Basic Solutions of Three Dimensional Elasticity

WX Zhang^{1,*}

¹Nantong Polytechnic College, Nantong, Jiangsu, 226002, China

Abstract. Elastic calculation method is an important research content of computational mechanics. The problems of elasticity include basic equations and boundary conditions. Therefore, the final solution consists of the general solutions of the basic equations and the special solutions satisfying the boundary conditions. Numerical method is often used in practical calculation, but the analytical solution is also an important subject for researchers. In this paper, the basic solution of three-dimensional elastic materials is given by theoretical derivation.

1 Introduction

Elasticity mainly studies the deformation and internal force of elastic objects under the action of external forces and other external factors [1-3]. Saint Venant's principle is the most commonly used method of elasticity, and its correctness has been fully verified. According to Saint Venant's principle, the boundary conditions can be replaced by equivalent loads, which makes the problem greatly simplified. However, the Saint Venant solution can only satisfy the average boundary conditions, ignoring the local effect, so it is an approximate solution. In many practical problems, researchers often ignore the local effect solution and only care about the overall effect of load on materials and structures. However, for some special cases, such as stress concentration caused by temperature or displacement constraints, the stress is not uniformly distributed, and the local solution plays a key role [4-6].

In this paper, on the basis of fully studying the traditional theory of elasticity, the dual form equations of three dimensional elastic body are constructed by using the methods of separation of variables and variational principle. Thus, the basic problem of elasticity is transformed into the solution of eigenvalues and general eigenvectors. Through strict theoretical derivation, we get the all eigenvectors in analytical form, and establish the closed form basic solution space composed of each order solution vectors, which provides a reliable basis for the numerical calculation of further engineering problems.

2 Solution method

A common calculation model of cylinder in elasticity. The displacement and stress components of elastic cylindrical solution can be described as follows

$$\begin{aligned}\bar{u} &= M_{1j} (C_j J_{n-1} + C_{j+3} Y_{n-1}) + N_{1j} (C_j J_n + C_{j+3} Y_n) \\ \bar{v} &= N_{2j} (C_j J_n + C_{j+3} Y_n) + K_{2j} (C_j J_{n+1} + C_{j+3} Y_{n+1}) \\ \bar{w} &= N_{3j} (C_j J_n + C_{j+3} Y_n) + K_{3j} (C_j J_{n+1} + C_{j+3} Y_{n+1})\end{aligned}\quad (1)$$

and

$$\begin{aligned}\bar{p}_1 &= N_{4j} (C_j J_n + C_{j+3} Y_n) + K_{4j} (C_j J_{n+1} + C_{j+3} Y_{n+1}) \\ \bar{p}_2 &= N_{5j} (C_j J_n + C_{j+3} Y_n) + K_{5j} (C_j J_{n+1} + C_{j+3} Y_{n+1}) \\ \bar{p}_3 &= M_{6j} (C_j J_{n-1} + C_{j+3} Y_{n-1}) + N_{6j} (C_j J_n + C_{j+3} Y_n)\end{aligned}\quad (2)$$

Respectively. Using the generalized variable method, we get

*Corresponding author's e-mail: Wxzhang697608@sina.com

$$\begin{cases} a_4 \partial_r \bar{u} + 2a_1 G^* (\partial_\theta \bar{v} + \bar{u}) + a_1 \bar{p}_3 = \bar{\sigma}_r^0(\theta, z) \\ \partial_r \bar{v} + \partial_\theta \bar{u} - \bar{v} = a_3 \bar{\tau}_{r\theta}^0(\theta, z) \\ \bar{p}_1 = \bar{\tau}_{rz}^0(\theta, z) \end{cases} \quad (r = a, b) \quad (3)$$

and

$$\begin{aligned} (-\mu J_0 + 2J_1 / r)(h_3 + h_4) + (-\mu Y_0 + 2Y_1 / r) &= 0 \\ 2a_7 \mu J_1 h_1 + 2a_7 \mu Y_1 h_2 + (J_1 - 2a_7 \mu r J_0)(h_3 - h_4) &= 0 \\ 2a_7 (\mu^2 J_0 - \mu J_1 / r) h_1 - 2a_7 (-\mu^2 Y_0 + \mu Y_1 / r) h_2 &= 0 \end{aligned} \quad (4)$$

The solution can be described as follows by Bessel function:

$$\boldsymbol{\eta} = \begin{bmatrix} (-a_4 s_1 \alpha_{13} + a_2 \alpha_{43}) \sin(\mu a s_1) & (-a_4 s_2 \alpha_{14} + a_2 \alpha_{44}) \sin(\mu a s_2) \\ \alpha_{33} \cos(\mu a s_1) & \alpha_{34} \cos(\mu a s_2) \end{bmatrix} \quad (5)$$

The solution of the stretching problem is

$$\begin{aligned} \bar{u}^{(0)} &= 2\mu a_8 r J_0(\mu r) - [1 + 2\mu a_8 M] J_1(\mu r) \\ \bar{w}^{(0)} &= (2\mu a_8 M - 1) J_0(\mu r) + 2\mu a_8 r J_1(\mu r) \\ \bar{p}_1^{(0)} &= 4\mu^2 a_8 r G^* [r J_0(\mu r) - M J_1(\mu r)] \\ \bar{p}_3^{(0)} &= 4\mu^2 a_8 r G^* [r(\mu M - 2) J_0(\mu r) - \mu r J_1(\mu r)] \\ \bar{v}^{(0)} = \bar{p}_2^{(0)} &= 0 \end{aligned} \quad (6)$$

Numerical example

All eigenvalues and solutions can be obtained by numerical calculation. The first five displacement components and stress components are shown in Fig. 1-3,

respectively. Obviously, these eigensolutions have local characteristics and must be considered in the study of stress concentration and deformation concentration.

Figure 1. Component u

Figure 2. Component w

Figure 3. Stress distribution

3 Conclusion

Using the dual solutions, all eigenvalues and eigensolutions can be obtained. These eigensolutions include Saint Venant's solution to describe global deformation and non-zero eigensolutions are important bases to describe local deformation.

Reference

1. Faghidian, S.A. (2016) Unified formulation of the stress field of saint-Venant's flexure problem for symmetric cross-sections. *Int. J. Mech. Sci.*, 111-112: 65-72.
2. Pipkin, A.C., Rogers, T.G. (1968) A nonlinear integral representation for viscoelastic behavior. *J. Mech. Phys. Solids*, 16: 59-72.

3. Beom, H.G., Atluri, S.N. (1996) Near-tip fields and intensity factors for interfacial cracks in dissimilar anisotropic piezoelectric media. *Int. J. Fract.*, 75: 163-183.
4. Perzyna, P. (1963) The constitutive equations for rate sensitive plastic models. *Quart. Appl. Math.*, 20: 321-332.
5. Argyris, J.H., Doltsinis, I.S.T., William, K.J. (1979) New developments in the inelastic analysis of quasistatic and dynamic problems. *Int. J. Numer. Meth. Eng.*, 14: 1813-1850.
6. Barbero, E.J., Luciano, R. (1995) Micromechanical formulas for the relaxation tensor of linear viscoelastic composites with transversely isotropic fibers. *Int. J. Solids Struct.*, 32: 1859-1872.