

Field theory in normal conical coordinates

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Abstract. In the scientific literature, the field theory is most fully covered in the cylindrical and spherical coordinate systems. This is explained by the fact that the mathematical apparatus of these systems is most well studied. When the source of field has a more complex structure than a point or a straight line, there is a need for new approaches to their study. The goal of this research is to adapt the field theory related to curvilinear coordinates in order to represent it in the normal conical coordinates. In addition, an important part of the research is the development of a geometrical modeling apparatus for scalar and vector field level surfaces using computer graphics. The paper shows the dependences of normal conical coordinates on rectangular Cartesian coordinates, Lamé coefficients. The differential characteristics of the scalar and vector fields in normal conical coordinates are obtained: Laplacian of scalar and vector fields, divergence, rotation of the vector field. The example case shows the features of the application of the mathematical apparatus of geometrical field modeling in normal conical coordinates. For the first time, expressions for the characteristics of the scalar and vector fields in normal conical coordinates are obtained. Methods for geometrical modeling of fields using computer graphics have been developed to provide illustration in their study.

1 Introduction

Many different processes and phenomena are modeled using the mathematical apparatus of the field theory. The elements of the field theory are most fully represented in the vector presentation in rectangular Cartesian coordinates [1-2].

When the field sources have different geometrical shapes, it is convenient to describe them using special coordinate systems. For example, if the field source is concentrated at a point, its differential characteristics are conveniently described in the spherical coordinates. The obtained level surfaces will be spheres with the center in the field source (concentric spheres). If the source of the field is a straight line, then the differential characteristics will be described in the cylindrical coordinates. In this case, the level surfaces will be coaxial cylinders.

The field theory with point and linear sources is most fully investigated and described in the scientific literature [3]. This is because the describing mathematical apparatus is based on the use of spherical and cylindrical coordinate systems.

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$$\begin{aligned}
 t &= \frac{\pi}{2}, \text{ if } x = 0, y > 0; \\
 t &= \frac{3\pi}{2}, \text{ if } x = 0, y < 0; \\
 t &= \arctg \frac{y}{x}, \text{ if } x > 0, y > 0; \\
 t &= \arctg \frac{y}{x} + \pi, \text{ if } x < 0; \\
 t &= \arctg \frac{y}{x} + 2\pi, \text{ if } x > 0, y < 0; \\
 u &= \sqrt{x^2 + y^2} \sin \alpha + z \cdot \cos \alpha; \\
 v &= \sqrt{x^2 + y^2} \cos \alpha - z \cdot \sin \alpha.
 \end{aligned} \tag{2}$$

The condition for the possibility of the unique solution of equations (1) for t, u, v and equations (2) for x, y, z

$$\frac{D(x, y, z)}{D(t, u, v)} = \frac{\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}} = u \sin \alpha + v \sin \alpha \neq 0, \tag{3}$$

where partial derivatives are used for functions (1):

$$\begin{aligned}
 \frac{\partial x}{\partial t} &= -(u \cdot \sin \alpha + v \cdot \cos \alpha) \sin t, & \frac{\partial y}{\partial t} &= (u \cdot \sin \alpha + v \cdot \cos \alpha) \cos t, & \frac{\partial z}{\partial t} &= 0, \\
 \frac{\partial x}{\partial u} &= \sin \alpha \cdot \cos t, & \frac{\partial y}{\partial u} &= \sin \alpha \cdot \sin t, & \frac{\partial z}{\partial u} &= \cos \alpha, \\
 \frac{\partial x}{\partial v} &= \cos \alpha \cdot \cos t, & \frac{\partial y}{\partial v} &= \cos \alpha \cdot \sin t, & \frac{\partial z}{\partial v} &= -\sin \alpha.
 \end{aligned} \tag{4}$$

Those points at which the expression of the Jacobian (3) is equal to zero,

$$u \cdot \sin \alpha + v \cdot \cos \alpha = 0 \tag{5}$$

make up a region of space whose coordination by normal conical coordinates is impossible. We define v from (5)

$$v = -u \operatorname{tg} \alpha. \tag{6}$$

The equation (6) corresponds to the axis of the cone, which coincides with the axis Oz .

Another paper will consider the definition of the existence region of the introduced system of normal conical coordinates (the region of the correct coordination of space by normal conical coordinates).

Differential-geometrical characteristics of fields in normal conical coordinates

We find the expressions of partial derivatives of functions

$$\begin{aligned} \frac{\partial x}{\partial t} &= -(u \sin \alpha + v \cos \alpha) \sin \left[t + \frac{(1 - \text{sign}(u \text{tg} \alpha + v))\pi}{2} \right]; \\ \frac{\partial y}{\partial t} &= (u \sin \alpha + v \cos \alpha) \cos \left[t + \frac{(1 - \text{sign}(u \text{tg} \alpha + v))\pi}{2} \right]; \\ \frac{\partial z}{\partial t} &= 0; \\ \frac{\partial x}{\partial u} &= \sin \alpha \cos \left[t + \frac{(1 - \text{sign}(u \text{tg} \alpha + v))\pi}{2} \right]; \\ \frac{\partial y}{\partial u} &= \sin \alpha \sin \left[t + \frac{(1 - \text{sign}(u \text{tg} \alpha + v))\pi}{2} \right]; \\ \frac{\partial z}{\partial u} &= \cos \alpha; \\ \frac{\partial x}{\partial v} &= \cos \alpha \cos \left[t + \frac{(1 - \text{sign}(u \text{tg} \alpha + v))\pi}{2} \right]; \\ \frac{\partial y}{\partial v} &= \cos \alpha \sin \left[t + \frac{(1 - \text{sign}(u \text{tg} \alpha + v))\pi}{2} \right]; \\ \frac{\partial z}{\partial v} &= -\sin \alpha. \end{aligned} \tag{7}$$

Lame coefficients for normal conical coordinates are determined by formulas (7):

$$H_t = u \sin \alpha + v \cos \alpha, \quad H_u = 1, \quad H_v = 1. \tag{8}$$

Elementary moves:

- when changing only t $dS_t = (u \sin \alpha + v \cos \alpha) dt$;
- when changing only u $dS_u = du$;
- when changing only v $dS_v = dv$.

Linear element:

$$dS^2 = (u \sin \alpha + v \cos \alpha)^2 dt^2 + du^2 + dv^2. \tag{10}$$

The element of volume:

$$d\tau = (u \sin \alpha + v \cos \alpha) dt du dv. \tag{11}$$

Calculations showed that the system of normal conical coordinates refers to triorthogonal systems. Since the coefficients H_t , H_u , H_v satisfy six second-order partial differential equations, all terms of which are zero. That is, in such a system, each pair of coordinate surfaces intersects each other along the lines of curvature.

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{H_t} \frac{\partial H_u}{\partial t} \right) + \frac{\partial}{\partial u} \left(\frac{1}{H_u} \frac{\partial H_t}{\partial u} \right) + \frac{1}{H_v^2} \frac{\partial H_t}{\partial v} \frac{\partial H_u}{\partial v} &= 0; \\ \frac{\partial}{\partial u} \left(\frac{1}{H_u} \frac{\partial H_v}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{H_v} \frac{\partial H_u}{\partial v} \right) + \frac{1}{H_t^2} \frac{\partial H_u}{\partial t} \frac{\partial H_v}{\partial t} &= 0; \\ \frac{\partial}{\partial v} \left(\frac{1}{H_v} \frac{\partial H_t}{\partial v} \right) + \frac{\partial}{\partial t} \left(\frac{1}{H_t} \frac{\partial H_v}{\partial t} \right) + \frac{1}{H_u^2} \frac{\partial H_v}{\partial u} \frac{\partial H_t}{\partial u} &= 0; \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 H_t}{\partial u \partial v} - \frac{1}{H_u} \frac{\partial H_u}{\partial v} \frac{\partial H_t}{\partial u} - \frac{1}{H_v} \frac{\partial H_v}{\partial u} \frac{\partial H_t}{\partial v} &= 0; \\ \frac{\partial^2 H_u}{\partial v \partial t} - \frac{1}{H_v} \frac{\partial H_v}{\partial t} \frac{\partial H_u}{\partial v} - \frac{1}{H_t} \frac{\partial H_t}{\partial v} \frac{\partial H_u}{\partial t} &= 0; \\ \frac{\partial^2 H_v}{\partial t \partial u} - \frac{1}{H_t} \frac{\partial H_t}{\partial u} \frac{\partial H_v}{\partial t} - \frac{1}{H_u} \frac{\partial H_u}{\partial t} \frac{\partial H_v}{\partial u} &= 0. \end{aligned}$$

Vector gradient of scalar field at points t, u, v , related to a moving local rapper

$$G = \text{grad}F = \frac{1}{(u \cdot \sin \alpha + v \cdot \cos \alpha)} \frac{\partial F}{\partial t} e_t + \frac{\partial F}{\partial u} e_u + \frac{\partial F}{\partial v} e_v \tag{12}$$

The derivative of the scalar field in the direction \bar{l}_0

$$\bar{l}_0 = \cos \bar{\alpha} e_t + \cos \bar{\beta} e_u + \cos \bar{\gamma} e_v, \tag{13}$$

where $\cos \bar{\alpha}, \cos \bar{\beta}, \cos \bar{\gamma}$ director cones of the unitary vector \bar{l}_0 with respect to e_t, e_u, e_v

$$\frac{dF}{dl} = \frac{G_t}{|G|} \cos \bar{\alpha} + \frac{G_u}{|G|} \cos \bar{\beta} + \frac{G_v}{|G|} \cos \bar{\gamma}. \tag{14}$$

where $G_t = \frac{1}{u \sin \alpha + v \cos \alpha} \frac{\partial F}{\partial t}, G_u = \frac{\partial F}{\partial u}, G_v = \frac{\partial F}{\partial v}, |G| = \sqrt{G_t^2 + G_u^2 + G_v^2}$.

Laplacian of a scalar field with the account of the expressions H_t, H_u, H_v (2.15)

$$\begin{aligned} \Delta F = \frac{1}{u \cdot \sin \alpha + v \cdot \cos \alpha} \left[\frac{\partial^2 F}{\partial t^2} \frac{1}{u \cdot \sin \alpha + v \cdot \cos \alpha} + \right. \\ \left. + \frac{\partial^2 F}{\partial u^2} (u \cdot \sin \alpha + v \cdot \cos \alpha) + \frac{\partial F}{\partial u} \sin \alpha + \frac{\partial^2 F}{\partial v^2} \cos \alpha \right]. \end{aligned} \tag{15}$$

Vector flux \bar{a} across the surface S is determined by the formula:

$$N = \iint_S [a_t(t, u, v) \cos(\bar{n}_t \hat{e}_t) + a_u(t, u, v) \cos(\bar{n}_u \hat{e}_u) + a_v(t, u, v) \cos(\bar{n}_v \hat{e}_v)] dS \tag{16}$$

where \bar{n} - normal vector to the elementary area dS .

Definition of the formula is possible only after the presentation of the field and surface.

The divergence of the vector field is calculated taking into account the expressions of the Lamé coefficients (8)

$$\begin{aligned} \text{div} \bar{a} = \frac{1}{u \cdot \sin \alpha + v \cdot \cos \alpha} \left[\frac{\partial a_t}{\partial t} + (u \cdot \sin \alpha + v \cdot \cos \alpha) \frac{\partial a_u}{\partial u} + a_u \sin \alpha + \right. \\ \left. + (u \cdot \sin \alpha + v \cdot \cos \alpha) \frac{\partial a_v}{\partial v} + a_v \cos \alpha \right]. \end{aligned}$$

After simplifications we get

$$\operatorname{div} \bar{a} = \frac{\frac{\partial a_t}{\partial t} + a_u \sin \alpha + a_v \cos \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} + \frac{\partial a_u}{\partial u} + \frac{\partial a_v}{\partial v}. \quad (17)$$

As we see the divergence of the vector field is a scalar.
 Laplacian of the vector field

$$\begin{aligned} \Delta \bar{a} = & \left[\frac{1}{(u \cdot \sin \alpha + v \cdot \cos \alpha)^2} \frac{\partial^2 a_t}{\partial t^2} + \frac{\partial^2 a_t}{\partial u^2} + \frac{\partial^2 a_t}{\partial v^2} + \frac{\sin \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_t}{\partial u} + \right. \\ & \left. + \frac{\cos \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_t}{\partial v} \right] e_t + \left[\frac{1}{(u \cdot \sin \alpha + v \cdot \cos \alpha)^2} \frac{\partial^2 a_u}{\partial t^2} + \frac{\partial^2 a_u}{\partial u^2} + \frac{\partial^2 a_u}{\partial v^2} + \right. \\ & \left. + \frac{\sin \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_u}{\partial u} + \frac{\cos \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_u}{\partial v} \right] e_u + \\ & \left[\frac{1}{(u \cdot \sin \alpha + v \cdot \cos \alpha)^2} \frac{\partial^2 a_v}{\partial t^2} + \frac{\partial^2 a_v}{\partial u^2} + \frac{\partial^2 a_v}{\partial v^2} + \frac{\sin \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_v}{\partial u} + \right. \\ & \left. + \frac{\cos \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_v}{\partial v} \right] e_v. \end{aligned} \quad (18)$$

The rotation (the curl) of the vector field is determined taking into account the expressions H_t, H_u, H_v (8)

$$\begin{aligned} W_t &= \frac{\partial a_v}{\partial u} - \frac{\partial a_u}{\partial v}; \\ W_u &= \frac{\partial a_t}{\partial v} + \frac{\cos \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} a_v - \frac{1}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_v}{\partial t}; \\ W_v &= \frac{1}{u \cdot \sin \alpha + v \cdot \cos \alpha} \frac{\partial a_u}{\partial t} - \frac{\partial a_t}{\partial u} - \frac{\sin \alpha}{u \cdot \sin \alpha + v \cdot \cos \alpha} a_t. \end{aligned}$$

$$\begin{aligned} \operatorname{rot} \bar{a} = & \left(\frac{\partial a_v}{\partial u} - \frac{\partial a_u}{\partial v} \right) e_t + \left[\frac{\partial a_t}{\partial v} + \frac{1}{u \cdot \sin \alpha + v \cdot \cos \alpha} \left(a_v \cos \alpha - \frac{\partial a_v}{\partial t} \right) \right] e_u + \\ & + \left[\frac{1}{u \cdot \sin \alpha + v \cdot \cos \alpha} \left(\frac{\partial a_u}{\partial t} - a_t \sin \alpha \right) - \frac{\partial a_t}{\partial u} \right] e_v. \end{aligned} \quad (19)$$

The above expressions of the differential-geometrical characteristics of the scalar and vector fields, presented in normal conical coordinates, make it possible to design a geometrical model of the field and involve visual computer graphics in its study.

Let us show in a specific context the features of the application of the mathematical apparatus of the geometrical field modeling in normal conical coordinates.

Task. Calculate the derivative of the scalar field

$$F = -0,25 \sin u \cos 6t - v \quad (20)$$

in the line of

$$\vec{l}^0 = 3e_t + 4,8e_u - 2e_v \quad (21)$$

at point $M (t=0,4, u=10, v=-5)$. The system cone-determiner parameter $\alpha = 0.52$.

To show scalar field level surfaces (20)

$$v = -0,25 \sin u \cos 6t + i \Delta C \tag{22}$$

with $i = 0, 1, 2, \Delta C = 0,4, 0 \leq t \leq 2\pi, 0,5 \leq u \leq 1,57$.

Solution. We find the direction cosines of the vector \vec{l}_0 in the basis e_t, e_u, e_v

$$\begin{aligned} \cos \bar{\alpha} &= \frac{3}{\sqrt{3^2 + 4,8^2 + 2^2}} = 0,5; \quad \cos \bar{\beta} = \frac{4,8}{\sqrt{3^2 + 4,8^2 + 2^2}} = 0,8; \\ \cos \bar{\gamma} &= -\frac{2}{\sqrt{3^2 + 4,8^2 + 2^2}} = -0,3317. \end{aligned}$$

We find the coordinates G_t, G_u, G_v of the scalar field gradient vectors (20) according to formulas (12)

$$G_t = -\frac{1}{u \sin \alpha + v \cos \alpha} 6(-0,25) \sin u \sin 6t, \quad G_u = -0,25 \cos u \cos 6t, \quad G_v = -1.$$

We substitute the values of the coordinates of the point M and the parameter α of the cone-determinant. We get

$$\begin{aligned} G_t &= \frac{-6(-0,25) \sin(10) \sin(6 \cdot 0,4)}{10 \sin(0,52) - 5 \cos(0,52)} = -0,8763; \\ G_u &= (-0,25) \cos(10) \cos(6 \cdot 0,4) = -0,1546; \\ G_v &= -1. \end{aligned}$$

Gradient vector module

$$G = \sqrt{(-0,8763)^2 + (-0,1546)^2 + 1^2} = \sqrt{0,7679 + 0,0239 + 1} = 1,3386.$$

We calculate the value of the derivative of the scalar field (20) in the direction (21) at the point M , using the formula (14) and the value of the direction cosines

$$\frac{dF}{dl} = -\frac{0,8763}{1,3386} \cdot 0,5 - \frac{0,1546}{1,3386} \cdot 0,8 - \frac{1}{1,3386} (-0,3317) = -0,1653.$$

We build level surfaces according to internal equations (22).

Figure 2 shows three level surfaces together; in Figure 3, 4, 5 these surfaces are shown separately.

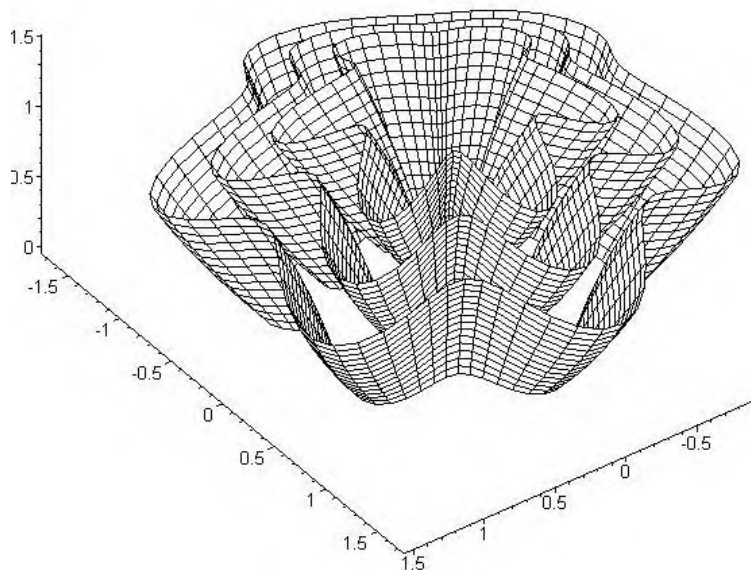


Fig. 2. Surfaces of the level $v = h \cdot \sin u \cdot \cos (n \cdot t) + i \Delta C$ of a scalar field

$F = h \cdot \sin u \cdot \cos (n \cdot t) - v$ with parameters $\Delta C=0,4$; $i=0, 1, 2$; $\alpha=0,52$; $h=-0,25$; $n=6$; $u=0,5 \dots 1,57$;
 $t=0 \dots 2\pi$

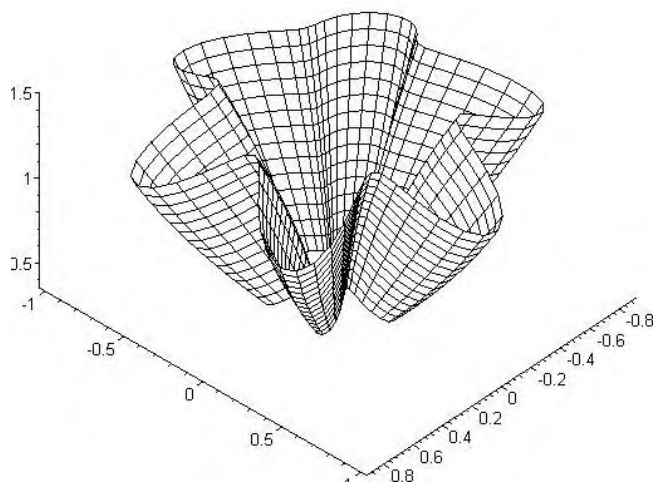


Fig. 3. Surfaces of the level $i=0$

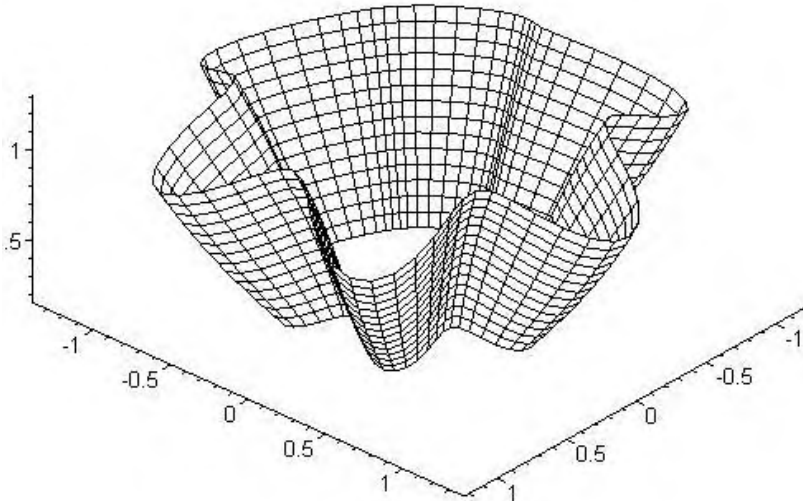


Fig. 4. Surfaces of the level $i=1$

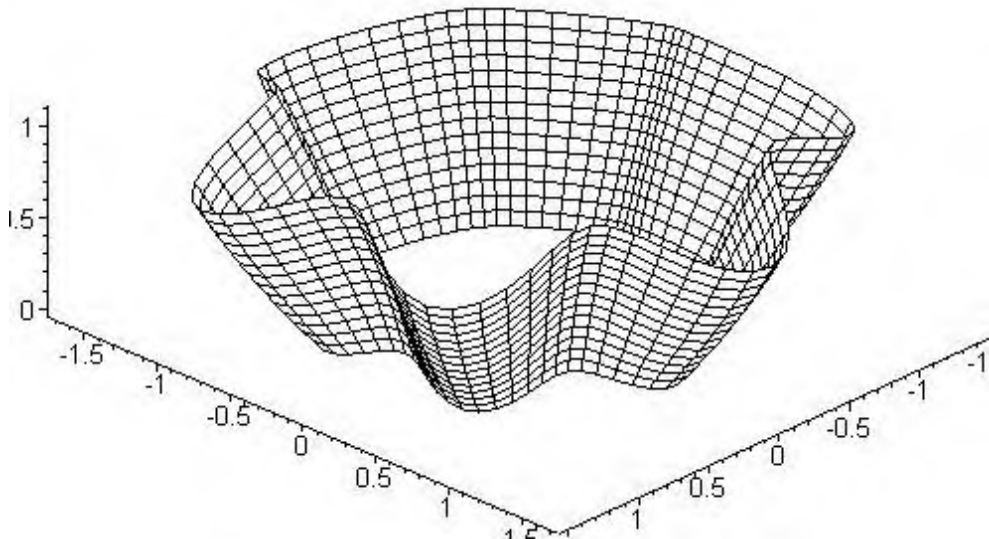


Fig. 5. Surfaces of the level $i=2$

3 Results

1. The paper gives the expressions of the differential characteristics of the scalar and vector fields in normal conical coordinates.
2. The methodology for calculating and constructing the level surfaces of the scalar fields represented in normal conical coordinates is shown.

4 Discussion

1. The obtained means of describing fields in normal conical coordinates allow visualizing, using computer graphics, the surfaces of a level with a complex structure.

2. The example shows the practical application of the results to describe the level surfaces of the scalar fields.
3. The results of the theoretical research can be applied to solve various problems:
 - obtaining the parameters of the scalar fields at a known point in space with the given parameters of the determinant cone and the subsequent visualization of level surfaces;
 - calculation of vector field parameters (divergence, Laplacian, rotation) in normal conical coordinates;
 - visualization of the field level surfaces in normal conical coordinates for various values of the parameter of the cone-determinant α .

5 Conclusion

Using special coordinates to describe fields of complex structure greatly simplifies the calculation of field characteristics. A further development of this topic is the study of the region of existence of the introduced system of normal conical coordinates, which can also be called the region of proper coordination of normal conical coordinates.

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