

Asymptotic approximation of the solution of a perturbed differential equation system

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Abstract. The article is devoted to the determination of second-order perturbations in rectangular coordinates and components of the body motion to be under study. The main difficulty in solving this problem was the choice of a system of differential equations of perturbed motion, the coefficients of the projections of the perturbing acceleration are entire functions with respect to the independent regularizing variable. This circumstance allows constructing a unified algorithm for determining perturbations of the second and higher order in the form of finite polynomials with respect to some regularizing variables that are selected at each stage of approximation. Special points are used to reduce the degree of approximating polynomials, as well as to choose regularizing variables. The problem of generation of an asymptotic approximation of the solution of a perturbed differential equation system is considered in the case where a bifurcation occurs in the “fast motions” equation when the parameter changes: two equilibrium positions merge, followed by a change in stability.

1 Introduction

One of the crucial tasks associated with trajectory measurements is the determination of the partial derivatives of rectangular coordinates that make up the body motion speed with respect to the initial conditions. In operations [1-2] added auxiliary functions, which are degree series with respect to the auxiliary variable. In operations [3-5] outlined ways of using universal variables in a number of tasks of mechanics to determine disturbances by the method of variation of arbitrary constants. In this case, it is convenient to consider the components of the initial values of the radius-vector and velocity as osculating variables. New methods for determining disturbances keep the standard features of the classical ones, while calculating the disturbances, the small parameter method is used, which makes it possible to obtain asymptotic decomposition of the solution

The error of the solution depends on the accuracy of the initial approximation of the perturbation function. General principles of the development of perturbation theory in coordinates were analyzed in operations [6-8] studied the use of regularizing variables for calculation of trajectories of motion. The results of this research show that the use of regularizing variables increases the computer-based accuracy of calculations and significantly reduces the calculation time.

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A crucial task of mechanics is to approximate the rectangular coordinates that make up the body speed and time in case of disturbed motion by algebraic polynomials of the lowest degree with respect to the auxiliary variable with a predetermined degree of accuracy.

One of the important problems in mechanics is the approximation of rectangular coordinates constituting the body velocity and time when the motion is perturbed by the lowest degree algebraic polynomials relative to the auxiliary variable with a predetermined degree of accuracy.

This paper describes a special system of differential equations of the perturbed moving body and this system is integrated through successive approximations method, which using the coordinates and constituents body velocity, take the form of polynomials in powers of some auxiliary variable. Its own independent variable is taken at each approximation step.

2 Problem specification and decision

The problem is set as follows. Spatial, almost parabolic, geocentric motion of a body is considered by a material point. The problem is considered as part of a limited three-body problem. The paper constructs an algorithm for calculating perturbations of any order in rectangular coordinates constituting body motion velocity, time and implements it on a computer to get the results in the form of corresponding polynomial coefficients. For the first time, the above functions are represented by finite polynomials at various stages of approximation with an accuracy high degree. An unperturbed parabolic or almost parabolic orbit is chosen as the initial approximation. The system of unperturbed two body problem regularized equations is chosen as the initial equations system [11]:

$$\begin{aligned} x'' &= 2hx - f_1; \quad x''' = 2hx'; \quad y'' = 2hy - f_2; \quad y''' = 2hy'; \quad z'' = 2hz - f_3; \\ & \quad z''' = 2hz'; \\ r'' &= 2hz + \mu; \quad t' = \frac{dt}{d\psi} = r; \quad h = \frac{\xi^{12} + \eta^{12} + \zeta^{12}}{2\rho^2} - \frac{\mu}{\rho}; \quad \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2} \end{aligned} \quad (1)$$

where the μ is the product of gravitational constant by the Earth mass, h – energy integral constant, f_1, f_2, f_3 are the Laplace integrals constants, ψ is the regularizing variable, ξ, η, ζ are the initial values of Cartesian coordinates.

Regularization is made to simplify the right-hand parts of spacecraft motion differential equations system and solution analytical properties. Let's set the Cauchy problem for the system (1). We need to find a solution of this system for the coordinates initial values ξ, η, ζ constituting the regularized velocity ξ', η', ζ' and the initial instant of time τ for a given value of the independent regularizing variable ψ . This value is chosen arbitrarily and considered a zero. We will find the general solution of the system (1) in the series form in powers of the following regularizing variable ψ .

$$x = \xi + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \xi^{(k)} \cdot \psi^k \quad (2)$$

$$x' = \xi' + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \xi^{(k+1)} \cdot \psi^k \quad (3)$$

$$x'' = \xi'' + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \xi^{(k+2)} \cdot \psi^k \quad (4)$$

Let's substitute the expressions (2), (4) in the first equation (1). In the left and right sides of the equation (1) we obtain series in powers of ψ . By equating the coefficients in both sides of the equation (1) with the same powers ψ , we obtain

$$\xi'' = 2 \cdot h \cdot \xi - f_1; \xi''' = 2 \cdot h \cdot \xi'; \xi^{IV} = 2 \cdot h \cdot \xi'', \dots, \xi^{(k+2)} = 2 \cdot h \cdot \xi^{(k)}, \quad (5)$$

if $k=1,2,\dots$

Here, it follows, that even coefficients are recurrently expressed in terms of even ones, odd coefficients – in terms of odd ones.

The article [11] describes the basic procedure used in compiling a special differential equation system for perturbed body motion. The main advantage of the perturbed motion differential equation system is that the coefficients at the projections of the perturbing acceleration X, Y, Z are entire functions of the independent regularizing variable ψ .

This makes it possible to effectively calculate not only first-order perturbances, but also those of higher orders, using the method of successive approximations. At each approximation, the degree of the generated polynomials is reduced using Chebyshev polynomials on the main interval $[-1,1]$.

The required functions in the constructed system of disturbed motion differential equations are the osculating initial values of the rectangular coordinates ξ, η, ζ constituting the regularized velocity ξ', η', ζ' of the regularized acceleration ξ'', η'', ζ'' , osculating initial values of the time instant τ , radius of the vector ρ and its derivatives ρ' and ρ'' by independent regularizing variable ψ . Normalization of the regularizing variable ψ gives a representation of the said perturbations in the form of lowest degree finite polynomials.

$$\frac{d}{d\psi} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = -r(\rho \cdot S_1 + \rho' \cdot S_2) \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + S_2 \cdot R \cdot \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} + R' \left[-S_2 \cdot \begin{pmatrix} x - \xi \\ y - \eta \\ z - \zeta \end{pmatrix} + S_3 \cdot \begin{pmatrix} \xi' \\ \eta' \\ \zeta' \end{pmatrix} \right] \quad (6)$$

$$\frac{d}{d\psi} \begin{pmatrix} \xi' \\ \eta' \\ \zeta' \end{pmatrix} = -r(\rho - \mu \cdot S_2) \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + S_1 \cdot R \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + R' \left[-S_1 \cdot \begin{pmatrix} x - \xi \\ y - \eta \\ z - \zeta \end{pmatrix} + S_3 \cdot \begin{pmatrix} \xi'' \\ \eta'' \\ \zeta'' \end{pmatrix} \right] \quad (7)$$

$$\frac{d}{d\psi} \begin{pmatrix} \xi'' \\ \eta'' \\ \zeta'' \end{pmatrix} = -r(\rho' + \mu \cdot S_1) \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + S_0 \cdot R \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + R' \left[-S_1 \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \psi \cdot \begin{pmatrix} \xi' \\ \eta' \\ \zeta' \end{pmatrix} \right] \quad (8)$$

$$\frac{d}{d\psi} \begin{pmatrix} \tau \\ -\rho \\ \rho' \\ -\rho'' \end{pmatrix} = r \begin{pmatrix} S_2 \\ S_1 \\ S_0 \\ \sigma \cdot S_1 \end{pmatrix} - R \cdot \left[(r + \rho) \cdot \begin{pmatrix} S_3 \\ S_2 \\ S_1 \\ S_0 \end{pmatrix} + \rho' \cdot \begin{pmatrix} \psi \cdot S_3 - 2 \cdot S_4 \\ S_3 \\ 0 \\ -\psi \end{pmatrix} + \rho'' \cdot \begin{pmatrix} \psi \cdot S_4 - 3 \cdot S_5 \\ 0 \\ -S_3 \\ S_2 \end{pmatrix} \right] \quad (9)$$

Let us consider the asymptotic approximation of the solution of the perturbed differential equation system. Let us introduce a perturbed differential equation system of the form:

$$\varepsilon \frac{dx}{dt} = f(x, y), \frac{dy}{dt} = g(x, y), \quad (10)$$

where the functions $f(x, y)$ and $g(x, y)$ are several times continuously differentiable in some domain of real variables (x, y) , $\varepsilon > 0$ – a small parameter. Together with system (10) let us consider a degenerated system, assuming in system (10) $\varepsilon = 0$:

$$f(x, y) = 0, \frac{dy}{dt} = g(x, y). \quad (11)$$

When investigating the qualitative behavior of the solutions to systems (10) and (11) at $\varepsilon \rightarrow 0$, the curve L plays an essential role. It is defined by the equation $f(x, y) = 0$, representing the set of all equilibrium positions of the equation:

$$\frac{dx}{d\tau} = f(x, y), \tau = \frac{t}{\varepsilon} \quad (12)$$

when changing the parameter y .

Let us consider the singular point (x_0, y_0) of the curve L , satisfying the following conditions:

$$\begin{aligned} f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0, f''_{xx}(x_0, y_0) \neq 0, \\ (f''_{xy}(x_0, y_0))^2 - f''_{xx}(x_0, y_0) \cdot f''_{yy}(x_0, y_0) = D^2 > 0. \end{aligned} \quad (13)$$

In the “fast motions” equation (12) at $y = y_0$, a bifurcation occurs – two equilibrium positions merge into one (x_0) , and then there is a stability change. Let us obtain an asymptotic approximation of the solution of the system (10) by the parameter ε near the singular point of the curve L , we will assume $x_0 = y_0 = 0$. In addition, we will assume that

$$g(0, 0) > 0. \quad (14)$$

Let us consider the geometric properties of the curve L at the point $(0, 0)$. Let us expand the function $f(x, y)$ by the Taylor formula at the point $(0, 0)$. Given the conditions (13), we obtain the curve L equation in the form

$$f''_{xx}(0, 0)x^2 + 2f''_{xy}(0, 0)xy + f''_{yy}(0, 0)y^2 + \dots = 0. \quad (15)$$

Given conditions (14), equation (15) has two solutions:

$$\begin{aligned} x = X_1(y) &= y(k_1 + m_1 y^l + \dots), \\ x = X_2(y) &= y(k_2 + m_2 y^l + \dots), l > 0, l \in \mathbb{Z}, \end{aligned} \quad (16)$$

where k_1, k_2 – the roots of the equation

$$f''_{xx}(0,0)k^2 + 2f''_{xy}(0,0)k + f''_{yy}(0,0) = 0.$$

$$\varepsilon X'_1(y) + \varepsilon^2 \alpha'(y) + \varepsilon^3 \beta'(y) = T(X_1(y), y) + T'_x(X_1(y), y) \cdot \varepsilon \alpha(y) + \quad (17)$$

$$+ T'_x(X_1(y), y) \cdot \varepsilon^2 \beta(y) + \dots$$

Let for certainty $f''_{xx}(0,0) > 0$. Assume that if k_1, k_2 are of the same sign, then this sign coincides with the sign $f''_{xx}(0,0)$. The functions $X_1(y), X_2(y)$ determine the equilibrium positions of equation (12) as the parameter y changes, and $X_1(y)$ – is stable at $y < 0$, $X_2(y)$ – is stable at $y > 0$. Note that

$$f'_x(X_1(y), y) = D \cdot y + o(y^2), f'_x(X_2(y), y) = -D \cdot y + o(y^2) \quad (18)$$

Since $g(0,0) \neq 0$, let us divide the first equation in system (10) by the second equation, and on the interval $-\delta < y < 0$, let us assume that y is an independent variable, δ is a small, positive number independent of ε . The result is an equation of the form

$$\varepsilon \frac{dx}{dy} = \frac{f(x, y)}{g(x, y)} = T(x, y). \quad (19)$$

On the interval $-\delta < y < 0$, let us represent the solution $x(y, \varepsilon)$ of equation (19) in the form

$$x(y, \varepsilon) = X_1(y) + \varepsilon \alpha(y) + \varepsilon^2 \beta(y). \quad (20)$$

Let's substitute the $x(y, \varepsilon)$ function into equation (19) by expanding the right part into a power series of ε .

$$\varepsilon X'_1(y) + \varepsilon^2 \alpha'(y) + \varepsilon^3 \beta'(y) = T(X_1(y), y) + T'_x(X_1(y), y) \cdot \varepsilon \alpha(y) + \quad (21)$$

$$+ T'_x(X_1(y), y) \cdot \varepsilon^2 \beta(y) + \dots$$

By equating the coefficients on the left and right with the same degrees ε in equality (21), we find

$\alpha(y)$ and $\beta(y)$. Let us consider the function $\alpha(y)$:

$$\alpha(y) = \frac{X'_1(y)}{T'_x(X_1(y), y)}, \quad (22)$$

where

$$T'_x(X_1(y), y) = \frac{f'_x(X_1(y), y)}{g(X_1(y), y)}, \quad (23)$$

Thus, $\alpha(y)$ can be represented in the form of

$$\alpha(y) = \frac{X_1'(y) \cdot g(X_1(y), y)}{f_x'(X_1(y), y)}. \quad (24)$$

Having regard to equalities (18), we obtain that $\alpha(y)$ has a first-order pole at $y=0$:

$$\alpha(y) = \frac{k_1 \cdot g(0, 0)}{D \cdot y} + \dots \quad (25)$$

Let us show that the function $X_\varepsilon(y) = X_1(y) + \varepsilon\alpha(y)$ is an asymptotic approximation of the solution of equation (19) on the interval $-\delta < y < -\Omega(\varepsilon)$.

Theorem. The solution of equation (19) $x = x(y, \varepsilon)$, satisfying the initial condition

$$|x(-\delta, \varepsilon) - X_\varepsilon(y)| = o(\varepsilon^2), \quad (26)$$

can be represented in the form of

$$x(y, \varepsilon) = X_\varepsilon(y) + R(y, \varepsilon), \quad (27)$$

where $|R(y, \varepsilon)| \leq \frac{M\omega(\varepsilon)}{y^4}$, $-\delta < y < -\Omega(\varepsilon)$, M is a sufficiently large value, independent of ε .

To prove it, let us consider two curves:

$$(L_1) \quad x = X_\varepsilon(y) + \frac{M\omega(\varepsilon)}{y^4}, \quad (28)$$

$$(L_2) \quad x = X_\varepsilon(y) - \frac{M\omega(\varepsilon)}{y^4} \quad (29)$$

and calculate at the points of curves (28) and (29) the derivative V by virtue of equation (19), multiplying it by ε :

$$\varepsilon V = \varepsilon \frac{dx}{dy} - \varepsilon X_\varepsilon'(y) \mp \varepsilon \left(\frac{M\omega(\varepsilon)}{y^4} \right)'. \quad (30)$$

Applying the Taylor formula and taking into account the specific type of function $\alpha(y)$, after a number of transformations, we obtain

$$\varepsilon V = \pm T_x'(X_1(y), y) \frac{M\omega(\varepsilon)}{y^4} - \varepsilon^2 \alpha'(y) \pm \frac{4M\varepsilon\omega(\varepsilon)}{y^5} + \dots \quad (31)$$

Using equations (18), (23), and (25), let us assume εV as

$$\varepsilon V = \frac{DM\omega(\varepsilon)}{y^3} \pm \frac{4M\varepsilon\omega(\varepsilon)}{y^5} + o\left(\frac{\varepsilon^2}{y^2}\right) + \dots \quad (32)$$

Suppose that $\omega(\varepsilon) = \varepsilon^2, \Omega(\varepsilon) = \varepsilon^{\frac{1}{6}}, -\delta \leq y \leq -\Omega(\varepsilon)$, we obtain $\varepsilon V > 0$ at points of the curve L_2 and $\varepsilon V < 0$ at points of the curve L_1 .

Thus, the solution of equation (19), the starting point of which is in the band between the curves L_1 and L_2 , cannot leave this band at the specified values of y .

If y varies on the interval $-\delta < y < -\varepsilon^{\frac{1}{6}}$, then the solution $x(y, \varepsilon)$ is

$$x(y, \varepsilon) = X_1(y) + \varepsilon \alpha(y) + R(y, \varepsilon), \quad (33)$$

where $R(y, \varepsilon)$ – is the value of the order, minimum $\varepsilon^{\frac{4}{3}}$.

3 Main results

The main paper conclusions are as follows:

- A special system of the perturbed body motion differential equations is constructed, which allows representing the rectangular coordinates constituting the regularized velocity and time of motion in the second and higher approximations by finite algebraic polynomials relative to a specially introduced regularizing variable, which is chosen at each approximation step.
- Possibility of representing the above functions by polynomials in powers of the regularizing variable is rigorously proven.
- The problem of obtaining an asymptotic approximation of the perturbed differential equation solution system in the case where a bifurcation occurs in the equation of “fast motions” when the parameter changes is considered: two equilibrium positions merge followed by a change in stability. The qualitative character of the behavior of solutions of an ordinary differential equation system when a small parameter tends to zero on the finite time interval at which the slow variable passes through some point corresponding to the bifurcation in the fast motion system is studied: a stable limit cycle merges with an unstable one and disappears.

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