DIFFERENTIAL QUADRATURE METHOD FOR VIBRATION ANALYSIS OF PRESTRESSED BEAMS

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Abstract. This paper investigates natural frequencies of free transverse vibrations of prestressed beams, and the governing equations and natural frequencies of the free vibration in related literatures are discussed and corrected. The differential quadrature methods (DQ) are applied directly to the corrected governing equations to get the values of natural frequency numerically. Under the simple supported boundary conditions, the natural frequencies of model beam are numerically studied, and the physical parameters of the beam are analyzed respectively. The numerical results show that the natural frequency values increase with the growth of concrete strength and eccentricity of prestressed steels. But with the increase of the span length of beam and values of original prestressing force, the natural frequency values decrease.

1 Introduction

Prestressed concrete technique can effectively improve the cracking of concrete structure, improve the stiffness and durability of the structure, and give full play to the high strength performance of materials, which has been widely used in civil engineering^[1-3]. In recent years, many researchers have studied the static performance of prestressed concrete structure, but the investigations on the complex dynamic performance are relatively insufficient.

The differential quadrature (DQ) method is a powerful tool for dealing with dynamical problems. Firstly it was introduced for structural analysis by Bert et al.^[4], and since then it has been successfully employed for the analysis of vibration of beams^[5-6].

For the vibration characteristics of prestressed beams, the formulas of different linear vibration frequencies are derived by applying the theory of material mechanics and structural dynamics^[3,7]. However, these formulas are based on simplified vibration control equations, and their accuracy is worth discussing. In this paper, the existing vibration control equations of prestressed beams are analyzed and corrected, and the differential quadrature method (DQ) is directly applied to the corrected equations. The differential equations are transformed to a generalized eigenvalue problem, and the numerical solutions of natural frequency are obtained based on the method of DQ. Under the simply supported boundary conditions, the first four order numerical solutions of natural frequency of model beam are given, and the different effects of physical parameters of prestressed beam and prestressed steel on natural frequency are analyzed.

The present paper is organized as follows. Section 2 establishes the governing equation for the transverse free vibration of a finite elastic prestressed beam. Section 3 develops the DQ schemes to discretize the governing equation. Section 4 presents numerical examples for investigations on the effects of the different parameters of the relevant beam and prestressed steel on the natural frequencies. Section 5 ends the paper with the concluding remarks.

2 Equation of motion

Fig. 1 The model of prestressed beam

Free transverse vibration of a prestressed beam can be described by transverse deflection of the cross section due to bending moment, depending on axial coordinate x and time t .

Fig.1 shows a finite elastic prestressed concrete beam with a length l , the bending moment and the axial force can be expressed as

$$\begin{cases} N_p = N_{p0} + \Delta N_p \\ M = N_p e = (N_{p0} + \Delta N_p) e \end{cases} \quad (1)$$

The free vibration equation of a prestressed beam is governed by the equation^[1]

$$\frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 y}{\partial x^2}) + \frac{\partial^2}{\partial x^2} (N_p y) - \frac{\partial^2}{\partial x^2} (M) + m \frac{\partial^2 y}{\partial t^2} = 0 \quad (2)$$

Substituting Eq. (1) into Eq. (2) yields

$$\frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 y}{\partial x^2}) + \frac{\partial^2}{\partial x^2} [(N_{p0} + \Delta N_p) y] - \frac{\partial^2}{\partial x^2} [(N_{p0} + \Delta N_p) e] + m \frac{\partial^2 y}{\partial t^2} = 0$$

(3)

where y , m and EI are vibration displacement, unit mass, and bending flexural rigidity of the beam. e , N_{p0} and ΔN_p are eccentricity, initial prestressing force and increased prestressing force due to flexural vibration.

It is generally accepted that y is much less than e , so the value of $\Delta N_{p,y}$ is negligibly small, and Eq.(3) can be rewritten as

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2}{\partial x^2} (N_{p0} y) - e \frac{\partial^2}{\partial x^2} (\Delta N_p) + m \frac{\partial^2 y}{\partial t^2} = 0 \quad (4)$$

Based on the assumption that increased prestressing forces ΔN_p is proportional to midspan deflection y , the horizontal displacement due to the applied midspan load F can be calculated from the equation

$$\delta = \int \frac{m_1 M}{EI} dx = \frac{Fl^2 e}{8EI} \quad (5)$$

The horizontal displacement caused by the unit force acting on the anchorage point can be obtained as

$$\delta_1 = \int \frac{m_1^2}{EI} dx + \int \frac{N_1^2}{EA} dx = \frac{le^2}{EI} + \frac{l}{EA} + \frac{l_t}{E_t A_t} \quad (6)$$

The change of anchoring force ΔN_p caused by load F is

$$\Delta N_p = \frac{\delta}{\delta_1} = \frac{\int \frac{m_1 M}{EI} dx}{\int \frac{m_1^2}{EI} dx + \int \frac{N_1^2}{EA} dx} = \frac{\frac{Fl^2 e}{8EI}}{\frac{le^2}{EI} + \frac{l}{EA} + \frac{l_t}{E_t A_t}} = \frac{Fle}{8(e^2 + \lambda)} \quad (7)$$

where $\lambda = \frac{l}{A} + \frac{EI}{E_t A_t} \frac{l_t}{l}$, A , I , l and EI are

cross-sectional area, cross-sectional moment of inertia, span length and bending flexural rigidity of the beam. l_t and $E_t A_t$ are length and flexural rigidity of prestressed reinforcements.

From Eq. (7), the load F can be given in the form

$$F = \frac{8(e^2 + \lambda)}{le} \Delta N_p \quad (8)$$

The deflection of the midpoint of a beam under the action of force F is

$$y_F = \frac{l^3}{48EI} F \quad (9)$$

Substituting Eq. (8) into Eq. (9) yields

$$y_F = \frac{l^2(e^2 + \lambda)}{6EIe} \Delta N_p \quad (10)$$

By the displacement reciprocal theorem, the upward displacement at midpoint due to ΔN_p is

$$y_{\Delta N_p} = \frac{l^2 e}{8EI} \Delta N_p \quad (11)$$

Deflection y can be calculated as

$$y = y_F - y_{\Delta N_p} = \frac{l^2(e^2 + \lambda)\Delta N_p}{6EIe} - \frac{l^2 e \Delta N_p}{8EI} = \frac{l^2(e^2 + 4\lambda)}{24EIe} \Delta N_p \quad (12)$$

Therefore increased prestressing forces ΔN_p can be expressed as

$$\Delta N_p = \frac{24EIe}{l^2(e^2 + 4\lambda)} y \quad (13)$$

Substitution of Eq. (13) into Eq. (4) yields

$$EI \frac{\partial^4 y}{\partial x^4} + \left[N_{p0} - \frac{24EIe^2}{l^2(e^2 + 4\lambda)} \right] \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} = 0 \quad (14)$$

Eq. (14) is the free vibration equation of prestressed beam with straight prestressed steel. It is derived by applying the theory of material mechanics and structural dynamics.

With method of separation of variables, the vibration solution to Eq. (14) can be assumed as

$$y = X_{(x)} \cdot e^{i\omega_n t} \quad (15)$$

Substituting Eq. (15) into Eq. (14) yields

$$EI \frac{d^4 X}{dx^4} - \left[N_{p0} - \frac{24EIe^2}{l^2(e^2 + 4\lambda)} \right] \frac{d^2 X}{dx^2} = m\omega^2 X \quad (16)$$

For a simply supported beam, $X(x)$ can be expressed as

$$X(x) = A \sin \frac{n\pi}{L} x, \quad (n = 1, 2, 3, 4L) \quad (17)$$

Substitution of Eq. (17) into Eq. (16) yields

$$EI \left(\frac{n\pi}{L} \right)^4 + \left[N_{p0} - \frac{24EIe^2}{l^2(e^2 + 4\lambda)} \right] \left(\frac{n\pi}{L} \right)^2 = m\omega^2 \quad (18)$$

Solving Eq. (18), the explicit formulae for natural circular frequencies can be obtained

$$\omega_n = \sqrt{\frac{EI}{m}} \cdot \left(\frac{n\pi}{l} \right)^2 \xi \quad (19)$$

where

$$\xi = \sqrt{1 - \left(\frac{l}{n\pi} \right)^2 \frac{N_{p0}}{EI} + \frac{24}{(n\pi)^2} \frac{e^2}{(e^2 + 4\lambda)}} \quad (20)$$

With Eq. (19), the natural circular frequencies of free vibration of a prestressed beam can be easily calculated. But it must be noticed that these formulas are based on simplified vibration control equations, and their accuracy is worth discussing.

In paper [2], Eq. (14) is rewritten as^[2]

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} + \left[N_{p0} - \frac{6EIe^2}{(e^2 + 4\lambda)(lx - x^2)} \right] \frac{\partial^2 y}{\partial x^2} + \frac{12EIe^2(l - 2x)}{(e^2 + 4\lambda)(lx - x^2)^2} \frac{\partial y}{\partial x} - \frac{12EIe^2(3x^2 - 3lx + l^2)}{(e^2 + 4\lambda)(lx - x^2)^3} y = 0 \quad (21)$$

It is a complex partial differential equation with variable coefficients, which cannot be directly solved to obtain the analytical expression of natural frequency.

In paper [3], considering ΔN_p is proportional to the midpoint ($x=l/2$) deflection of the beam, the simplified equation is given by

$$EI \frac{\partial^4 y}{\partial x^4} + \left[N_{p0} - \frac{24EIe^2}{l^2(e^2 + 4\lambda)} \right] \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} = 0 \quad (22)$$

Solving the equation with method of separation of variables, the explicit formulae for natural circular frequencies can be obtained^[3]. Comparing two equations, it can be found that the last two items of Eq. (21) disappear in Eq. (22). However, when $x=l/2$ is substituted into Eq. (21), the coefficient before y term is not zero,

and this term mainly reflects the influence of eccentricity on vibration performance. So the accuracy of the analytical expression of natural frequency obtained by Eq. (22) is also worth discussing.

Considering the coefficient before y term of Eq. (21) is not zero, this paper corrected the Eq. (22) as

$$EI \frac{\partial^4 y}{\partial x^4} + \left[N_{p^0} - \frac{24EIe^2}{l^2(e^2 + 4\lambda)} \right] \frac{\partial^2 y}{\partial x^2} - \frac{192EIe^2}{l^4(e^2 + 4\lambda)} y + m \frac{\partial^2 y}{\partial t^2} = 0 \quad (23)$$

To simplify the expression, the Eq. (23) can be rewritten as

$$\rho A y_{,tt} + EI y_{,xxxx} + k_p y_{,xx} - k_e y = 0 \quad (24)$$

where

$$k_p = N_{p^0} - \frac{24EIe^2}{l^2(e^2 + 4\lambda)}, \quad k_e = \frac{192EIe^2}{l^4(e^2 + 4\lambda)} \quad (25)$$

Introducing the non-dimensional quantities into Eq. (24) results in the non-dimensional form

$$y_{,tt} + k_f^2 y_{,xxxx} + k_1 y_{,xx} - k_2 y = 0 \quad (26)$$

For simply supported beams, the dimensionless boundary conditions can be expressed by

$$y(0, t) = y(1, t) = 0, \quad y''(0, t) = y''(1, t) = 0 \quad (27)$$

3 Differential quadrature method

Eq.(26) is a complex partial differential equation, and the analytical expression of natural frequency cannot be obtained directly. In here, the method of DQ is employed to numerically solve the problem. This method requires to discretize the domain of the problem into N points. Then the derivatives at any point are approximated by a weighted linear summation of all the functional values along the discretized domain.

Consider the computational domain $0 \leq x \leq 1$ of the beam. The number of sampling point is N in the x direction. Introduce N sampling point as [5]

$$x_1 = 0, x_N = 1, \quad x_i = \frac{1}{2} \left[1 - \cos \left(\frac{i-1}{N-1} \pi \right) \right] \quad (i = 2, 3, \dots, N-1) \quad (28)$$

The quadrature rules for the derivatives of a function at the sampling points yield

$$y_{,xx}(x_i, t) = \sum_{j=1}^N A_{ij}^{(2)} y(x_j, t), \quad (29)$$

$$y_{,xxxx}(x_i, t) = \sum_{j=1}^N A_{ij}^{(4)} y(x_j, t)$$

Substitution of Eq. (29) into Eq. (26) yields

$$\ddot{y}_i + \sum_{k=1}^N \left(k_f^2 A_{ik}^{(4)} + k_1 A_{ik}^{(2)} \right) y_k - k_2 y_i = 0 \quad (30)$$

where

$$y_i(t) = y(x_i, t) \quad (31)$$

under simply supported boundary conditions

$$y_1 = y_N = 0, \quad \sum_{k=1}^N A_{1k}^{(2)} y_k = \sum_{k=1}^N A_{Nk}^{(2)} y_k = 0 \quad (32)$$

where the weight coefficient is defined by

$$A_{ik}^{(1)} = \begin{cases} \frac{\prod_{\mu=1, \mu \neq i}^N (x_i - x_\mu)}{\prod_{\mu=1, \mu \neq k}^N (x_i - x_\mu)} & (i, k = 1, 2, \dots, N, i \neq k) \\ \sum_{\mu=1, \mu \neq i}^N \frac{1}{x_i - x_\mu} & (i = 1, 2, \dots, N, i = k) \end{cases} \quad (33)$$

and in case of $r=2, 3, \dots, N-1$

$$A_{ik}^{(r)} = \begin{cases} r \left(A_{ii}^{(r-1)} A_{ik}^{(1)} - \frac{A_{ik}^{(r-1)}}{x_i - x_k} \right) & (i, k = 1, 2, \dots, N, i \neq k) \\ - \sum_{\mu=1, \mu \neq i}^N A_{i\mu}^{(r)} & (i = 1, 2, \dots, N, i = k) \end{cases} \quad (34)$$

To overcome difficulties in the implementation of the boundary conditions, the modified weighting coefficient matrix are introduced as

$$[\tilde{A}^{(2)}] = \begin{bmatrix} 0 & 0 & \dots & 0 \\ A_{21}^{(2)} & A_{22}^{(2)} & \dots & A_{2N}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(N-1)1}^{(2)} & A_{(N-1)2}^{(2)} & \dots & A_{(N-1)N}^{(2)} \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (35)$$

$$[\tilde{A}^{(r)}] = [A^{(1)}][\tilde{A}^{(r-1)}] \quad (r = 3, 4, \dots) \quad (36)$$

Substituting matrix (35) and (36) into Eq. (30), one obtain

$$\ddot{y}_i + \sum_{k=1}^N \left(k_f^2 \tilde{A}_{ik}^{(4)} + k_1 \tilde{A}_{ik}^{(2)} \right) y_k - k_2 y_i = 0 \quad (37)$$

Eq. (37) leads to a generalized eigenvalue problem that can be numerically solved.

4 Numerical results

Consider a finite elastic prestressed concrete model beam, the parameters of the beam are listed in Table 1. Under pinned boundary conditions, from Eq. (37), the values of natural frequency can be found by numerical method.

Table 1 Properties of the beam

Item	Notation	Value
Young's modulus of concrete	E_c	31.5 GPa
Young's modulus of prestressed steel	E_t	195 GPa
Mass density	ρ	$2.6 \times 10^3 \text{ kg/m}^3$
Cross sectional area	A	$8 \times 10^4 \text{ mm}^2$
Second moment of area	I	$2.67 \times 10^8 \text{ mm}^4$
Eccentricity	e_0	30mm
Length	L	7 m

For the free motion of the model beam, the effects of parameters on the natural frequencies are respectively studied. In Table 2, the first four natural frequencies of the model beam with various concrete strength are given. For same concrete strength, with the growth of the orders, the natural frequencies increase greatly, but for the beam with higher concrete strength, the natural frequencies increase slightly. In Table 3, it shows that for same span length, with the growth of the orders, the natural frequencies increase significantly, but with the increase of span length, the natural frequencies decrease a little,

and the natural frequency values of higher orders decrease more quickly.

Table 2 Natural frequencies with various concrete strength

	f_1	f_2	f_3	f_4
C20	19.381	74.476	166.271	294.780
C30	20.700	79.988	178.778	317.080
C35	21.085	81.602	182.441	323.612
C40	21.308	82.542	184.577	327.423

Table 3 Natural frequencies with various beam length

L	f_1	f_2	f_3	f_4
6.5	24.335	94.517	211.465	375.190
7.0	21.085	81.602	182.441	323.612
7.5	18.461	71.182	159.024	282.000
8.0	16.312	62.652	139.858	247.942

Table 4 Natural frequencies with various original prestressing force

N_{p0}	f_1	f_2	f_3	f_4
600	21.394	81.923	182.764	323.936
800	21.240	81.763	182.603	323.774
1000	21.085	81.602	182.441	323.612
1200	20.929	81.441	182.279	323.450

Table 5 Natural frequencies with various eccentricity

e_0	f_1	f_2	f_3	f_4
20	20.976	81.490	182.328	323.499
30	21.085	81.602	182.441	323.612
50	21.326	81.889	182.793	324.051
80	21.684	82.265	183.171	324.432

In Table 4 and Table 5, with the increase of the values of the original prestressing force, the natural

frequencies decrease slightly, but with the growth of eccentricity, the natural frequencies increase a little.

5 Conclusions

This paper investigates natural frequencies of free transverse vibrations of a finite prestressed beams under simply supported boundary conditions, and the free vibration governing equations in related literatures are discussed and rechecked. As for the complex partial differential equation, the analytical expression of natural frequency cannot be obtained directly, the differential quadrature methods are applied to get the numerical results, and the effects of the physical parameters of the beam on the natural frequencies are respectively analyzed. The numerical results show that the natural frequency values increase with the growth of concrete strength and eccentricity of prestressed steels. But with the increase of the beam span length and values of original prestressing force, the natural frequency values decrease.

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