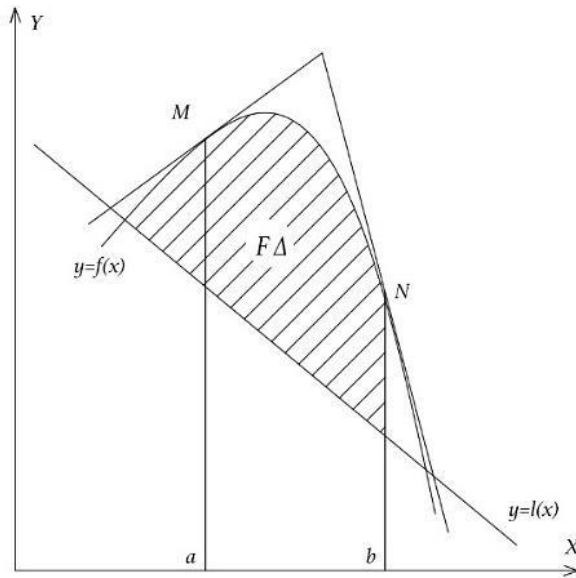


Fig. 1. Parabolic triangle in the general case.

Let $M(a; f(a))$ and $N(b; f(b))$ be two random points of the parabola $y = f(x)$, where $b > a$ (Fig. 1). We construct a triangle formed by the segments of the tangent lines drawn to the parabola at points M and N, and the segment of the line $y = l(x)$. We call such a triangle circumscribed near the parabolic triangle F_{Δ} introduced above and denote \widetilde{F}_{Δ} . Since the points $M(a; f(a))$ and $N(b; f(b))$ are random points of the parabola $y = f(x)$, it is obvious that there are infinitely many such triangles. The condition for the existence of such triangles is that $a < x_B$, $b > x_B$ where the point $B(x_B; y_B)$ – is the intersection point of the tangents constructed to the quadratic function $y = f(x)$ at points M and N respectively. We investigate the extremum function $S(a, b)$ of the area of the circumscribed triangle.

Suppose that the linear function $l(x)$ is not constant, that is, $l(x) = \alpha x + \beta$, where $\alpha \neq 0, \beta \neq 0$. For the convenience of calculations, we place the described triangle \widetilde{F}_{Δ} on the plane so that the axis OY is the axis of symmetry of the quadratic function $y = f(x)$,

presented in the form $f(x) = px^2 + r$, where $p < 0, r > 0$ (Fig. 2).

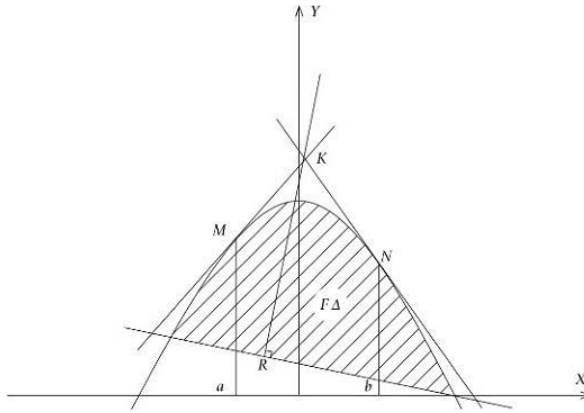


Fig. 2. A parabolic triangle bounded by $y = f(x), l(x) = \alpha x + \beta$.

The function $S(a, b)$ does not reach its maximum value, since choosing a tangent to the parabola $y = f(x)$ parallel to the straight line $y = l(x)$, using the geometric meaning of the Lagrange theorem [3], we obtain an infinite value of the area of the function $S(a, b)$. Therefore, we study the question of the minimum area of the described parabolic triangle $S(a, b) \rightarrow \min$.

The equations of the tangent at the points $M(a; f(a))$ and $N(b; f(b))$ respectively have the form

$$\begin{cases} y_M = (pa^2 + r) + 2pa(x - a) \\ y_N = (pb^2 + r) + 2pb(x - b) \end{cases} \quad \begin{cases} y_M = 2pax + (r - pa^2) \\ y_N = 2pbx + (r - pb^2) \end{cases} \quad (1)$$

moreover $a \neq 0, b \neq 0$.

Find the coordinates of the intersection point of the tangents (1) $K(x_k; y_k)$, where $x_k = (a + b)/2; y_k = pab + r$, where x_k – is the arithmetic mean of a and b .

Find the point $K_1(x_{K_1}; y_{K_1})$ as the intersection point of the tangent to the parabola $y = f(x)$ at point M and the line $l(x) = \alpha x + \beta$ from the condition $\alpha x + \beta = 2pax + (r - pa^2)$:

$$\begin{cases} x_{K_1} = \frac{pa^2 + \beta - r}{2pa - \alpha} \\ y_{K_1} = \frac{\alpha(pa^2 + \beta - r)}{2pa - \alpha} + \beta \end{cases}$$

Therefore, the second vertex of the triangle \widetilde{F}_Δ described near the parabolic triangle F_Δ has the coordinates:

$$K_1 \left(\frac{pa^2 + \beta - r}{2pa - \alpha}; \frac{\alpha(pa^2 + \beta - r)}{2pa - \alpha} + \beta \right).$$

Finally, the third vertex \widetilde{F}_Δ of the triangle is found by analogy as the point of intersection of the tangent to the parabola at point N and the line $y = l(x)$:

$$\left\{ \begin{array}{l} x_{K_2} = \frac{pb^2 + \beta - r}{2pb - \alpha} \\ y_{K_2} = \frac{\alpha(pb^2 + \beta - r)}{2pb - \alpha} + \beta \end{array} \right., \quad K_2 \left(\frac{pb^2 + \beta - r}{2pb - \alpha}; \frac{\alpha(pb^2 + \beta - r)}{2pb - \alpha} + \beta \right).$$

The length of the side of the triangle $\widetilde{F}_\Delta K_1 K_2$ is equal to:

$$K_1 K_2 = \sqrt{x_{K_1 K_2}^2 + y_{K_1 K_2}^2} = \frac{p(b-a)(2pba - 2\beta + 2r - \alpha b - \alpha a)}{(2pb - \alpha)(2pa - \alpha)} \cdot \sqrt{1 + \alpha^2}.$$

Here

$$\left\{ \begin{array}{l} x_{K_1 K_2} = \frac{p(b-a)(2pba - 2\beta + 2r - \alpha b - \alpha a)}{(2pb - \alpha)(2pa - \alpha)} \\ y_{K_1 K_2} = \alpha x_{K_1 K_2} \end{array} \right. \quad (2)$$

To find the area \widetilde{F}_Δ , we introduce the line $l_1(x)$ perpendicular to the line $l(x) = \alpha x + \beta$. Let $R(x_R, y_R)$ be the intersection point of these lines. We find its coordinates from the relationship between the angular coefficients of these lines

$k_1 \cdot k = -1$, we get

$$\left\{ \begin{array}{l} x_R = \frac{a+b+2apab+2ar-2\alpha\beta}{2(\alpha^2+1)} \\ y_R = \alpha \left(\frac{a+b+2apab+2ar-2\alpha\beta}{2(\alpha^2+1)} \right) + \beta \end{array} \right. \quad (3)$$

Finding length of KR :

$$KR = \sqrt{x_{KR}^2 + y_{KR}^2} = \frac{(-2pab - 2r + 2\beta + \alpha b + \alpha a)}{2(\alpha^2 + 1)} \cdot \sqrt{1 + \alpha^2} \quad (4)$$

here

$$\left\{ \begin{array}{l} x_{KR} = \frac{\alpha(2pab + 2r - 2\beta - \alpha b - \alpha a)}{2(\alpha^2 + 1)} \\ y_{KR} = \frac{(-2pab - 2r + 2\beta + \alpha b + \alpha a)}{2(\alpha^2 + 1)} \end{array} \right.$$

then

$$S(a, b) = \frac{1}{2} \cdot KR \cdot K_1 K_2 = \frac{1}{4} \cdot \frac{p(b-a)(2pab + 2r - 2\beta - \alpha b - \alpha a)^2}{(\alpha - 2pb)(2pa - \alpha)} \quad (5)$$

We study the function $S(a, b)$ to the extremum. To do this, we find the partial derivatives of this function with respect to the variables a and b :

$$\left\{ \begin{array}{l} S'_a(a, b) = -\frac{p(2pab + 2r - 2\beta - \alpha b - \alpha a)}{4(2pa - \alpha)^2} \cdot (2pba - \alpha b + 3\alpha a - 2r + 2\beta - 4pa^2) \\ S'_b(a, b) = \frac{p(2pab + 2r - 2\beta - \alpha b - \alpha a)}{4(2pb - \alpha)^2} \cdot (2pba + 3\alpha b - \alpha a - 4pb^2 - 2r + 2\beta) \end{array} \right. \quad (6)$$

We find points that are suspicious of an extremum. To do this, we solve the system of equations:

$$\left\{ \begin{array}{l} S'_a(a, b) = 0 \\ S'_b(a, b) = 0 \end{array} \right.$$

From where, taking into account the signs of the coefficients p and r , $a \neq b$, $b > a$, we obtain

$$\begin{cases} a^* = \frac{\alpha - \sqrt{\alpha^2 - 4p(r - \beta)}}{2p} \\ b^* = \frac{\alpha + \sqrt{\alpha^2 - 4p(r - \beta)}}{2p} \end{cases} \quad (7)$$

Note that for $\alpha = 0$ and $\beta = 0$, a^* and b^* take the values

$$a^* = -(-r/p)^{1/2} ; b^* = (-r/p)^{1/2} \quad (8)$$

The point (a^*, b^*) is the desired one and delivers the minimum value to the function $S(a, b)$.

$$S_{min} = S(a^*, b^*) = -\frac{1}{4p^2} \cdot (\alpha^2 - 4p(r - \beta))^{\frac{3}{2}} \quad (9)$$

Finding the abscissas of intersection points of $y = f(x)$ and $y = l(x)$:

$$\begin{cases} x_1 = \frac{\alpha + \sqrt{\alpha^2 - 4p(r - \beta)}}{2p} \\ x_2 = \frac{\alpha - \sqrt{\alpha^2 - 4p(r - \beta)}}{2p} \end{cases} \quad (10)$$

Let

$$S_{min}(F_{\Delta}) = \int_{x_1}^{x_2} ((px^2 + r) - (\alpha x + \beta)) dx = -\frac{(\alpha^2 - 4p(r - \beta))^{\frac{3}{2}}}{6p^2} \quad (11)$$

Consider the special case for this problem, when $l(x) = const$. Place the figure \widetilde{F}_{Δ} on the plane in the same way as in the general case (Fig. 3).

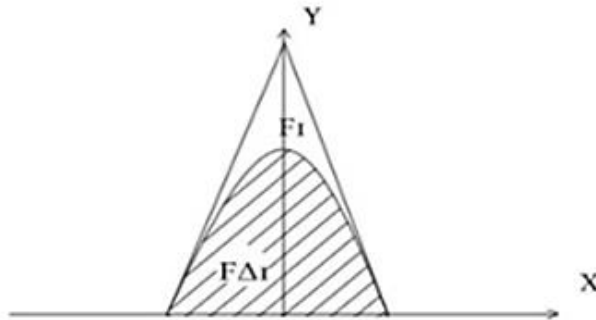


Fig. 3. Parabolic quadrangle

Therefore, the equation of the tangents and the coordinates of their intersection points are the same as in the general case. The two other coordinates of the vertices of the triangle \widetilde{F}_{Δ} are as follows:

$$E\left(\frac{pa^2-r}{2pa}; 0\right) \text{ и } L\left(\frac{pb^2-r}{2pb}; 0\right).$$

Length of EL in \widetilde{F}_{Δ} is equal to:

$$EL = \frac{pb^2 - r}{2pb} - \frac{pa^2 - r}{2pa} = \frac{(b-a)(r+pba)}{2pab} \quad (12)$$

So the area of \widetilde{F}_Δ may be written down like:

$$S(a, b) = \left(\frac{1}{4pa} - \frac{1}{4pb} \right) (bpa + r)^2 \quad (13)$$

Investigating $S(a, b)$ on extremums in the area $a \in (-\infty; 0)$ and $b \in (0; +\infty)$. To do it let us study the derivatives of this function of two variables (13) :

$$\begin{cases} S'_a(a, b) = \frac{(pba + r)(pab - 2pa^2 - r)}{4pa^2} \\ S'_b(a, b) = \frac{(pba + r)(2pb^2 - bpa + r)}{4pb^2} \end{cases} \quad (14)$$

By analogy with the general case, we obtain

$$\begin{cases} \frac{(pba + r)(pab - 2pa^2 - r)}{4pa^2} = 0 \\ \frac{(pba + r)(2pb^2 - bpa + r)}{4pb^2} = 0 \end{cases}$$

Taking into account the signs of the coefficients p and r , we obtain the values a^* , b^* are the same as (8).

Then the points $M(a^*, f(a^*))$ and $N(b^*, f(b^*))$ are the intersection points of the parabola $y = f(x)$ with the axis OX . By virtue of its uniqueness, the critical point (a^*, b^*) is the desired one and delivers the minimum value to the function $S(a, b)$.

$$S_{min} = S(a^*, b^*) = \left(\frac{1}{4pa^*} - \frac{1}{4pb^*} \right) (b^*pa^* + r)^2 = -\frac{2r^2}{p} \left(\frac{1}{\sqrt{\frac{-r}{p}}} \right) \quad (15)$$

So, bringing a or b closer to zero on the corresponding side, we have $S(a, b) \rightarrow \infty$. That is, the function $S(a, b)$ still does not reach its maximum.

The area of the parabolic triangle $F_\Delta \Delta$ is defined as an integral of the form:

$$S(F_\Delta) = \int_{-\sqrt{\frac{-r}{p}}}^{\sqrt{\frac{-r}{p}}} (px^2 + r) dx = \frac{4r\sqrt{\frac{-r}{p}}}{3} \quad (16)$$

Let $f_1(x)$ and $f_2(x)$ be two quadratic functions defined on the coordinate plane, and the direction of their branches is the opposite. These parabolas have exactly two points in common. The part of the XOY plane bounded by the graphs of the functions $f_1(x)$ and $f_2(x)$ will be called a parabolic quadrangle and denoted by F_\blacksquare .

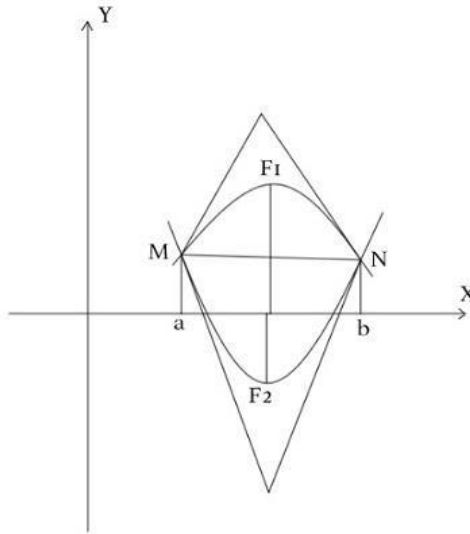


Fig. 4. Parabolic quadrangle with $l = \text{const}$

We construct the triangles described above about parabolic triangles as parts of F_{\blacksquare} , the triangles \widetilde{F}_1 и \widetilde{F}_2 . We get a quadrangle, which we will call described near and denote $\widetilde{F}_{\blacksquare}$. Consider the special case when $l(x) = \text{const}$. Let $f_1(x) = px^2 + r$, $f_2(x) = qx^2 + d$, with $q < 0, r < 0$.

In order for the common points of the parabolas to have an ordinate equal to zero, we require that the sets of roots of the corresponding quadratic equations coincide. Namely, to satisfy the condition: $\sqrt{\frac{-r}{p}} = \sqrt{\frac{-q}{d}}$ or, what is the same, $rq = qd$ as a second-order determinant:

$$\begin{vmatrix} p & r \\ q & d \end{vmatrix} = 0$$

We carry out the arguments similarly to the previous case, we find the abscissas of the points of intersection of the parabola $f_1(x) = px^2 + r$ and the line l . To do this, we divide the parabolic quadrangle into two parabolic triangles along the line $l(x)$. In turn, the described triangles $F_{\Delta 1}$ and $F_{\Delta 2}$ in total form the described quadrangle $\widetilde{F}_{\blacksquare}$. We use the results obtained above for the case

$p < 0, r > 0$ and $q > 0, d < 0$, when such parabolic triangles exist.

3 Results of the study

The dimensionless coefficient for the particular case when the linear function is given in the form $l(x) = 0$, taking into account the values (15) and (16), is equal to:

$$\delta = \frac{S_{min}}{F_{\Delta}} = \frac{-\frac{2r^2}{p} \left(\frac{1}{\sqrt{\frac{-r}{p}}} \right)}{4r \frac{\sqrt{-r}}{\sqrt{p}}} = \frac{3}{2} = \text{const.}$$

Thus, this coefficient takes a constant value, independent of the coefficients q and r , which determine the quadratic function for the studied parabolic triangles, and also δ that does not depend on α and β , which determine the linear function.

We define a dimensionless coefficient for the general case when the linear function is given in the form $l(x) = \alpha x + \beta$ as the ratio of the minimum area of the described triangle to the area of the parabolic triangle. Taking into account the obtained values (9) and (11), we have:

$$\delta = \frac{S_{min}}{F_{\Delta}} = \frac{-\frac{1}{4p^2} \cdot (\alpha^2 - 4p(r - \beta))^{\frac{3}{2}}}{-\frac{(\alpha^2 - 4p(r - \beta))^{\frac{3}{2}}}{6p^2}} = \frac{3}{2} = const.$$

If for a parabolic quadrilateral S_1 and S_2 are the areas of triangles \widetilde{F}_1 and \widetilde{F}_2 соответственно, respectively, then the dimensionless parameter is calculated by the formula

$$\delta = \frac{S_1 + S_2}{F_{\blacksquare}} \quad (17)$$

For a parabolic quadrilateral in the general case, we have

$$\delta = \frac{S_1 + S_2}{F_{\blacksquare}} = \frac{\widetilde{F}_{\Delta 1} + \widetilde{F}_{\Delta 2}}{F_{\Delta 1} + F_{\Delta 2}} \quad (18)$$

Remembering that $\delta_1 = \frac{S_1}{F_{\Delta 1}} = \frac{\widetilde{F}_{\Delta 1}}{F_{\Delta 1}} = \frac{3}{2} = const$,

$$\delta_2 = \frac{S_2}{F_{\Delta 2}} = \frac{\widetilde{F}_{\Delta 2}}{F_{\Delta 2}} = \frac{3}{2} = const, \text{ TO } \frac{\widetilde{F}_{\Delta 1}}{F_{\Delta 1}} = \frac{\widetilde{F}_{\Delta 2}}{F_{\Delta 2}}.$$

$$\text{Then area of } \widetilde{F}_{\Delta 1} = \frac{\widetilde{F}_{\Delta 2} \cdot F_{\Delta 1}}{F_{\Delta 2}} \quad (19)$$

Substituting (19) into the dimensionless parameter taking into account (18), we obtain:

$$\delta = \frac{\frac{\widetilde{F}_{\Delta 2} \cdot F_{\Delta 1}}{F_{\Delta 2}} + \widetilde{F}_{\Delta 2}}{F_{\Delta 1} + F_{\Delta 2}} = \frac{\widetilde{F}_{\Delta 2}}{F_{\Delta 2}} = \frac{3}{2} = const.$$

We obtain exactly the same result for the special case when the parabolic quadrilateral intersects the line $l(x) = 0$. Thus, the dimensionless coefficient takes a constant value $\delta = \frac{3}{2}$.

We study the possibility of describing $\widetilde{F}_{\blacksquare}$ around a circle for such coefficients p, r, q and d so that the condition for the existence of the circumscribed circle contained in the Ptolemy theorem [10] is satisfied. Suppose that there exists a quadrangle around which a circle can be described, then the abscissas of two common points have the form:

$$x_1 = \sqrt{\frac{d-r}{p-q}} \quad ; \quad x_2 = -\sqrt{\frac{d-r}{p-q}}, \text{ where } d \neq r, \quad p \neq q, \quad \frac{d-r}{p-q} > 0$$

The coordinates of points A, B, M, N satisfy the equation of the circumscribed circle (Fig. 5).

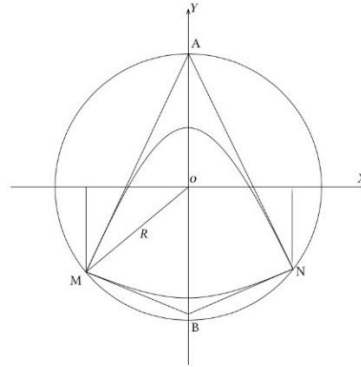


Fig. 5. The ability to describe a circle around a parabolic quadrilateral

So $(a; pa^2 + r)$, $N(b; pb^2 + r)$. Let R be the radius of a given circle, then

$$\begin{cases} a^2 + (pa^2 + r)^2 = R^2 \\ b^2 + (pb^2 + r)^2 = R^2 \end{cases} \quad (20)$$

Solving the system (20), get

$$\begin{cases} p = \frac{\sqrt{R^2 - a^2}}{a^2} - \frac{a^2\sqrt{R^2 - b^2} - b^2\sqrt{R^2 - a^2}}{(a^2 - b^2)a^2} \\ r = \frac{a^2\sqrt{R^2 - b^2} - b^2\sqrt{R^2 - a^2}}{a^2 - b^2} \end{cases} \quad (21)$$

On the other hand $M(a; qa^2 + d)$, $N(b; qb^2 + d)$ lies on the same circumscribed circle, so:

$$\begin{cases} a^2 + (qa^2 + d)^2 = R^2 \\ b^2 + (qb^2 + d)^2 = R^2 \end{cases} \quad (22)$$

From where we find

$$\begin{cases} q = \frac{\sqrt{R^2 - a^2}}{a^2} - \frac{a^2\sqrt{R^2 - b^2} - b^2\sqrt{R^2 - a^2}}{(a^2 - b^2)a^2} \\ d = \frac{a^2\sqrt{R^2 - b^2} - b^2\sqrt{R^2 - a^2}}{a^2 - b^2} \end{cases} \quad (23)$$

Note that from (21) and (23) it follows $p = q, r = d$, which leads to a contradiction. Therefore, a circle cannot be described around the quadrangle $\widehat{F}_{\blacksquare}$, that is, the condition for the existence of the circumscribed circle contained in the Ptolemy theorem is not satisfied.

4 Discussion and conclusions

In the study of the relationship between the areas of parabolic figures and the polygons described near them, the ratio of the areas of these figures was found, and it was also proved that, regardless of the configuration of the given parabola and the lines forming the polygon described near the parabolic figure, the coefficient will always be constant and equal to . In addition, it was proved that it is impossible to describe a circle around a

quadrangle described near a parabolic quadrangle, regardless of the functions that specify its parameters.

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