The use of hyperbolic and Asaoka's methods to estimate the moisture diffusivity and the hydraulic conductivity of unsaturated soils from tests to determine the SWRC

Enrique Asanza^{1*}, Fernando Pardo de Santayana¹, and Miriam Martín-Ruiz¹

¹Centro de Estudios y Experimentación de Obras Públicas (CEDEX), Laboratorio de Geotecnia, c/Alfonso XII, Madrid, Spain,

Abstract. The equation governing the unsaturated transient flow in a soil sample when subjected to suction (ψ) at its base and that of the classical 1-D consolidation are exactly alike (equation of diffusion). The former can be arrived at by Richards' equation, being in this case the moisture diffusivity (D), instead of the coefficient of consolidation (c_v), the governing parameter. D need not be constant, but rather a function of the volumetric water content, $D=D(\theta)$, and is defined as the ratio of the hydraulic conductivity, $k(\theta)$, over the specific water capacity, $C(\theta)=d\theta/d\psi$, i.e., the slope of the SWRC. The hyperbolic method has been used for several geotechnical purposes and, most importantly, as an alternative to Asaoka's method for predicting the final settlement and c_v of soft soils undergoing consolidation, improved by preloading. This paper shows that both methods prove to be very useful as a means of obtaining $D(\theta)$ and $k(\theta)$ at a certain range of θ , provided that a reduced number of water contents at known elapsed times are determined over the medium stage of this transient flow. It is addressed in the paper both by theoretical grounds and on the light of experimental data of 4 soils.

1 Introduction

When the base of an unsaturated soil sample is subjected to a source of suction, irrespective of the technique used, an unsteady flow of water towards a new equilibrium is established. This process is governed by the equation of diffusion, just like the heat transfer or, in the scope of soil mechanics itself, a process of consolidation, where the coefficient of consolidation (c_v) plays the same role as the moisture diffusivity (D).

Due to the difficulty of assessing a global value of c_v , and the final settlement (δ_∞) of a site on soft soils subjected to a surcharge, a few predictive methods are proven to be useful, as long as a number of measurements (settlement at known elapsed times) over the medium stage are available. The hyperbolic [1-4] and Asaoka's methods [5] are perhaps the most widespread in this field. The former has even been successfully extended to other problems in Geotechnics.

Assuming the identity of the governing equation of the described unsaturated flow and a 1-D consolidation process, such predictive methods can be extended, *mutatis mutandis*, to predict the final water content of the sample subjected to certain suction, and more importantly, its moisture diffusivity (D). A reduced number of water contents at different elapsed times would suffice for these two methods to be applicable.

2 Theoretical backgrounds

2.1 Unsaturated soils

2.1.1 Suction, SWRC and hydraulic conductivity

Suction is the intensity that an unsaturated soil exerts in holding back or capturing water from its surrounding, so that the lower the water content is, the higher is its suction. In fact, suction is a potential (energy per unit mass of the water kept within the soil): it is just a negative water head. However, in this case it is strongly dependent upon the mineral constituents of the soil and its pore size distribution [6].

The soil water retention curve (SWRC) is defined as the relationship between the suction at which a soil is subjected and its water content held back in equilibrium. Note that in soil physics, rather than the gravimetric (ordinary) water content, volumetric water content (θ) is preferred. Then, the SWRC can be denoted as $\theta = \theta(\psi)$ or $\psi = \psi(\theta)$. Moreover, it will be assumed, for the sake of simplicity, that it is not a hysteretic relationship. The specific water capacity is defined as $C(\theta)=d\theta/d\psi$, that is, the natural slope of the SWRC.

With regard to hydraulic conductivity, when a soil loses its saturation, the connectivity of the saturated pore network is then strongly reduced, as air bubbles occupy voids. Therefore, permeability becomes a function of the

* Corresponding author: esianza@cedex.es

degree of saturation (S_R), and then it is called hydraulic conductivity: $k=k(S_R)$. Alternatively, it can be expressed as a function of the volumetric water content or suction: $k=k(\theta) \rightarrow k(\theta(\psi)) \rightarrow k=k(\psi)$.

Conversion among parameters should be done with care if changes in water content entail measurable volume changes. Besides, assuming a double-porosity model [7], the residual water content (that adsorbed by the clay platelets, removable only by oven-drying) is normally neglected, leading to a redefinition of the degree of saturation (S_R^*).

Direct laboratory methods for measuring the hydraulic conductivity of fine-grained soils is costly, time-consuming or, to some extent, unreliable, all the more when handling swelling-shrinking soils. Masroui et al. [8] summarized the available methods, including their advantages and disadvantages.

The method described in this paper (sect. 3.2) resembles somehow the unsteady state method by Gardner [9] with a membrane pressure plate, yet in the present case, instead of readings of flow measurements, the weight of soil sample (due to intake or loss of water) are measured at several time intervals.

Indirect techniques embrace statistical methods and the inverse problem, together with pore size distribution and the integration of the SWRC (see [8]). To the authors' knowledge, the expressions proposed by Brooks & Corey, van Genuchten and Fredlund & Xing [10-12] are the most widespread in Geomechanics. Nonetheless, some authors suggest on theoretical grounds that the hydraulic conductivity is closely dependent upon S_R^* raised to an exponent n which, unless proven otherwise, $n=3$ is a decent estimation [13-17].

As shown later, predictive methods are helpful as well to reach more accurate formulae of the hydraulic conductivity.

2.1.2 Richards' equation (1931)

Let a soil sample subjected to certain suction at its base, being the sample thickness negligible in terms of its water head. Depending whether the initial suction of the sample is lower or higher than that imposed at its base, the unsteady process established will lead to an inflow or an outflow. Anyway, Darcy's law holds, so the unit flow is expressed as:

$$q = -k(\psi) \cdot \frac{dh}{dz} \cong -k(\psi) \cdot \frac{d\psi}{dz} \quad (1)$$

On the other hand, the flow must meet the continuity principle, i.e., changes in water content must be counterbalanced by the net vertical flow:

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z} \quad (2)$$

Then, the combination of (1) and (2) yields the following expression, also known as the Richards' equation [18]:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[k(\psi) \cdot \frac{\partial \psi}{\partial z} \right] \quad (3)$$

In equation (3), t (time) and z (height) are the independent parameters; with minor mathematical

manipulation, the Richards' equation can be solely expressed as a function of the volumetric water content:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[\frac{k(\theta)}{c(\theta)} \cdot \frac{\partial \theta}{\partial z} \right] = \frac{\partial}{\partial z} \left[D(\theta) \cdot \frac{\partial \theta}{\partial z} \right] = \frac{\partial^2 \theta}{\partial z^2} [D(\theta)] \quad (4)$$

where:

$$\frac{k(\theta)}{c(\theta)} = D(\theta) \quad (5)$$

In this case, the governing parameter $D(\theta)$, the moisture (or hydraulic) diffusivity, unlike the coefficient of consolidation (c_v), need not be constant. A few closed-form solutions of the Richards' equations can be found in treatises of soil physics [19, 20]. Leaving aside commercial codes, software such as Hydrus-1D [21], available for public-domain, is suitable for solving unsteady flow.

Note that, as the determination of SWRC is rather simple, so $C(\theta)$ is. Accordingly, the hydraulic conductivity $k=k(\theta)$ could be inferred if $D(\theta)$ were obtained just as c_v is from predictive methods.

2.2 Predictive methods

Two predictive methods well established in the field of soil improvement by preloading are described below, including their adaptation to unsaturated soils.

2.2.1 Hyperbolic method

A rectangular (or equilateral) hyperbola can be transformed into a straight line (Fig. 1), leading to useful relationships, especially when the hyperbola is defined just by a limited number of points.

The 1-D consolidation equation (Terzaghi-Frölich) and, therefore, any other diffusion phenomena, fit extremely well a hyperbola in the range of $U=0,6$ to $0,9$. Such partial fitting, the prognosis of δ_∞ and the derivation of c_v are referred in the literature [1-2]. Figure 1 shows the relationship between the plotted straight line and δ_∞ and c_v . This method has gained insight and been validated even for more complex consolidation processes and for case studies with Prefabricated Vertical Drains (PVD) in preloaded soils [1-2].

The hyperbolic method had been used earlier in Geotechnics for purposes other than consolidation: modelling the stress-strain relationship in triaxial tests to account for the loss of shear stiffness [22, 23]; reproducing the load-settlement relationship for pile load tests [24-26]. In both cases the asymptotic values (deviator or pile bearing capacity) can be readily predicted well before failure.

2.1.2 Asaoka's method

Asaoka derived a useful method for computing δ_∞ and c_v . Magnan & Deroy [27] extended the method for 3-D consolidation with PVD.

Fig. 1. Principles of the hyperbolic method

The easiest explanation of this method lies on the fact that discrete values (f_{i-1} ; f_i ; f_{i+1} ,...) computed from constant intervals Δt of functions $f(t)$ with the pattern shown in Figure 2 (top) follow a linear recurrence law (Fig. 2). Therefore, if a limited number of values $f(i \cdot \Delta t)$ is known or measured, the coefficients of the function can be inferred just as shown in Fig. 2 (bottom). Accordingly, the 1-D consolidation equation is governed by a function $f(t)$ with that pattern when U (degree of consolidation $> 0,6$).

Therefore, the final settlement (i.e. $t_i=t_{i+1}$) can be graphically obtained as the intersection of the line formed by the plotted data and a reference line at 45°. The coefficient of consolidation (c_v) is a function of the slope (M), the drainage length (H) and the time interval of measurement (Δt).

Note again that any diffusion process (beyond a degree of equilibrium of 60 %) meet the previous requirements. Table 1 summarizes the equations for computing $D(\theta)$ with both methods.

Fig. 2. Principles of Asaoka's method; equations (top) and plotting (bottom). Axes are dimensionless.

Table 1. Summary of the equations for computing D and θ_{∞}

Hyperbolic		Asaoka	
Consolidation			
$\delta_{\infty} = \frac{0.821}{M}$ $c_v = 0.29 \frac{M}{N} H^2$		$\delta_{\infty} = \frac{N}{1 - M}$ $c_v = - \frac{4H^2}{\pi \Delta t} \ln(M)$	
Legend	M = slope of the plotted line N = ordinate at origin of the plotted line H = drainage length Δt = time intervals (Asaoka's method)		
Conversion for unsaturated flow			
$\delta_{\infty} \leftrightarrow \theta_{\infty}$ final settlement $\leftrightarrow$ final water content $c_v \leftrightarrow D(\theta)$ coefficient of consolidation $\leftrightarrow$ moisture diffusivity			

3 Experimental works

3.1 Soils tested

On one hand, some experimental results from the pioneering research on unsaturated soils at the Laboratory of Geotechnics of CEDEX with Madrid clay [28, 29] and further research [30] have been reanalysed; and, on the other, samples of shale rock have been tested at CEDEX as well, after been reduced to powder.

Juca [29] studied the strength and deformability of two unsaturated clays from Madrid: "grey clay", containing mica as main mineral, and to a lesser extent, kaolinite, smectite and palygorskite (Atterberg Limits: LL=71 and PL=36); and "red clay" formed basically by palygorskite, (Atterberg Limits: LL=33 and PL=19). The initial suctions of the samples compacted at the Standard Proctor (SP) conditions were: 800 kPa and 280 kPa, respectively.

Asanza [30] studied a mixture of Madrid "grey clay" and Na-montmorillonite (7%), reaching LL=74 and LL=37, while its initial suction (at SP) was 1,9 MPa and the swelling pressure reached 300 kPa.

In addition, powdered specimens of shale rock retrieved at a depth of 100m from a shallow well in Pennsylvania were tested at CEDEX, among other, for the purpose of the present research. It is a very hard rock, with unconfined compression strength of ~100 MPa on average and very low porosity ($n=0,05$). Its principal minerals are phyllosilicates (~60%) and quartz (30%). The specimens belong to the Marcellus Formation (Appalachian Basin, between the States of Virginia and Pennsylvania), the largest source of natural gas in the United States, extracted by hydraulic fracturing ("fracking"). The Total Organic Carbon (TOT) of the tested shale specimens yielded 2,6 % approximately. Villamor-Lora et al. [31] carried out a detailed study of the anisotropic strain-stress relationships at several temperatures with similar specimens of this shale rock.

The powdered sample reached LL=25. The powdered samples were compacted at dry conditions (hygroscopic water content was lower than 1 %), reaching a dry unit weight of 17 kN/m³.

3.2 Description of the tests and results

All four soils described above were tested at CEDEX with pressure cells, used to determine the SWRC. These apparatuses are based on the axis translation technique, i.e., a source of water at the cell base is kept at atmospheric pressure, while the air phase is risen in the cell with pressurized N₂ to a target value. Hence, cavitation is prevented, leading to a value of matric suction ($u_{\text{water}} - u_{\text{air}}$) that just equals the target pressure of the air phase. A membrane of cellulose enables the coexistence of pressurized air and atmospheric water on either side of the membrane and, ultimately, in the soil sample. Note that dissolved salts can flow freely through the membrane, so osmotic suction is cancelled out.

All these tests share that the samples were weighed at intervals, as a means of measuring the rate of intake or loss of water over the equilibrium process.

3.2 Tests results

Figure 3 shows the evolution of the water content of 4 samples of Madrid clays tested by Juca [29] Note that imposed values higher than the initial suction cause a drying process; else, a wetting one. Figure 4 depicts the evolution of the water content of 4 samples subjected to 25, 100, 500 and 1000 kPa (all wetting processes) tested by Asanza (2009). Finally, the tests with the powdered dry Marcellus shale rock conducted "ad hoc" at CEDEX are shown in Figure 5. The samples were subjected to the following suctions: 20, 100, 600 and 1250 kPa. As they were compacted at dry conditions, they all underwent a wetting process.

As powdered samples of Marcellus shale were thinner ($H=7\text{mm}$) than usual ($H=20\text{mm}$), the wetting flow became much faster than expected; measurements at 5 days would have improved the analysis with Asaoka's method.

Values are plotted in Figures 3, 4 and 5 as increments of gravimetric water contents (w), despite the described theoretical framework is addressed with the volumetric water content (θ). Thus, negligible volumetric changes due to changes in water content should be assumed when applying the predictive methods.

4 Data analysis with predictive methods

Figures 6 and 7 are shown as explanatory examples. They depict the results, respectively, of the powdered Marcellus shale using the hyperbolic method, and the clay tested by Asanza (2009) using the Asaoka's method.

Table 2 includes the moisture diffusivity $D(\theta)$ arrived at by both methods, using the equations in Table 1.

Fig. 3. Change of water content; Madrid (red & grey) clay [29].

Richards' equation assumes that the bulk volume of the soil is constant. Even though clays undergo changes of volume when wetted or dried, in this case the samples maintained a certain range of suctions throughout the tests. Thus, the higher the suctions and the lower both the change of water content and the LL, then, the closer the allowance of such assumption. Concerning this research, except the sample at 25 kPa, volume changes measured by Asanza [30] (LL=74) seemed to be negligible.

Fig. 4. Evolution of water content; clays tested by Asanza [30]

Fig. 5. Evolution of water content of powdered Marcellus shale

Fig. 6. Data from Marcellus shale; hyperbolic method

It is worth noting that as both methods apply just when the increments of water contents exceed 60 % of the total increment, values of $D(\theta)$ in Table 2 seem to be averaged values of such range of water contents. Final water contents (w_∞) are also presented in Table 2. Asaoka's method tends to yield values of $D(\theta)$ slightly higher. Nevertheless, $D(\theta)$ does not vary much (most in the range of 1×10^{-6} to 6×10^{-6} cm^2/s), despite the wide range of suctions applied to the samples (20 kPa – 2 MPa).

3 Conclusions

The two predictive methods described above are useful for processes governed by the equation of diffusion. In particular, they prove to be reliable for assessing the

moisture diffusivity (D) of unsaturated flows. A limited number of weight determinations of the sample undergoing the flow due to a imposed suction at its base suffices to obtain an estimation of D . Finally, $k(\theta)$ can be assessed as long as the SWRC is available.

Fig. 6. Data from Asanza (2009); Asaoka's method

Table 2. Prediction of D (cm^2/s) and w_∞ (%)

Soil	Hyperbolic	Asaoka
Here, e refers to 10 (i.e.: $e-3 \equiv 10^{-3}$)		
grey clay Juca (1990)	Drying, 2 MPa; $\Delta t=4$ days	
	$M=0.17; 0.16$ $N=31; 29$ $D(\theta)=1.8e-6$ $w_\infty=5.8; 6.1$	$M=0.729; 0.697$ $N=1.499; 1.516$ $D(\theta)=4.7e-6; 5.3e-6$ $w_\infty=5.5; 5.0$
	wetting, 30 kPa	
red clay Juca (1990)	$M=0.21; N=22;$ $D(\theta)=3.1e-6; w_\infty=4.8$	$M=0.527; N=1.992$ $D(\theta)=9.4e-6; w_\infty=4.2$
	Drying, 1 MPa; $\Delta t=4$ days	
	$M=0.08; N=37;$ $D(\theta)=7.0e-7; w_\infty=2.1$	$M=0.296; N=1.250$ $D(\theta)=1.8e-6 w_\infty=1.8$
Madrid clay Asanza (2009) (all, wetting)	25; 100; 500; 1000 kPa; $\Delta t=5$ days	
	$M=0.06; N=4$ $D(\theta)=4.8e-6; w_\infty=16.4$	$M=0.702; N=4.592$ $D(\theta)=4.2e-6; w_\infty=15.4$
	$M=0.11; N=5$ $D(\theta)=7.1e-6; w_\infty=8.8$	$M=0.654; N=3.030$ $D(\theta)=5.0e-6; w_\infty=8.8$
	$M=0.22; N=10$ $D(\theta)=7.1e-6; w_\infty=4.5$	$M=0.631; N=1.619$ $D(\theta)=5.4e-6; w_\infty=4.5$
	$M=0.67; N=255$ $D(\theta)=8.5e-7; w_\infty=1.5$	$M=0.862; N=0.200$ $D(\theta)=1.7e-6; w_\infty=1.5$
Powdered Marcellus shale	20; 100; 600; 1250 kPa; $\Delta t=4$ days	
	$M=0.04; N=2$ $D(\theta)=7.9e-7; w_\infty=22.9$	(only reliable: 100 kPa) $M=0.098;$ $N=15.01$ $D(\theta)=1.7e-5$ $w_\infty=15.4$
	$M=0.06; N=2$ $D(\theta)=1.2e-6; w_\infty=16.7$	
	$M=0.16; N=2$ $D(\theta)=3.2e-6; w_\infty=6.3$	
	$M=0.19; N=5$ $D(\theta)=1.5e-6; w_\infty=5.2$	

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