Mathematical modeling of 3D current flows for narrow shallow water bodies of complicated forms

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Abstract. This article is devoted to the modeling of three-dimensional currents for narrow shallow water systems like Kerch straight. Model, which is presented in this article, is based on previously constructed 3D discrete model which has used cell filling function and rectangular uniform grids. The effect of rising free surface function has been detected in narrowest part of straight in numerical modelling. The proposed discrete models remain stable at depth differences tens of times, which is an important factor for coastal systems. Also this approach may be applied for wave evolution prediction in narrow straits of complicated bottom relief and coastal line.

1 Introduction

The article deals with the mathematical modeling of three-dimensional currents for narrow shallow water bodies of complex shape. The assumed effect of the free surface elevation function in the narrowest part of the computational domain was confirmed by numerical simulation.

To describe wave processes in shallow water bodies, a system of Navier-Stokes equations is used, which includes three equations of motion in areas with dynamically changing geometry of the computational domain. Discrete models remain stable, in contrast to known ones, with significant depth differences, which is an important factor for coastal systems. Software has been developed that makes it possible to predict three-dimensional currents in narrow places of shallow water bodies of complex shape, such as the Azov Sea, including the Kerch Strait, where it is possible to increase the amplitude of the free surface rise function due to the narrowing and difference in depths.

2 Three-dimensional mathematical model of shallow water bodies hydrodynamics

3D mathematical model of the wave hydrodynamics of a shallow water body includes [6-8]:

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– equations of motion (Navier-Stokes):

$$\begin{aligned} u'_t + uu'_x + vv'_y + ww'_z &= -\frac{1}{\rho} p'_x + (\mu u'_x)'_x + (\mu u'_y)'_y + (v u'_z)'_z, \\ v'_t + uv'_x + vv'_y + ww'_z &= -\frac{1}{\rho} p'_y + (\mu v'_x)'_x + (\mu v'_y)'_y + (v v'_z)'_z, \\ w'_t + uw'_x + vv'_y + ww'_z &= -\frac{1}{\rho} p'_z + (\mu w'_x)'_x + (\mu w'_y)'_y + (v w'_z)'_z + g; \end{aligned} \quad (1)$$

– continuity equation:

$$\rho'_t + (\rho u)'_x + (\rho v)'_y + (\rho w)'_z = 0, \quad (2)$$

where $\mathbf{V} = \{u, v, w\}$ is the velocity vector of the water flow of a shallow water body; ρ is the density of the aquatic environment; p is the hydrodynamic pressure; g is the gravitational acceleration; μ ν are coefficients of turbulent exchange in the horizontal and vertical directions; n is the normal vector to the surface describing the boundary of the computational domain.

Fig. 1. Initial geometry of the computational domain.

Fig. 2. Simplified geometry and contours of the bottom surface depth function.

Add boundary conditions to system (1)-(2):

$$\text{the entrance (left border): } \mathbf{V} = \mathbf{V}_0, p'_n = 0,$$

$$\text{the bottom border: } \rho\mu(\mathbf{V}_\tau)'_n = -\tau, \mathbf{V}_n = 0, p'_n = 0,$$

$$\text{the lateral border: } (\mathbf{V}_\tau)'_n = 0, \mathbf{V}_n = 0, p'_n = 0,$$

$$\text{the upper border: } \rho\mu(\mathbf{V}_\tau)'_n = -\tau, w = -\omega - p'_t / \rho g, p'_n = 0, \quad (3)$$

$$\text{the surface of the structure: } \rho\mu(\mathbf{V}_\tau)'_n = -\tau, w = 0, p'_n = 0,$$

where ω is the intensity of evaporation of a liquid, $\mathbf{V}_n, \mathbf{V}_\tau$ are the normal and tangential component of the velocity vector, $\tau = \{\tau_x, \tau_y, \tau_z\}$ is the vector of tangential stress [1-3].

Let $\tau = \rho_a C d_s |\mathbf{w}| \mathbf{w}$ is the vector of tangential stress for the free surface, $C d_s = 0,0026$, $\mathbf{w}$ is the wind velocity relative to water, ρ_a is the atmosphere density, $C d_s$ is the dimensionless surface resistance coefficient, which depends on wind speed [4].

Let us set the tangential stress vector for the bottom taking into account the movement of water as follows: $\tau = \rho C d_b |\mathbf{V}| \mathbf{V}$, $C d_b = g k^2 / h^{1/3}$, where $k = 0.04$ is the group roughness coefficient in the Manning formula, considered in the range of 0.025 – 0.2; $h = H + \eta$ is the depth of the water area, [m]; H is the depth to the undisturbed surface, [m]; η is the elevation of the free surface relative to the geoid (sea level), [m].

Let use an approximation that allows us to build a non-uniform in depth depth vertical turbulent exchange coefficient on the basis of measured pulsations of the water flow velocity [5]:

$$\nu = C_s^2 \Delta^2 \frac{1}{2} \sqrt{\left(\frac{\partial \bar{U}}{\partial z}\right)^2 + \left(\frac{\partial \bar{V}}{\partial z}\right)^2}, \quad (4)$$

where C_s is the dimensionless empirical constant, determined on the basis of the calculation of the attenuation process of homogeneous isotropic turbulence; Δ is the characteristic scale of the grid; $\bar{U}, \bar{V}$ are the time-averaged ripple components of the velocity of the water flow in the horizontal direction.

The computational domain approximation

The computational domain inscribed in a parallelepiped. For the numerical realization of the discrete mathematical model of the hydrodynamic problem posed, a uniform grid is introduced:

$$\begin{aligned} \bar{w}_h &= \{t^n = n\tau, x_i = ih_x, y_j = jh_y, z_k = kh_z; \\ n &= \overline{0..N_t}, i = \overline{0..N_x}, j = \overline{0..N_y}, k = \overline{0..N_z}; \\ N_t \tau &= T, N_x h_x = l_x, N_y h_y = l_y, N_z h_z = l_z \}, \end{aligned}$$

where τ is the step by the time, h_x, h_y, h_z are steps in space, N_t is the number of time layers, T is the upper bound on the time coordinate, N_x, N_y, N_z are the number of nodes by spatial coordinates, l_x, l_y, l_z are the boundaries along the parallelepiped in the direction of the axes Ox, Oy and Oz accordingly.

The method of correction to pressure was used to solve the hydrodynamic problem. The variant of this method in the case of a variable density will take the form [6-8]:

$$\begin{aligned} \frac{\tilde{u}-u}{\tau} + u\bar{u}'_x + v\bar{u}'_y + w\bar{u}'_z &= (\mu\bar{u}'_x)'_x + (\mu\bar{u}'_y)'_y + (\nu\bar{u}'_z)'_z, \\ \frac{\tilde{v}-v}{\tau} + u\bar{v}'_x + v\bar{v}'_y + w\bar{v}'_z &= (\mu\bar{v}'_x)'_x + (\mu\bar{v}'_y)'_y + (\nu\bar{v}'_z)'_z, \\ \frac{\tilde{w}-w}{\tau} + u\bar{w}'_x + v\bar{w}'_y + w\bar{w}'_z &= (\mu\bar{w}'_x)'_x + (\mu\bar{w}'_y)'_y + (\nu\bar{w}'_z)'_z + g, \\ p''_{xx} + p''_{yy} + p''_{zz} &= \frac{\hat{\rho}-\rho}{\tau^2} + \frac{(\hat{\rho}\tilde{u})'_x}{\tau} + \frac{(\hat{\rho}\tilde{v})'_y}{\tau} + \frac{(\hat{\rho}\tilde{w})'_z}{\tau}, \\ \frac{\hat{u}-\tilde{u}}{\tau} &= -\frac{1}{\rho}\hat{p}'_x, \quad \frac{\hat{v}-\tilde{v}}{\tau} = -\frac{1}{\rho}\hat{p}'_y, \quad \frac{\hat{w}-\tilde{w}}{\tau} = -\frac{1}{\rho}\hat{p}'_z, \end{aligned} \quad (5)$$

where $V = \{u, v, w\}$ are the components of the velocity vector, $\{\hat{u}, \hat{v}, \hat{w}\}, \{\tilde{u}, \tilde{v}, \tilde{w}\}$ are the components of the velocity vector fields on the «new» and intermediate time layers, respectively, $\bar{u} = (\tilde{u} + u)/2$, $\hat{\rho}$ and ρ are the distribution of the density of the aqueous medium on the new and previous time layers, respectively.

In the construction of discrete mathematical models of hydrodynamics, the fullness of the control cells was taken into account, which makes it possible to increase the real accuracy of the solution in the case of a complex geometry of the investigated region by improving the approximation of the boundary.

Through $o_{i,j,k}$ marked the volume of fluid (VOF) of the cell (i, j, k) [9-10]. VOF is determined by the pressure of the liquid column inside this cell. If the average pressure at the nodes that belong to the vertices of the cell in question is greater than the pressure of the liquid column inside the cell, then the cell is considered to be full ($o_{i,j,k} = 1$). In the general case, VOF can be calculated by the following formula:

$$o_{i,j,k} = \frac{p_{i,j,k} + p_{i-1,j,k} + p_{i,j-1,k} + p_{i-1,j-1,k}}{4\rho gh_z} \quad (6)$$

We introduce the coefficients $q_0, q_1, q_2, q_3, q_4, q_5, q_6$, describing VOF of regions located in the vicinity of the cell. The value q_0 characterizes the VOF of the area

$$D_0: \{x \in (x_{i-1}, x_{i+1}), y \in (y_{j-1}, y_{j+1}), z \in (z_{k-1}, z_{k+1})\},$$

$$q_1 _ D_1 = D_0 \cap \{x > x_i\}, q_2 _ D_2 = D_0 \cap \{x < x_i\}, q_3 _ D_3 = D_0 \cap \{y > y_j\},$$

$$q_4 _ D_4 = D_0 \cap \{y < y_j\}, q_5 _ D_5 = D_0 \cap \{z > z_k\}, q_6 _ D_6 = D_0 \cap \{z < z_k\}.$$

The filled parts of the regions D_m will be called Ω_m , where $m = \overline{0..6}$. In accordance with this, the coefficients q_m can be calculated using the formulas:

$$(q_m)_{i,j,k} = \frac{S_{\Omega_m}}{S_{D_m}}, \quad (q_1)_{i,j,k} = \frac{o_{i+1,j,k} + o_{i+1,j+1,k} + o_{i+1,j,k+1} + o_{i+1,j+1,k+1}}{4},$$

$$(q_2)_{i,j,k} = \frac{o_{i,j,k} + o_{i,j+1,k} + o_{i,j,k+1} + o_{i,j+1,k+1}}{4},$$

$$(q_3)_{i,j,k} = \frac{o_{i+1,j+1,k} + o_{i,j+1,k} + o_{i+1,j+1,k+1} + o_{i,j+1,k+1}}{4},$$

$$(q_4)_{i,j,k} = \frac{o_{i,j,k} + o_{i+1,j,k} + o_{i,j,k+1} + o_{i+1,j,k+1}}{4},$$

$$(q_5)_{i,j,k} = \frac{o_{i,j,k+1} + o_{i+1,j,k+1} + o_{i+1,j+1,k+1} + o_{i,j+1,k+1}}{4},$$

$$(q_6)_{i,j,k} = \frac{o_{i,j,k} + o_{i+1,j,k} + o_{i+1,j+1,k} + o_{i,j+1,k}}{4}, \quad (q_0)_{i,j,k} = \frac{1}{2}((q_1)_{i,j,k} + (q_2)_{i,j,k}).$$

In the case of boundary conditions of the third kind $c'_n(x, y, t) = \alpha_n c + \beta_n$, the discrete analogues of the convective uc'_x and diffusion $(\mu c'_x)'_x$ transfer operators, taking into account the VOF, can be written in the following form [11, 12]:

$$(q_0)_i uc'_x \sqcap (q_1)_i u_{i+1/2} \frac{c_{i+1} - c_i}{2h_x} + (q_2)_i u_{i-1/2} \frac{c_i - c_{i-1}}{2h_x},$$

$$(q_0)_i (\mu c'_x)'_x \sqsupset (q_1)_i \mu_{i+1/2} \frac{c_{i+1} - c_i}{h_x^2} - (q_2)_i \mu_{i-1/2} \frac{c_i - c_{i-1}}{h_x^2} - |(q_1)_i - (q_2)_i| \mu_i \frac{\alpha_x c_i + \beta_x}{h_x}.$$

Similarly, approximations for the remaining coordinate directions will be recorded. The error in approximating the mathematical model is equal to $O(\tau + \|h\|^2)$, where $\|h\| = \sqrt{h_x^2 + h_y^2 + h_z^2}$. The conservation of the flow at the discrete level of the developed hydrodynamic model is proved, as well as the absence of non-conservative dissipative terms obtained as a result of discretization of the system of equations. A sufficient condition for the stability [13] and monotony of the developed model is determined on the basis of the maximum principle [14], with constraints on the step with respect to the spatial coordinates:

$h_x < |2\mu / u|$, $h_y < |2\mu / v|$, $h_z < |2\nu / w|$ or $Re \leq 2N$, where $Re = |V| \cdot l / \mu$ is the Reynolds number [10], l is the characteristic size of the region $N = \max \{N_x, N_y, N_z\}$. Discrete analogs of the system of equations (5) are solved by an adaptive modified alternating-triangular method of variational type [15].

Results of numerical experiments based on hydrodynamic model

In places with «bottleneck» geometry, frequent occurrence is the free surface rise in the narrowest part of the computational domain.

To simulate 3D current flows for narrow shallow water bodies, the bottom geometry of the «bottleneck» type was used with depths as close as possible to the real geometry of the bottom of the Kerch Strait (maximum depth 18 m from the Black Sea).

The source of disturbances is set at a given distance from the coastline. The modeling area is 50 by 50 km, the depth is 18 m; the peak rises 2 m above sea level.

When digitizing the surface relief, it is assumed that the average level is zero. The distances between the grid nodes are 10 cm vertically, 500 m horizontally, that is, 500×500 m.

A modern software package has been created, adapted for modeling 3D current flows for narrow shallow water bodies, used in a wide range of parameters for calculating the velocity and pressure fields of the water environment, as well as for assessing the impact on the coast.

Fig. 3. Simplified geometry: the velocity vector field of the aquatic environment movement.

Fig. 4. Initial geometry: the velocity vector field of the aquatic environment movement.

The developed software package allows experiments with both simplified geometry (bottleneck type) and geometry close to the real strait. Visualization of the results shows that a free surface elevation does indeed appear on the simplified geometry, and the results obtained for the simplified geometry are consistent with the results for the bottom geometry close to the real strait. The developed numerical algorithms for solving model problems and implementing a complex of programs can be used to study 3D current flows for narrow

shallow water bodies of complicated forms, such as the Azov and White Seas, the North Caspian, as well as to find the velocity and pressure fields of the aquatic environment.

Figure 3, 4 show the results of the forecast of hydrodynamic processes in the flow of liquid in the strait, taking into account the simplified and real geometry of the bottom, the figure shows the vector field of the velocity of the water medium (cut from the XOZ plane) at the same time.

Figure 5 shows the pressure fields for the simplified and original bottom geometry.

Fig. 5. Bottom relief graphs of simplified and initial geometry.

3 Conclusion

The article describes mathematical modeling of 3D current flows for narrow shallow water bodies of complicated forms. This paper presents a continuous three-dimensional mathematical model of the hydrodynamics of shallow water bodies, its discretization. When constructing discrete mathematical models of hydrodynamics, the filling of control cells is taken into account, which makes it possible to increase the accuracy of the solution in a complex region of the investigated area by improving the approximation of the boundary. The article discusses the approximation of the problem of calculating the velocity field at an intermediate time layer in terms of spatial variables. The software package allows you to calculate the free surface elevation function, provided that discrete models are stable with dozens of times depth differences, which is an important element for coastal systems.

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