Curvature Identities for Generalized Kenmotsu Manifolds

Abu-Saleem Ahmad[1], *Ivan* Kochetkov[2,*], and *Aligadzhi* Rustanov[2]

[1]Al al-Bayt University, 25113, Mafraq, 130040, Jordan
[2]Moscow State University of Civil Engineering, 129377, Yaroslavskoye Shosse, 26, Moscow, Russia

Abstract. In the present paper we obtained 2 identities, which are satisfied by Riemann curvature tensor of generalized Kenmotsu manifolds. There was obtained an analytic expression for third structure tensor or tensor of *f*-holomorphic sectional curvature of *GK*-manifold. We separated 2 classes of generalized Kenmotsu manifolds and collected their local characterization.

1 Introduction

Let M be a connected smooth manifold of (*2n+1*) dimension, $C^\infty(M)$ is the algebra of smooth functions on M, $\mathcal{X}(M)$ - C^∞ - module of smooth vector fields on M, d is the operator of exterior differentiation. If M preserves Riemannian metric $\langle\cdot,\cdot\rangle$, then corresponding Riemannian connection is expressed as ∇. In the future, all manifolds, tensor fields (tensors, and similarly objects are assumed to be smooth of class C^∞.

Differential 1-form of maximal rank on an odd-dimensional Riemannian manifold produce a special differential-geometric structure called contact metric structure that naturally generalizes to the so-called almost contact metric structure.

We recall [1] that a contact form or contact structure on an odd-dimensional manifold M, $\dim M = 2n+1$, is called 1-form η on M, which in each point of the manifold is $\eta \wedge \underbrace{(d\eta)\wedge \dots \wedge (d\eta)}_{n\ time} \neq 0$, i.e. $rg\eta = \dim M$ is in each point of M. Manifold with a fixed contact form on it is called a contact manifold [2]. Distributions $\mathcal{L} = \ker\eta$, $\dim\mathcal{L} = 2n$, and $\mathcal{M} = \ker(d\eta)$, $\dim\mathcal{M} = 1$ are determined on such manifold internally. It is easy to derive from Darboux theorem [1] that $\mathcal{L} \cap \mathcal{M} = \{0\}$, consequently $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$. Let us take $\xi \in \mathcal{M}$ such that $\eta(\xi) = 1$. Then, it is possible to determine mutually complementing projections $m = \xi \otimes \eta$ and $\ell = id - \xi \otimes \eta$ on distributions $\mathcal{M}$ and $\mathcal{L}$ respectively.

Let Riemannian metric h be fixed on M. Based on the metric $h|\mathcal{L}$, it is not difficult to build metric $\langle\cdot,\cdot\rangle$ on $\mathcal{L}$ such that the operator $I: \mathcal{L} \to \mathcal{L}$, which is determined with the identity $\langle X, IY\rangle = d\eta(X,Y)$; $X, Y \in \mathcal{L}$, will be involutory, i.e. $I^2 = -id$, and thus $\langle IX, IY\rangle = \langle X, Y\rangle$. Hence vector ξ, covector η, operator $f = I \circ \ell$ and Riemannian metric $\langle X, Y\rangle = \langle \ell X, \ell Y\rangle + [h(\xi,\xi)]^{-1} h(mX, mY)$ on M, clearly, possess the following properties:

* Corresponding author: idkochetkov@mail.ru

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$$1)\ f(\xi) = 0;\ 2)\ \eta \circ f = 0;\ 3)\ \eta(\xi) = 1;\ 4)\ f^2 = -id + \eta \otimes \xi;\ 5)\ \langle X, fY \rangle = d\eta(X,Y);\ 6)\ \eta(X) = \langle \xi, X \rangle;\ 7)\ \langle fX, fY \rangle = \langle X, Y \rangle - \eta(X)\eta(Y); X, Y \in \mathcal{X}(M). \quad (1.1)$$

A contact manifold *M* that is provided with Riemannian metric $\langle \cdot,\cdot \rangle$, for which relations (1) are true, is called a contact metric manifold.

The given construction makes the following definition natural.

Definition 1.1 [3]. Almost a contact metric (or almost Gray's) structure on a manifold *M* is a set of $\{g, f, \xi, \eta\}$ tensor fields on *M*, where $g = \langle \cdot,\cdot \rangle$ is a (pseudo-) Riemannian metric, *f* is a tensor of (1.1) kind called structure endomorphism, ξ is a vector field, which is called characteristic, η is a differential 1-form, which is called a contact form of structure. Here:

$$1. f(\xi) = 0;\ 2. \eta \circ f = 0;\ 3. \eta(\xi) = 1;\ 4. f^2 = -id + \eta \otimes \xi;\ 5. \langle fX, fY \rangle = \langle X, Y \rangle - \eta(X)\eta(Y); X, Y \in \mathcal{X}(M). \quad (1.2)$$

Note that these relations are not independent; for example, (1.2:1) and (1.2:2) result from (1.2:3) and (1.2:4) [4]. In addition, it follows from (1.2:1), (1.2:3) and (1.2:5) that $\eta(X) = \langle \xi, X \rangle, X \in \mathcal{X}(M)$; and from (1.2:2), (1.2:4) and1.2:5) that tensor $\Omega(X,Y) = \langle X, \Phi Y \rangle$ is skew-symmetric; this tensor is called the fundumental form of the structure. A triple $\{f, \xi, \eta\}$, which meets the conditions (1.2:3) and1.2:4), is called an almost contact structure; it was introduced by S. Sasaki in [3] as such. Its assignment matches the assignment of *G*-structure on *M* with the structural group $\boldsymbol{R}^* \times GL(n, \boldsymbol{C})$, where $\boldsymbol{R}^*$ is a multiplicative group of positive real numbers. In fact, this is how J. Gray introduced it in 1959 [2]. Assignment of the almost contact structure $\{\Phi, \xi, \eta\}$ on manifold *M* induces a canonical hyper-distribution $\mathcal{L} = \ker \eta$ on this manifold, called contact, which is invariant in relation to *f*. Due to (1.2:4), operator *f* induces an almost complex structure on $\mathcal{L}$. This gave grounds for J. Bouzon to assume an almost contact structure as an almost complex [5]. If $\{g, f, \xi, \eta\}$ is an almost contact metric structure, then by the virtue of (1.2:5), the pair $\{f|\mathcal{L}, g|\mathcal{L}\}$ sets an almost Hermitian structure on this hyper-distribution, due to which an almost contact metric structure can be naturally called a metric almost cocomplex structure. Obviously, its assignment equally matches the assignment of *G*-structure on *M* with a structural group $\{e\} \times U(n)$. We note that the concept of an almost contact structure has historically evolved in the following sequence: S.S. Chern [6] found that contact manifold admits *G*-structure with structural group $\{e\} \times U(n)$; J. Gray called manifolds that admit such structure as almost contact manifolds [2]. S. Sasaki noticed that such a *G*-structure generates a triple $\{f, \xi, \eta\}$ that has the above mentioned conditions (1.2:3) and (1.2:4), from which it is to derive (1.2:1) and (1.2:2). Moreover, based on an arbitrary Riemannian metric *h* on such manifold, he constructed Riemannian metric $\langle X, Y \rangle = h(fX, fY) + h(f^2X, f^2Y) + \eta(X)\eta(Y)$, complementing $\{f, \xi, \eta\}$ to almost contact metric structure [3].

The most important example of almost contact metric structures, which largely determines their role in differential geometry, is the structure induced on the hypersurface *N* of the manifold *M* equipped with an almost Hermitian structure $\{J, \langle \cdot,\cdot \rangle\}$. Let us recall this construction. Let n^0 be a unit normal field to *N*. Then vector is $\xi = J(n^0) \in \mathcal{X}(N)$, where its orthogonal complement $\mathfrak{L}$ on *N* is invariant to *J*. We define in $\mathcal{X}(N)$ a linear operator $\Phi = J|\mathfrak{L} \oplus 0|\mathfrak{M}$, where $\mathfrak{M}$ is the linear hull of the vector ξ, and 1-form is $\eta(X) = \langle \xi, X \rangle$. Then $\{\langle \cdot,\cdot \rangle, f, \xi, \eta\}$ is the almost contact metric structure on *N*. In particular, such a structure is induced on the odd-dimensional sphere S^{2n-1} considered as a hypersurface in the realification of the space C^n. This is the most important and, apparently, historically the first concrete example of such a structure.

Then $\{g = \langle \cdot,\cdot \rangle, f, \xi, \eta\}$ is the almost contact metric structure on manifold *M*. It is well-known [1] that in this case almost Hermitian structure $\{J, h\}$ induces on the manifold

$M \times R$, where $J = f|\mathfrak{L} \oplus J_1, h = g|\mathfrak{L} \oplus g_1, J_1$ is a canonical almost complex structure on the 2-dimensional distribution $\mathfrak{M} \times R$, g_1 is a metric on this distribution being a direct sum of metric $g|\mathfrak{M}$ and the canonical metric on R. An almost contact structure $\{f, \xi, \eta\}$ is called normal if the structure $\{J, h\}$ is integrable [4]; a necessary and sufficient condition for the structure to be normal has the form $N + \frac{1}{2}\xi \otimes d\eta = 0$, where N is the Nijenhuis tensor of the operator f [4].

Today, there active studies of the geometry of almost contact metric structures on manifolds. One of the most pressing issues in this section of geometry is the study of individual classes of almost contact metric manifolds. In 1972, Kenmotsu [7] introduced a class of almost contact metric structures characterized by an identity $\nabla_X(f)Y = \langle fX, Y\rangle\xi - \eta(Y)fX;\ X, Y \in \mathcal{X}(M)$. Kenmotsu structures, for example, naturally arise in the Tanno classification of connected almost contact metric manifolds such that automorphism group has maximum dimension [8]. They have a number of remarkable properties. For example, Kenmotsu structures are normal and integrable. They are not contact structures, so they are not Sasakian. There are known examples of Kenmotsu structures on odd-dimensional Lobachevsky spaces of curvature (-1). Such structures are obtained using the construction of warped product $\boldsymbol{R} \times_f \boldsymbol{C}^n$ in the sense of Bishop and O'Neill [9] of complex Euclidean space and a real line, where $f(t) = ce^t$ (see [7]).

Polarizing the identity characterizing the Kenmotsu manifolds, S.V. Umnova [10] identified in his thesis paper a class of almost contact metric manifolds; this class was a generalization of Kenmotsu manifolds and was called the class of generalized (in short, *GK*-) Kenmotsu manifolds. In [10], it was proved that generalized Kenmotsu manifolds of constant curvature are Kenmotsu manifolds of constant curvature -1.

In [11], this class of manifolds is called as a class of nearly Kenmotsu manifolds. The authors prove that a second-order symmetric closed recurrent tensor, recurrence covector of which annihilates the characteristic vector ξ, is a multiple of the metric tensor g. In addition, the authors consider the *f*-recurrent nearly Kenmotsu manifolds. It is proved that *f*-recurrent nearly Kenmotsu manifolds are Einstein manifolds, and locally *f*-recurrent nearly Kenmotsu manifolds are manifolds of constant curvature -1.

In [12], M.B. Banaru studied hypersurfaces of almost Hermitian manifolds of class W_3 with the Kenmotsu structure and obtained interesting properties of Kenmotsu manifolds.

In our papers [13-14], Einstein's generalized Kenmotsu manifolds were studied; contact analogs of Gray identities were obtained, three classes of this type of manifolds were distinguished; a local characterization of the distinguished classes of manifolds was obtained. In [15], the curvature identities for the Riemannian curvature tensor were considered for the particular case of generalized Kenmotsu manifolds, called special generalized Kenmotsu manifolds of second kind [10].

The paper [16] examines the integrability properties of generalized Kenmotsu manifolds. In this paper, we investigate *GK*-manifolds, the first fundamental distribution of which is completely integrable. It is shown that an almost Hermitian structure induced on integral manifolds of maximum dimension of the first distribution of a *GK*-manifold is nearly Kahlerian. Local structure of a *GK*-manifold with a closed contact form is obtained, expressions for the first and second structure tensors are given. The components of the Nijenhuis tensor of a *GK*-manifold are also calculated. Since defining the Nijenhuis tensor is equivalent to defining four tensors $N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)}$, the geometric meaning of vanishing of these tensors is studied. Local structure of an integrable and normal *GK*-structure is obtained. It is proved that characteristic vector of *GK*-structure is not a Killing vector.

It is clear from the given reviews of works on generalized Kenmotsu manifolds that the interest in studying of this class of manifolds does not fade, but rather grows.

In this paper, we continue the study of generalized Kenmotsu manifolds and investigate the geometry of the Riemannian curvature tensor for this class of manifolds.

This paper is organized as follows. In paragraph 2, we give preliminary information needed in the further presentation and construct the space of the adjoint *G*-structure. In paragraph 3, we give a definition of generalized Kenmotsu manifolds, provide a complete group of structure equations, and give the components of the Riemann-Christoffel tensor on space of the adjoint *G*-structure. In paragraph 4 we, using the procedure of restoring the identity of [17- 18], obtain some identities, which are satisfied by the Riemannian curvature tensor of generalized Kenmotsu manifolds and, on their basis, we separate two classes of generalized Kenmotsu manifolds. In addition, we get a local characterization of these classes.

2 Preliminaries

Let M be a smooth manifold of dimension $2n+1$, $\mathcal{X}(M)$, C^∞ be a module of smooth vector fields on manifold *M*. Further, all manifolds, tensor fields and similarly objects are assumed to be smooth of class C^∞.

Definition 2.1 ([17-18]). *Almost contact structure* on a manifold *M* is a triple (η, ξ, f) tensor fields on this manifold, where η is a differential 1-form called the *contact form of structure*, ξ is a vector field called characteristic, *f* is endomorphism of module $\mathcal{X}(M)$ called the *structure endomorphism*. Here

$$1)\ \eta(\xi) = 1;\ \ 2)\ \eta \circ f = 0;\ \ 3)\ f(\xi) = 0;\ \ 4)\ f^2 = -id + \eta \otimes \xi. \tag{2.1}$$

Moreover if Riemannian structure $g = \langle \cdot,\cdot \rangle$ is fixed on *M* such that

$$\langle fX, \mathrm{f}Y \rangle = \langle \mathrm{X}, \mathrm{Y} \rangle - \eta(X)\eta(Y),\ X, Y \in \mathcal{X}(M), \tag{2.2}$$

four $(\eta, \xi, f, g = \langle \cdot,\cdot \rangle)$ is called an *almost contact metric* (*AC*-) *structure*. Manifold with a fixed almost contact (metric) structure is called an *almost contact* (*metric* (*AC*-)) *manifold*.

Skew-symmetric tensor $\Omega(X,Y) = \langle X, fY \rangle, X, Y \in \mathcal{X}(M)$ is called the *fundamental form of AC-structure* ([17], [18]).

Let (η, ξ, f, g) be the almost contact metric structure on manifold M^{2n+1}. In the module $\mathcal{X}(M)$, two mutually complementary projections $m = \eta \otimes \xi$ and $\ell = id - m = -f^2$ are defined internally [18]; thus, $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L} = Im(\Phi) = ker\eta$ is the so-called *contact* (or *first fundamental*) *distribution*, $dim\mathcal{L} = 2n, \mathcal{M} = Im\, m = ker(f) = L(\xi)$ is the linear hull of structure vector or the so-called *second fundamental distribution* (moreover, ℓ and m are projections onto submodules $\mathcal{L}, \mathcal{M}$ respectively) [17-18]. Obviously, distribution $\mathcal{L}$ and $\mathcal{M}$ are invariant to f and mutually orthogonal. It is also clear that $\tilde{f}^2 = -id, \langle \tilde{f}X, \tilde{f}Y \rangle = \langle X, Y \rangle, X, Y \in \mathcal{X}(M)$, where $\tilde{f} = f|\mathcal{L}$. Consequently, $\{\tilde{f}_p, g_p|\mathcal{L}\}$ is the Hermitian structure on manifold $\mathcal{L}_p$.

Complexification $\mathcal{X}(M)^{\mathbf{C}}$ of the module $\mathcal{X}(M)$ decomposes into a direct sum $\mathcal{X}(M)^{\mathbf{C}} = D_f^{\sqrt{-1}} \oplus D_f^{-\sqrt{-1}} \oplus D_f^0$ of proper subspaces of structure endomorphism *f*, corresponding to proper values $\sqrt{-1}, \sqrt{-1}$ and 0 respectively. Moreover, the projections onto the summands of this direct sum will, respectively, be endomorphisms [17-18] $\pi = \sigma \circ \ell = -\frac{1}{2}(f^2 + \sqrt{-1}f), \bar{\pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(-f^2 + \sqrt{-1}f),\ m = id + f^2$, where $\sigma = \frac{1}{2}(id - \sqrt{-1}f), \bar{\sigma} = \frac{1}{2}(id + \sqrt{-1}f)$.

The mappings $\sigma_p : \mathcal{L}_p \longrightarrow D_f^{\sqrt{-1}}$ and $\bar{\sigma}_p : \mathcal{L}_p \longrightarrow D_f^{-\sqrt{-1}}$ are, respectively, isomorphisms and anti-isomorphisms of Hermitian spaces. Therefore, to each point $p \in M^{2n+1}$ one can attach a family of frames of the space $T_p(M)^C$ of the form $(p, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}})$, where

$\varepsilon_a = \sqrt{2}\sigma_p(e_a), \varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}_p(e_a), \varepsilon_0 = \xi_p$; where $\{e_a\}$ is an orthonormal basis of Hermitian space $\mathcal{L}_p$. This frame is called an *A-frame* [18]. It is easy to see that matrices of the tensor components f_p and g_p in the *A*-frame have the form:

$$\left(f_j^i\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}, \tag{2.3}$$

where I_n is the identity matrix of size *n*. It is well-known [17, 18] that the set of such frames defines the *G*-structure on *M* with the structure group $\{1\} \times U(n)$ represented by matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$, where $A \in U(n)$. This *G*-structure is called *adjoint* [17, 18].

Let $(M^{2n+1}, \Phi, \xi, \eta, g = \langle\cdot,\cdot\rangle)$ be an almost contact metric manifold. Let us agree that throughout the entire paper, unless otherwise stated, the indices $i, j, k, l, \ldots$ take values from 1 to $2n$, the indices $a, b, c, d, \ldots$ take values from 1 to *n*, and let $\hat{a} = a + n, \hat{\hat{a}} = a, \hat{0} = 0$. Let (U, φ) be a local chart on the manifold *M*. According to the Main Theorem of Tensor Analysis [17, p. 243], the assignment of a structural endomorphism *f* and Riemannian structure $g = \langle\cdot,\cdot\rangle$ on the manifold *M* induces the assignment on the total space *BM* of a frame bundle over *M* of a system of functions $\{f_j^i\}, \{g_{ij}\}$, satisfying in the coordinate neighborhood $W = \pi^{-1}(U) \subset BM$ the system of differential equations of the form

$$df_j^i + f_j^k\theta_k^i - f_k^i\theta_j^k = f_{j,k}^i\omega^k,\ dg_{ij} - g_{kj}\theta_i^k - g_{ik}\theta_j^k = g_{ij,k}\omega^k \tag{2.4}$$

where $\{\omega^i\}, \{\theta_j^i\}$ are the components of solder form and Riemannian connection ∇, respectively, $f_{j,k}^i, g_{ij,k}$ are the components of covariant differential of tensors f and *g* in this connection. In addition, by the definition of Riemannian connection $\nabla g = 0$ and, therefore,

$$g_{ij,k} = 0. \tag{2.5}$$

Relations (2.4) on the space of the adjoint *G*-structure can be written in the form [17-18]

$$\begin{gathered} f_{b,k}^a = 0, f_{\hat{b},k}^{\hat{a}} = 0, f_{0,k}^0 = 0, \\ \theta_{\hat{b}}^a = \frac{\sqrt{-1}}{2} f_{\hat{b},k}^a\omega^k, \theta_b^{\hat{a}} = -\frac{\sqrt{-1}}{2} f_{b,k}^{\hat{a}}\omega^k, \\ \theta_0^a = \sqrt{-1}f_{0,k}^a\omega^k, \theta_0^{\hat{a}} = -\sqrt{-1}f_{0,k}^{\hat{a}}\omega^k, \\ \theta_a^0 = -\sqrt{-1}f_{a,k}^0\omega^k, \theta_{\hat{a}}^0 = \sqrt{-1}f_{\hat{a},k}^0\omega^k, \\ \theta_j^i + \theta_{\hat{j}}^{\hat{i}} = 0, \theta_0^0 = 0. \end{gathered} \tag{2.6}$$

On top of that, note that, since the corresponding forms and tensors are real, $\overline{\omega^i} = \omega^{\hat{i}}, \overline{\theta_j^i} = \theta_{\hat{j}}^{\hat{i}}, \overline{\nabla f_{j,k}^i} = \nabla f_{\hat{j},\hat{k}}^{\hat{i}}$, where $t \to \bar{t}$ is the complex conjugation operator.

The first group of structure equations for Riemannian connection $d\omega^i = -\theta_j^i \wedge \omega^j$ on the space of the adjoint *G*-structure of an almost contact metric manifold, can be written in the following form, called the *first group of structure equations for an almost contact metric manifold* [17-18]:

$$\begin{gathered} d\omega = C_{ab}\omega^a \wedge \omega^b + C^{ab}\omega_a \wedge \omega_b + C_a^b\omega^a \wedge \omega_b + C_a\omega \wedge \omega^a + C^a\omega \wedge \omega_a; \\ d\omega^a = -\theta_b^a \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^{ab}\omega \wedge \omega_b + B^a{}_b\omega \wedge \omega^b; \\ d\omega_a = \theta_a^b \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B_{ab}\omega \wedge \omega^b + B_a{}^b\omega \wedge \omega_b, \end{gathered} \tag{2.7}$$

where $\omega = \omega^0 = \pi^*(\eta)$; π is the natural projection of the space of the adjoint *G*-structure on the manifold *M*, $\omega_i = g_{ij}\omega^i$,

$$B^{ab}{}_c = -\frac{\sqrt{-1}}{2} f^a_{\hat{b},c}, B_{ab}{}^c = \frac{\sqrt{-1}}{2} f^{\hat{a}}_{b,\hat{c}}, B^{abc} = \frac{\sqrt{-1}}{2} f^a_{\hat{b},\hat{c}},$$
$$B_{abc} = -\frac{\sqrt{-1}}{2} f^{\hat{a}}_{b,c}, B^a{}_b = \sqrt{-1} f^a_{0,b}, B_a{}^b = -\sqrt{-1} f^{\hat{a}}_{0,\hat{b}},$$
$$B^{ab} = \sqrt{-1}\left(f^a_{0,\hat{b}} - \tfrac{1}{2} f^a_{\hat{b},0}\right), B_{ab} = -\sqrt{-1}\left(f^{\hat{a}}_{0,b} - \tfrac{1}{2} f^{\hat{a}}_{b,0}\right), \tag{2.8}$$
$$C^{ab} = \sqrt{-1} f^0_{[\hat{a},\hat{b}]}, C_{ab} = -\sqrt{-1} f^0_{[a,b]}, C^b_a = -\sqrt{-1}\left(f^0_{\hat{b},a} + f^0_{a,\hat{b}}\right) = B^b{}_a - B_a{}^b,$$
$$C^a = -\sqrt{-1} f^0_{\hat{a},0}, C_a = \sqrt{-1} f^0_{a,0}.$$

Let us introduce the following notation [17]

$$C^{abc} = \tfrac{\sqrt{-1}}{2} f^a_{\hat{b},\hat{c}}, C_{abc} = -\tfrac{\sqrt{-1}}{2} f^{\hat{a}}_{b,c}, F^{ab} = \sqrt{-1} f^0_{\hat{a},\hat{b}}, F_{ab} = -\sqrt{-1} f^0_{a,b}. \tag{2.9}$$

Consider the following function families on the space of adjoint *G*-structure [17]:

- $B = \{B^i{}_{jk}\}$; $B^a{}_{\hat{b}c} = B^{ab}{}_c, B^{\hat{a}}{}_{b\hat{c}} = B_{ab}{}^c$; all other components of family *B* are zero;
- $C = \{C^i{}_{jk}\}$; $C^a{}_{\hat{b}\hat{c}} = C^{abc}$; $C^{\hat{a}}{}_{bc} = C_{abc}$; all other components of family *C* are zero;
- $D = \{D^i{}_j\}$; $D^a{}_{\hat{b}} = B^{ab}$; $D^{\hat{a}}{}_b = B_{ab}$; all other components of family *D* are zero;
- $E = \{E^i{}_j\}$; $E^a{}_b = B^a{}_b$; $E^{\hat{a}}{}_{\hat{b}} = B_a{}^b$; all other components of family *D* are zero;
- $F = \{F^i{}_j\}$; $F^a{}_{\hat{b}} = F^{ab}$; $F^{\hat{a}}{}_b = F_{ab}$; all other components of family *F* are zero;
- $G = \{G^i\}$; $G^a = C^a$; $G^a = C_a$; all other components of family *D* are zero.

These systems of functions define tensors of the corresponding kinds on the manifold *M*; these tensors are called *the first, second,.., sixth structure tensors* of the *AC*-structure, respectively. The following takes place

Proposition 2.1 [17]. Structure tensors of the *AC*-structure have the following properties:

$$1)\ f \circ B(X,Y) = -B(fX,Y) = B(X,fY);$$
$$2)\ \langle\langle B(X,Y),Z\rangle\rangle + \overline{\langle\langle Y,B(X,Z)\rangle\rangle} = 0;$$
$$3)\ f \circ C(X,Y) = -C(fX,Y) = -C(X,fY);$$
$$4)\ \langle\langle C(X,Y),Z\rangle\rangle + \overline{\langle\langle Y,C(X,Z)\rangle\rangle} = 0;$$
$$5)\ f \circ D = -D \circ f;$$
$$6)\ f \circ E = E \circ f;$$
$$7)\ f \circ F = -F \circ f;$$
$$8)\ G \in \mathcal{L};\ \text{were}\ \langle\langle X,Y\rangle\rangle = \langle X,Y\rangle + \sqrt{-1}\langle X,fY\rangle, \left(X,Y,Z \in \mathcal{X}(M)\right).$$

3 Generalized Kenmotsu manifolds

Let $(M^{2n+1}, f, \xi, \eta, g = \langle\cdot,\cdot\rangle)$ be an almost contact metric manifold.

Definition 3.1 ([10], [11]). A class of almost contact metric manifolds characterized by the identity

$$\nabla_X(f)Y + \nabla_Y(f)X = -\eta(Y)fX - \eta(X)fY; X,Y \in \mathcal{X}(M), \tag{3.1}$$

is called *generalized Kenmotsu manifolds* (*GK-manifolds*).

Note that this class of manifolds appears as *nearly* Kenmotsu manifolds ([11] and others). We will call these manifolds, as in [10], *generalized Kenmotsu manifolds*, and in short *GK-manifolds*.

The following theorem takes place.

Theorem 3.1 [13]. The complete group of structure equations for *GK*-manifolds on the space of the adjoint *G*-structure has the form:

$$1)\ d\omega = F_{ab}\omega^a \wedge \omega^b + F^{ab}\omega_a \wedge \omega_b;$$
$$2)\ d\omega^a = -\theta^a_b \wedge \omega^b + C^{abc}\omega_b \wedge \omega_c - \tfrac{3}{2}F^{ab}\omega \wedge \omega_b + \delta^a_b\omega \wedge \omega^b;$$
$$3)\ d\omega_a = \theta^b_a \wedge \omega_b + C_{abc}\omega^b \wedge \omega^c - \tfrac{3}{2}F_{ab}\omega \wedge \omega^b + \delta^b_a\omega \wedge \omega_b;$$
$$4)\ d\theta^a_b = -\theta^a_c \wedge \theta^c_b + \left(A^{ad}_{bc} - 2C^{adh}C_{hbc} - \frac{3}{2}F^{ad}F_{bc}\right)\omega^c \wedge \omega_d +$$
$$+\left(-\frac{1}{3}\delta^a_b F_{cd} + \frac{2}{3}\delta^a_c F_{db} + \frac{2}{3}\delta^a_d F_{bc}\right)\omega^c \wedge \omega^d +$$
$$+\left(\tfrac{1}{3}\delta^a_b F^{cd} - \tfrac{2}{3}\delta^c_b F^{da} - \tfrac{2}{3}\delta^d_b F^{ac}\right)\omega_c \wedge \omega_d;$$
$$5)\ dC^{abc} + C^{dbc}\theta^a_d + C^{adc}\theta^b_d + C^{abd}\theta^c_d = C^{abcd}\omega_d - 2\delta^{[a}_d F^{bc]}\omega^d - C^{abc}\omega;$$
$$6)\ dC_{abc} - C_{dbc}\theta^d_a - C_{adc}\theta^d_b - C_{abd}\theta^d_c = C_{abcd}\omega^d - 2\delta^d_{[a}F_{bc]}\omega_d - C_{abc}\omega;$$
$$7)\ dF^{ab} + F^{cb}\theta^a_c + F^{ac}\theta^b_c = -2F^{ab}\omega;$$
$$8)\ dF_{ab} - F_{cb}\theta^c_a - F_{ac}\theta^c_b = -2F_{ab}\omega; \tag{3.2}$$
$$9)\ dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{ad}_{bch}\omega^h + A^{adh}_{bc}\omega_h + A^{ad}_{bc0}\omega,$$

where

$$C^{[abc]} = C^{abc},\ C_{[abc]} = C_{abc},\ \overline{C^{abc}} = C_{abc},\ F^{ab} + F^{ba} = 0,\ F_{ab} + F_{ba} = 0,\ \overline{F^{ab}} = F_{ab},$$
$$A^{ad}_{[bc]} = A^{[ad]}_{bc} = 0,\ C^{a[bcd]} = \tfrac{3}{2}F^{a[b}F^{cd]},\ C_{a[bcd]} = \tfrac{3}{2}F_{a[b}F_{cd]},\ F_{ad}C^{dbc} = 0. \tag{3.3}$$

Corollary 1 [10]. If $C^{abc} = C_{abc} = 0$ and $F^{ab} = F_{ab} = 0$ then *GK*-manifold is a Kenmotsu manifold.

Definition 3.2 [10]. GK-structure is called: a special generalized Kenmotsu structure of the first kinds (in short SGK-structure of the first kind) if $\mathrm{C}^{\mathrm{dbc}} = \mathrm{C}_{\mathrm{dbc}} = 0$; a special generalized Kenmotsu structure of the second kind (in short SGK-structure of the second kind) if $F_{ad} = F^{ad} = 0$.

Identity

$$\left(A^{ag}_{b[c} - 2C^{agf}C_{fb[c}\right)C_{|g|dh]} = 0 \tag{3.4}$$

is called the first fundamental identity.

Identity

$$\left(A^{ah}_{b[c} - \tfrac{3}{2}F^{ah}F_{b[c}\right)F_{|h|d]} = 0 \tag{3.5}$$

is called the second fundamental identity.

Identity

$$2F^{ab}F^{cd} = F^{ac}F^{db} + F^{ad}F^{bc}; \tag{3.6}$$

is called the third fundamental identity.

Let *M* be a *GK*-manifold. Let us recall the following theorems from [13-14].

Theorem 3.2 [13]. The nonzero essential components of the Riemann-Christoffel tensor on the space of the adjoint *G*-structure have the form:

$$1)\ R^a_{00b} = F^{ac}F_{cb} + \delta^a_b;\ \ R^{\hat{a}}_{00\hat{b}} = F_{ac}F^{cb} + \delta^b_a;$$

2) $R^{a}_{bcd} = -\frac{2}{3}\delta^{a}_{b}F_{cd} + \frac{1}{3}\delta^{a}_{c}F_{db} + \frac{1}{3}\delta^{a}_{d}F_{bc}$; $R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} = -\frac{2}{3}\delta^{b}_{a}F^{cd} + \frac{1}{3}\delta^{c}_{a}F^{db} + \frac{1}{3}\delta^{d}_{a}F^{bc}$;

3) $R^{a}_{bc\hat{d}} = A^{ad}_{bc} - C^{adh}C_{hbc} - \frac{1}{2}F^{ad}F_{bc} - \delta^{a}_{c}\delta^{d}_{b}$; $R^{\hat{a}}_{\hat{b}c\hat{d}} = -A^{bd}_{ac} + C^{bdh}C_{hac} + \frac{1}{2}F^{bd}F_{ac} + \delta^{b}_{c}\delta^{d}_{a}$;

4) $R^{a}_{\hat{b}cd} = 2C^{abh}C_{hcd} + F^{ab}F_{cd} - 2\delta^{a}_{[c}\delta^{b}_{d]}$; $R^{\hat{a}}_{b\hat{c}\hat{d}} = 2C_{abh}C^{hcd} + F_{ab}F^{cd} - 2\delta^{[c}_{a}\delta^{d]}_{b}$;

5) $R^{a}_{\hat{b}\hat{c}\hat{d}} = C^{acdb} - \frac{1}{2}(F^{ab}F^{cd} + F^{ac}F^{db} + F^{ad}F^{bc})$; $R^{\hat{a}}_{bcd} = C_{acdb} - \frac{1}{2}(F_{ab}F_{cd} + F_{ac}F_{db} + F_{ad}F_{bc})$. (3.7)

4 Curvature identities for GK-manifolds

In [19], we obtained several identities for the Riemannian curvature tensor of generalized Kenmotsu manifolds and separated two subclasses of *GK*-manifolds. In addition, there were obtained the local structure of separated classes for *GK*-manifolds. In [20], two classes of generalized Kenmotsu manifolds were separated, called the class of *f*-holomorphic and the class of *f*-paracontact manifolds; a complete classification of the separated classes was obtained. In this paragraph we will also, as in [19] and [20], consider two classes of generalized Kenmotsu manifolds.

Definition 4.1. Let us call *AC*-manifold as a manifold of class R_3 if its curvature tensor satisfies the equation:

$$R(f^2X, f^2Y)f^2Z - R(f^2X, fY)fZ - R(fX, f^2Y)fZ - R(fX, fY)f^2Z = 0;\ \forall X, Y, Z \in \mathcal{X}(M). \quad (4.1)$$

Theorem 4.1. A *GK*-manifold is a manifold of classR_3 if and only if on the space of the adjoint *G*-structure $R^{0}_{abc} = R^{d}_{abc} = R^{\hat{d}}_{abc} = 0$.

Corollary. Let *GK*-manifold be a manifold of class R_3. Then its Riemannian curvature tensor satisfies the identity (4.1). Writing out identity (4.1) on the space of the adjoint *G*-structure, we obtain: $R^{i}_{jkl}f^{k}_{m}f^{m}_{s}X^{s}f^{l}_{p}f^{p}_{q}Y^{q}f^{j}_{r}f^{r}_{t}Z^{t} - R^{i}_{jkl}f^{k}_{m}f^{m}_{s}X^{s}f^{l}_{p}Y^{p}f^{j}_{q}Z^{q} - R^{i}_{jkl}f^{k}_{m}X^{m}f^{l}_{p}f^{p}_{q}Y^{q}f^{j}_{r}Z^{r} - R^{i}_{jkl}f^{k}_{m}X^{m}f^{l}_{p}Y^{p}f^{j}_{r}f^{r}_{s}Z^{s} = 0$, i.e. $R^{0}_{abc} = R^{d}_{abc} = R^{\hat{d}}_{abc} = 0$. Conversely, let the components of the Riemannian curvature tensor of the *GK*-manifold satisfy the equations $R^{0}_{abc} = R^{d}_{abc} = R^{\hat{d}}_{abc} = 0$. Applying the identity reconstruction procedure to the equation $R^{i}_{abc} = 0$, we obtain the identity (4.1). □

Theorem 4.2. A *GK*-manifold is a manifold of class R_3 if and only if it is a special generalized Kenmotsu manifold of the second kind such that $C_{abcd} = 0$.

Let *M* be a GK-manifold of class R_3, then it follows from theorems 4.1 and (3.7) that:

$$1)\ R^{a}_{bcd} = 2\delta^{a}_{[c}F_{|b|d]} + 2\left(-\frac{1}{3}\delta^{a}_{b}F_{cd} + \frac{2}{3}\delta^{a}_{c}F_{db} + \frac{2}{3}\delta^{a}_{d}F_{bc}\right) = 0;$$
$$2)\ R^{\hat{a}}_{bcd} = -2C_{ab[cd]} + F_{ab}F_{cd} - 2F_{a[c}F_{|b|d]} = 0. \quad (4.2)$$

The equation (4.2:1) we rewrite as $2\delta^{a}_{b}F_{cd} - \delta^{a}_{c}F_{db} - \delta^{a}_{d}F_{bc} = 0$. Let us contract the latter equation by indices *a* and *b*, then we get $2(n+1)F_{cd} = 0$. Hence, we obtain that $F_{cd} = 0$, i.e. the manifold is a special generalized Kenmotsu manifold of the second kind.

Now let us consider the equation (4.2:2). By virtue of $F_{cd} = 0$, then, taking into account equation (3.3), we obtain that $C_{abcd} = 0$. Since for a special generalized Kenmotsu manifold of the second kind $F_{cd} = 0$, then $R^{a}_{bcd} = 0$. And by virtue of $C_{abcd} = 0$, we have $R^{\hat{a}}_{bcd} = 0$. Hence, according to Theorem 4.1, the manifold *M* is a *GK*-manifold of class R_3. □

Sinc $R^0_{ab\hat{c}} = A^{0c}_{ab} - C^{0ch}C_{hab} - \frac{1}{2}F^{0c}F_{ab} - \delta^0_b\delta^c_a = 0, R^d_{ab\hat{c}} = A^{ad}_{bc} - C^{adh}C_{hbc} - \frac{1}{2}F^{ad}F_{bc} - \delta^a_c\delta^d_b, R^{\hat{d}}_{ab\hat{c}} = -\delta^d_aF_{bc} + \delta^d_bF_{ac} + 2\delta^d_{[a}F_{bc]}$, i.e. $R^i_{ab\hat{c}} = A^{ic}_{ab} - C^{ich}C_{hab} - \frac{1}{2}F^{ic}F_{ab} - \delta^c_b\delta^i_a$. At fixed point $p \in M$ it is obvious equivalent to the relation $R(\varepsilon_b, \varepsilon_{\hat{c}})\varepsilon_a = A(\varepsilon_a, \varepsilon_b, \varepsilon_{\hat{c}}) + \nabla_{\varepsilon_{\hat{c}}}(C)(\varepsilon_a, \varepsilon_b) - \varepsilon_b\langle\varepsilon_a, \varepsilon_{\hat{c}}\rangle$. Since vectors $\{\varepsilon_a\}$ form a basis for the subspace $\left(D_f^{\sqrt{-1}}\right)_p$, while vectors $\{\varepsilon_{\hat{a}}\}$ form a basis for the subspace $\left(D_f^{-\sqrt{-1}}\right)_p$, and projections of the module $\mathcal{X}(M)^C$ onto submodules $D_f^{\sqrt{-1}}, D_f^{-\sqrt{-1}}$ are endomorphisms $\pi = \sigma \circ \ell = -\frac{1}{2}(f^2 + \sqrt{-1}f), \bar{\pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(-f^2 + \sqrt{-1}f)$, the identity $R(\varepsilon_b, \varepsilon_{\hat{c}})\varepsilon_a = A(\varepsilon_a, \varepsilon_b, \varepsilon_{\hat{c}}) + \nabla_{\varepsilon_{\hat{c}}}(C)(\varepsilon_a, \varepsilon_b) - \varepsilon_b\langle\varepsilon_a, \varepsilon_{\hat{c}}\rangle$ can be rewritten as $R\left(f^2X + \sqrt{-1}fX, -f^2Y + \sqrt{-1}fY\right)\left(f^2Z + \sqrt{-1}fZ\right) = A\left(f^2Z + \sqrt{-1}fZ, f^2X + \sqrt{-1}fX, -f^2Y + \sqrt{-1}fY\right) + \nabla_{-f^2Y+\sqrt{-1}fY}(C)\left(f^2Z + \sqrt{-1}fZ, f^2X + \sqrt{-1}fX\right) - \left(f^2X + \sqrt{-1}fX\right)\langle -f^2Y + \sqrt{-1}fY, f^2Z + \sqrt{-1}fZ\rangle;\ \forall X, Y, Z \in \mathcal{X}(M)$. Developing this relation by linearity, splitting the real and imaginary parts of the resulting equality, and taking into account the properties of tensors *A* and ∇C (see [20] and [21]), we obtain an equivalent identity:

$$R(f^2X, f^2Y)f^2Z + R(f^2X, fY)fZ - R(fX, f^2Y)fZ + R(fX, fY)f^2Z = -4A(Z, X, Y) + \nabla_{f^2Y}(C)(f^2Z, f^2X) - \nabla_{f^2Y}(C)(fZ, fX) + \nabla_{fY}(C)(f^2Z, fX) + \nabla_{fY}(C)(fZ, f^2X) - 2f^2X\langle fY, fZ\rangle - 2fX\langle Y, fZ\rangle;\ \forall X, Y, Z \in \mathcal{X}(M). \quad (4.3)$$

We call identity (4.3) the fourth complementary curvature identity for a *GK*-manifold.

Let us introduce the following definition.

Definition 4.2. Let us call *AC*-manifold as a manifold of class R_4 if its curvature tensor satisfies the equation:

$$R(f^2X, f^2Y)f^2Z + R(f^2X, fY)fZ - R(fX, f^2Y)fZ + R(fX, fY)f^2Z = 0;\ \forall X, Y, Z \in \mathcal{X}(M). \quad (4.4)$$

The following results immediately from (4.3) and Definition 4.2.

Theorem 4.3. A *GK*-manifold is a manifold of class R_4 if and only if $A(Z, X, Y) = \frac{1}{4}\{\nabla_{f^2Y}(C)(f^2Z, f^2X) - \nabla_{f^2Y}(C)(fZ, fX) + \nabla_{fY}(C)(f^2Z, fX) + \nabla_{fY}(C)(fZ, f^2X) - 2f^2X\langle fY, fZ\rangle - 2fX\langle Y, fZ\rangle\};\ \forall X, Y, Z \in \mathcal{X}(M)$.

Theorem 4.3 provides an analytic expression for the third structure tensor or tensor of *f*-holomorphic sectional curvature of *GK*-manifold of class R_4. This tensor was introduced in [20]. An analytical expression for this tensor was also obtained in that work, when the manifold is a manifold of pointwise constant *f*-holomorphic sectional curvature. And properties of this tensor were proved in [21].

Theorem 4.4. A *GK*-manifold is a manifold of class R_4 if and only if on the space of the adjoint *G*-structure $R^d_{ab\hat{c}} = R^{\hat{d}}_{ab\hat{c}} = 0$.

Proof. Let *M be a GK*-manifold that is a manifold of class R_4. Hence, Riemannian curvature tensor of such manifold satisfies the condition (4.4), which on the space of adjoint *G*-structure takes the form:

$$R^i_{jkl}(f^2X)^k(f^2Y)^l(f^2Z)^j + R^i_{jkl}(f^2X)^k(fY)^l(fZ)^j - R^i_{jkl}(fX)^k(f^2Y)^l(fZ)^j + R^i_{jkl}(fX)^k(fY)^l(f^2Z)^j = 0.$$

Taking into account (3.7) and the form of the matrix *f*, the latter equation will be written in the form: $R^d_{ab\hat{c}}X^bY^cZ^a + R^{\hat{d}}_{ab\hat{c}}X_bY^cZ_a = 0$. The resulting equation is fulfilled if and only if $R^d_{ab\hat{c}} = R^{\hat{d}}_{ab\hat{c}} = 0$.

Conversely, if $R^{d}_{ab\hat{c}} = R^{\hat{d}}_{ab\hat{c}} = 0$ fulfilled, then, since the equality $R^{0}_{ab\hat{c}} = 0$ is also satisfied for the *GK*-manifold, applying the identity restoration procedure to the equations $R^{i}_{ab\hat{c}} = 0$, we obtain identity (4.4), i.e. the manifold is the manifold of class R_4. □

Theorem 4.5. GK-manifold of class R_4 is an SGK-manifold of the second kind.

Proof. Let *M be a GK*-manifold of class R_4. Then, according to Theorem 4.4 $R^{\hat{d}}_{ab\hat{c}} = 0$ and considering (3.7), we have:

$$R^{\hat{a}}_{bc\hat{d}} = R^{d}_{cba} = -\frac{2}{3}\delta^{d}_{c}F_{ba} + \frac{1}{3}\delta^{d}_{b}F_{ac} + \frac{1}{3}\delta^{d}_{a}F_{cb} = 0. \quad (4.5)$$

Let us contract this equation by indices *d* and *a*; then we obtain $(n+1)F_{bc} = 0 \Longrightarrow F_{bc} = 0$, i.e. the manifold is a *SGK*-manifold of the second kind. □

Theorem 4.6. Let *M be a GK*-manifold of class R_4. Then on the space of adjoint *G*-structure: 1) $A^{cd}_{ab} = \frac{1}{2}\tilde{\delta}^{cd}_{ab}$; 2) $C^{dch}C_{hab} = \delta^{cd}_{ab}$.

Proof. Let *M be a GK*-manifold that is a manifold of class R_4. Then, according to Theorem 4.4 $R^{d}_{ab\hat{c}} = R^{\hat{d}}_{ab\hat{c}} = 0$ and considering (3.7), we have:

$$1)\ R^{a}_{bc\hat{d}} = A^{ad}_{bc} - C^{adh}C_{hbc} - \frac{1}{2}F^{ad}F_{bc} - \delta^{a}_{c}\delta^{d}_{b} = 0;\ \ 2)\ R^{\hat{a}}_{bc\hat{d}} = R^{d}_{cba} = -\frac{2}{3}\delta^{d}_{c}F_{ba} + \frac{1}{3}\delta^{d}_{b}F_{ac} + \frac{1}{3}\delta^{d}_{a}F_{cb} = 0. \quad (4.6)$$

Let us contract the second equation by indices *d* and *a*; then we obtain $(n+1)F_{bc} = 0 \Longrightarrow F_{bc} = 0$, i.e. the manifold is a *SGK*-manifold of the second kind. Hence the equation (4.6:1) take the form: $A^{dc}_{ab} - C^{dch}C_{hab} - \delta^{c}_{b}\delta^{d}_{a} = 0$. Symmetrizing the latter equation, firstly, by indices *a* and *b*, and then by indices *c* and *d*, we get $A^{(dc)}_{(ab)} = \delta^{c}_{b}\delta^{d}_{a} = \frac{1}{2}\tilde{\delta}^{cd}_{ab}$. Due to the symmetry of tensor A^{cd}_{ab} in lower and upper pair of indices, the resulting identity can be rewritten as: $A^{cd}_{ab} = \frac{1}{2}\tilde{\delta}^{cd}_{ab}$. Hence the equation $A^{dc}_{ab} - C^{dch}C_{hab} - \delta^{c}_{b}\delta^{d}_{a} = 0$ is rewritten as $C^{dch}C_{hab} = \delta^{cd}_{ab}$. □

Definition 4.5. Let us call *AC*-manifold as a manifold of class R_5 if its curvature tensor satisfies the equation:

$$R(f^2X, f^2Y)f^2Z + R(f^2X, fY)fZ + R(fX, f^2Y)fZ - R(fX, fY)f^2Z = 0;\ \forall X, Y, Z \in \mathcal{X}(M). \quad (4.7)$$

Theorem 4.7. *GK*-manifold is a manifold of class R_5 if and only if this manifold is a *SGK*-manifold of the second kind and is on the space of adjoint *G*-structure $C^{adh}C_{hbc} = \delta^{a}_{[b}\delta^{d}_{c]}$.

Proof. Let *M be a GK*-manifold of the second kind that is a manifold of class R_5. Hence, Riemannian curvature tensor of such manifold satisfies the condition (4.7), which on the space of adjoint *G*-structure takes the form:

$$R^{i}_{jkl}(f^2X)^k(f^2Y)^l(f^2Z)^j + R^{i}_{jkl}(f^2X)^k(fY)^l(fZ)^j + R^{i}_{jkl}(fX)^k(f^2Y)^l(fZ)^j - R^{i}_{jkl}(fX)^k(fY)^l(f^2Z)^j = 0.$$

Taking into account (3.7) and the form of the matrix *f*, the latter equation will be written in the form: $R^{d}_{\hat{a}bc}X^bY^cZ_a + R^{\hat{d}}_{a\hat{b}\hat{c}}X_bY_cZ^a + R^{d}_{a\hat{b}\hat{c}}X_bY_cZ^a + R^{\hat{d}}_{\hat{a}bc}X^bY^cZ_a = 0$. The resulting equation is fulfilled if and only if $R^{d}_{a\hat{b}\hat{c}} = R^{\hat{d}}_{a\hat{b}\hat{c}} = 0$. According to (3.7):

$$1)\ C^{adh}C_{hbc} = \delta^{a}_{[b}\delta^{d}_{c]} - \frac{1}{2}F^{ad}F_{bc};\ \ 2)\ \frac{2}{3}\delta^{d}_{a}F^{bc} - \frac{1}{3}\delta^{b}_{a}F^{cd} - \frac{1}{3}\delta^{c}_{a}F^{db} = 0. \quad (4.8)$$

Let us contract (4.8:2) by indices *d* and *a*; then we obtain $\frac{2}{3}(n+1)F^{bc}=0 \Longrightarrow F^{bc}=0$, i.e. the manifold is a *SGK*-manifold of the second kind. Hence the equation (4.8:1) take the form $C^{adh}C_{hbc}=\delta^a_{[b}\delta^d_{c]}$.

Conversely, if relations (4.8) are satisfied for a *SGK*-manifold of the second kind, then the components of the Riemannian curvature tensor meet the conditions $R^d_{a\hat{b}\hat{c}}=R^{\hat{d}}_{a\hat{b}\hat{c}}=0$. Thus, the manifold is a manifold of class R_5 [15]. Since a *SGK*-manifold of the second kind is a *GK*-manifold, we get the required condition. □

Corollary. *GK*-manifold of class R_5 of 3 dimension is a Kenmotsu manifold.

Proof. Contract the equation $C^{adh}C_{hbc}=\delta^a_{[b}\delta^d_{c]}$, firstly, by indices *a* and *b*, and then by indices *c* and *d*, we get $\sum_{a,b,c}|C_{abc}|^2=\frac{1}{2}n(n-1)$. From the obtained equation it follows that for $n=1$ we have $C_{abc}=0$, i.e. the manifold is the Kenmotsu manifold. □

References

1. S. H. Kobayasi, K. Nomidzu, Osnovy differentsial'noy geometrii, **2**, 414 (Moscow, Nauka, 1981)
2. J. W. Gray, Some global properties of contact structures, Ann. Math., **69(2)**, 421 – 450 (1959)
3. S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structures*, Tôhoku Math. J., **12(3)**, 456 – 476 (1960)
4. D. E. Blair, *Contact manifolds in Riemannian geometry*, Lect. Notes Math., **509**, 146 (1976)
5. J. Bouzon, *Structures Presque cocomplexes*, Univ. et Politechn. Torino. Rend. Sem. Nat., **65(24)**, 53 – 123 (1964)
6. S. S. Chern, Pseudo-groupes continus infinis. Colloq. Internat. Centre nat. rech. Scient. 52, 119 – 136 (Strasbourg, Paris, 1953)
7. K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J., **24**, 93 – 103 (1972)
8. S. Tanno, *The automorphisms groups of almost contact Riemannian manifolds* Tôhoku Math. J., **21**, 21 – 38 (1969)
9. R. L. Bishop, B. O'Neil, *Manifolds of negative curvature*. Trans. Amer. Math. Soc., **145**, 1 – 50 (1969)
10. S. V. Umnova, Geometriya mnogoobraziy Kenmotsu i ikh obobshcheniy: Dis. … kand. fiz.-mat. nauk., 88 (Moscow, MPGU, 2002)
11. B. Najafi, N. H. Kashani, *On nearly Kenmotsu manifolds*. Turkish Journal of Mathematics, **37**, 1040 – 1047 (2013) http://journals.tubitak.gov.tr/ma th/ (Last accessed 12.12.2020)
12. M. B. Banaru. *O giperpoverkhnostyakh Kenmotsu spetsial'nykh ermitovykh mnogoobraziy* Sib. matem. zhurn., **45(1)**, 11–15 (2004)
13. A. Abu-Salem, A. R. Rustanov, *Some aspects of the geometry of Generalized Kenmotsu manifolds*. Far East Journal of Mathematical Sciences (FJMS), **103(9)**, 1407-1432 (2018) http://www.pphmj.com (Last accessed 22.12.2020) http://dx.doi.org/10.17654/MS103091407.
14. A. Abu-Salem, A. R. Rustanov, *Analogs of Gray Identities for the Riemannian Curvature Tensor of Generalized Kenmotsu Manifolds*, International Mathematical

Forum, **12(2)**, 87–95 HIKARI Ltd (2017) www.m-hikari.com (Last accessed 22.12.2020) https://doi.org/10.12988/imf.2017.611149.

15. A. Abu-Saleem, A. R. Rustanov, *Curvature Identities Special Generalized Manifolds Kenmotsu Second Kind*, Malaysian Journal of Mathematical Sciences, **9(2)**, 187-207 (2015) http://einspem.upm.edu.my/journal (Last accessed 25.12.2020)
16. A. Abu-Salem, A. R. Rustanov, S. V. Kharitonova, *Svoystva integriruyemosti obobshchennykh mnogoobraziy Kenmotsu*, Vladikavkazskiy matematicheskiy zhurnal,, **20(3)**, 4-20 (2018) DOI 10.23671/VNC.2018.3.13829.
17. V. F. Kirichenko, Differentsial'no-geometricheskiye struktury na mnogoobraziyakh. Izdaniye vtoroye, dopolnennoye. Odessa: «Pechatnyy Dom», **458** (2013)
18. V. F. Kirichenko, A. R. Rustanov, *Differentsial'naya geometriya kvazi-sasakiyevykh mnogoobraziy*, Matematicheskiy sbornik, **193(8)**, 71-100 (2002)
19. A. Abu-Saleem, I. D. Kochetkov, A. R. Rustanov, *O nekotorykh podklassakh obobshchennykh mnogoobraziy Kenmotsu*, IOP Conf. Series: Materials Science and Engineering, **918**, 012062 (2020) doi:10.1088/1757-899X/918/1/012062.
20. A. Abu-Salem, A. R. Rustanov, S. V. Kharitonova, F. Aksioma, *golomorfnykh (2r+1)-ploskostey dlya obobshchennykh mnogoobraziy Kenmotsu*. Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika., **66**, 5 – 3 (2020)
21. A. Abu-Saleem, I. D. Kochetkov, A. R. Rustanov, *Strukturnyye tenzory obobshchennykh mnogoobraziy Kenmotsu*, IOP Conf. Series: Materials Science and Engineering, **918**, 012062 (2020) doi:10.1088/1757-899X/918/1/012063