Temperature distribution in an elliptical body with an internal heat source with partial adiabatic isolation

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Abstract. The article calculates the temperature field in an elliptical body with internal heat dissipation. In this case, the boundary conditions are boundary conditions of the third kind. The solution is located at the transition to the elliptic coordinate system. The author has obtained an analytical solution for the distribution of the temperature field in a body with an elliptical cross-section of infinite length with zero ambient temperature with partial adiabatic isolation in the form of a functional series using hypergeometric functions.

1 Introduction

The processes of heat transfer and associated mass transfer play an exceptional role in nature and technology. Indeed, the temperature regime of the environment depends on them. The flow of the working process in a variety of technological installations depends on them.

In connection with the improvement of thermal equipment of energy-consuming and producing devices, a more accurate calculation of heat transfer processes in heat networks is required. Therefore, it seems advisable to improve the methods of calculating heat transfer in such systems. It is known that for better cooling of fuel elements (electrical conductors, rods of nuclear reactors, etc.), it is necessary to have a large heat transfer surface. The increase in the surface can be achieved either by finning, or by replacing the rods of circular cross-section, which have a minimum surface of the heat sink, with rods of other cross-sections, for example, oval or elliptical. A special place is occupied by bodies with an elliptical cross-section. Their peculiarity is that by manipulating the change in the length of the semi-axes of the ellipse, it is possible to obtain accurate analytical solutions to stationary problems of thermal conductivity for a very wide range of shape changes: from a cylinder to a thin plate.

Several papers are devoted to the calculation of temperature fields in bodies of elliptical cross-section in the presence of internal heat sources under various conditions. This article is a continuation of the work [1].

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2 Main Part

The main task of this paper is to find the distribution of the temperature field in a body with an elliptical cross-section of infinite length under boundary conditions of the third kind. In power plants, heat transfer between two media (heat carriers) through a solid wall separating them is often found, which is called heat transfer. In this case, the heat from the more heated heat carrier is transferred to the wall by heat transfer and heat radiation, inside the wall heat exchange occurs due to thermal conductivity, and from the opposite wall surface is carried out by heat transfer to the less heated heat carrier.

In the theory of thermal conductivity, the process of heat transfer is understood as the thermal conductivity of the wall under boundary conditions of the third kind. The boundary condition of the third kind consists in setting a linear combination of the transport potential and its derivative with respect to the normal at the boundary of the considered region. Boundary conditions of this type play an important role in the theory of heat and mass transfer, since they are a mathematical formulation of the conditions of convective heat and mass transfer.

We will look for the temperature distribution in an infinitely long body, the cross section of which is an ellipse with semi-axes a and b (Fig. 1), half of the surface of which is adiabatically isolated. The body in question is located in a zero-temperature environment. The same heat source qv operates inside the body.

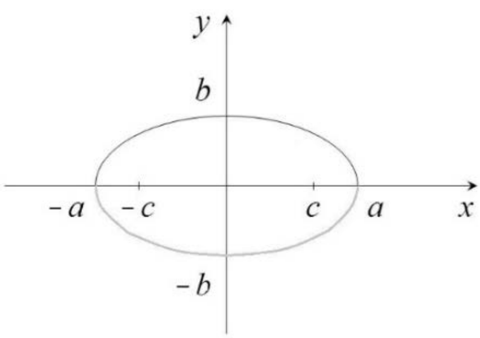

Fig. 1. Elliptical section of the body in the section.

To find the temperature distribution, it is necessary to solve the Poisson equation, which is an elliptic partial differential equation

$$\Delta T + \frac{q_v}{\lambda} = 0 \tag{1}$$

which describes a stationary temperature field with boundary conditions on the body surface: the first half is the heat transfer

$$-\lambda gradT = hT \tag{2}$$

the second heat transfer is not

$$-\lambda gradT = 0 \tag{3}$$

To obtain the formula describing the temperature field, we use the elliptic coordinate system α, β, $0 \le \alpha < \infty$, $-\pi \le \beta \le \pi$. If $\alpha = \alpha_0$ is the equation of the body surface, then

$$a = cch\alpha_0, b = chsh\alpha_0, c = \sqrt{a^2 + b^2} \tag{4}$$

The Poisson equation in elliptic coordinates has the form

$$\frac{1}{c^2(ch^2\alpha - \cos^2\beta)}\left(\frac{\partial^2 T}{\partial\alpha^2} + \frac{\partial^2 T}{\partial\beta^2}\right) = -\frac{q_v}{\lambda} \tag{5}$$

and the boundary conditions are given by the dependence for α=α0 0≤β≤π

$$-\lambda\frac{1}{c\sqrt{ch^2\alpha_0 - \cos^2\beta}}\frac{\partial T}{\partial\alpha} = hT \tag{6}$$

for α=α0 -π≤β≤0

$$\frac{\partial T}{\partial\alpha} = 0 \tag{7}$$

the substitution of the form

$$T = U(\alpha,\beta) - \frac{q_v c^2}{4\lambda}\left(sh^2\alpha + \cos^2\beta\right) \tag{8}$$

transforms equation (1) into Laplace equation

$$\frac{\partial^2 U}{\partial\alpha^2} + \frac{\partial^2 U}{\partial\beta^2} = 0 \tag{9}$$

the boundary condition (6) takes the form

$$\frac{1}{\sqrt{ch^2\alpha_0 - \cos^2\beta}}\left(\frac{\partial U}{\partial\alpha} - \frac{q_\upsilon}{4\lambda}c^2 sh2\alpha_0\right) = Bi\left[\frac{q_\upsilon}{4\lambda}c^2(sh^2\alpha_0 + \cos^2\beta) - U\right],$$

$$Bi = \frac{hc}{\lambda} \tag{10}$$

where: Bi – the number of Bio, which characterizes the intensity of heat exchange between the surface of the body and the environment, h-the coefficient of heat transfer between the medium and the surface of the body.

And the boundary condition (7) will take the form

$$\frac{\partial U}{\partial\alpha} - \frac{q_\upsilon}{4\lambda}c^2 sh2\alpha_0 = 0 \tag{11}$$

The solution of equation (8) is given by the dependence

$$U(\alpha,\beta) = \sum_{n=0}^{\infty}\left(A_n chn\alpha\cos n\beta + B_n shn\alpha\sin n\beta\right) \tag{12}$$

We find the constants B_n from the boundary condition (10)

$$\sum_{n=0}^{\infty}\left(nA_n shn\alpha_0\cos n\beta - nB_n chn\alpha_0\sin n\beta\right) = \frac{q_\upsilon}{4\lambda}c^2 sh2\alpha_0 \tag{13}$$

and integrating from - π to 0, we get

$$2B_{2n+1}ch(2n+1)\alpha_0 = \frac{\pi q_\upsilon}{4\lambda}c^2 sh\,2\alpha_0 \tag{14}$$

from

$$B_{2n+1} = \frac{q_\upsilon}{2\lambda}\frac{\pi c^2 sh\,2\alpha_0}{ch\,(2n+1)\alpha_0} \tag{15}$$

We find the constants A_n from the boundary condition (9)

$$\frac{1}{\sqrt{ch^2\alpha_0 - \cos^2\beta}}\left[\sum_{n=0}^{\infty}\left(nA_n shn\alpha_0 \cos n\beta - nB_n chn\alpha_0 \sin n\beta\right) - \frac{q_\upsilon}{4\lambda}c^2 sh\,2\alpha_0\right] =$$

$$= Bi\left[\frac{q_\upsilon}{4\lambda}c^2(sh^2\alpha_0 + \cos^2\beta) - \sum_{n=0}^{\infty}\left(A_n chn\alpha \cos n\beta + B_n shn\alpha \sin n\beta\right)\right] \tag{16}$$

The function

$$f(\beta) = \sqrt{ch^2\alpha_0 - \cos^2\beta} \tag{17}$$

is even with respect to β, we decompose it in a series of cosines

$$\sqrt{ch^2\alpha_0 - \cos^2\beta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\beta \tag{18}$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sqrt{ch^2\alpha_0 - \cos^2\beta}\cos n\beta d\beta \tag{19}$$

Since

$$\cos n\beta = \cos^n\beta - C_n^2\cos^{n-2}\beta\sin^2\beta + C_n^4\cos^{n-4}\beta\sin^4\beta - ... \tag{20}$$

then all with odd n vanish. Therefore

$$\sqrt{ch^2\alpha_0 - \cos^2\beta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n}\cos 2n\beta \tag{21}$$

Rewriting (15) as

$$\sum_{n=0}^{\infty} nA_n shn\alpha_0 \cos n\beta - \sum_{n=0}^{\infty} nB_n shn\alpha_0 \sin nB - \frac{q_\upsilon}{4\lambda}c^2 sh2\alpha_0 =$$

$$= Bi\frac{q_v}{4\lambda}c^2(sh^2\alpha_0\sqrt{ch^2\alpha_0 - \cos^2\beta} + \sqrt{ch^2\alpha_0 - \cos^2\beta}\cos^2\beta) -$$

$$- Bi(\frac{a_n}{2}\sum_{n=0}^{\infty} A_n chn\alpha_0 \cos n\beta + \frac{a_0}{2}\sum_{n=0}^{\infty} B_n sh\alpha_0 \sin n\beta +$$

$$+\sum_{n=1}^{\infty} a_{2n}\cos 2n\beta \sum_{n=0}^{\infty} A_n chn\alpha_0 \cos n\beta +$$

$$+\sum_{n=1}^{\infty} a_{2n}\cos 2n\beta \sum_{n=0}^{\infty} B_n shn\alpha_0 \cos n\beta) \qquad (22)$$

and integrating by β from 0 to π, we get

$$-B_{2n+1}ch(2n+1)\alpha_0 - \frac{q_u\pi}{4\lambda}c^2 sh2\alpha_0 = Bi\frac{q_u}{4\lambda}c^2(sh^2\alpha_0\int_0^{\pi}\sqrt{ch^2\alpha_0-\cos^2\beta}\,d\beta +$$

$$+\int_0^{\pi}\sqrt{ch^2\alpha_0-\cos^2\beta}\cos^2\beta d\beta) - Bi\frac{\pi}{2}(a_0A_0 + ch2n\alpha_0 a_{2n}A_{2n}) \qquad (23)$$

from where

$$A_{2n} = \left[\frac{q_u}{2\lambda}c^2 sh2a_0\pi + 4B_{2n+1}ch(2n+1)a_0 + Bi\frac{q_u}{2\lambda}c^2(sh^2a_0\int_0^{\pi}\sqrt{ch^2a_0-\cos^2\beta}\,d\beta +\right.$$

$$\left.+\int_0^{\pi}\sqrt{ch^2\alpha_0-\cos^2\beta}\cos^2\beta d\beta)\right] : Bi\pi ch2n\alpha_0 a_{2n}, n = 0,1,2... \qquad (24)$$

elliptic integrals are used to calculate the integrals in the last formula

$$\int_0^{\pi}\sqrt{ch^2a_0-\cos^2\beta}\,d\beta = \int_{-\pi/2.}^{\pi/2} ch.a_0\sqrt{1-\frac{\sin^2 x}{ch^2a_0}}\,d(-x) =$$

$$= 2ch.a_0 E\left(\frac{1}{ch.a_0},\frac{\pi}{2}\right) \qquad (25)$$

where E is a complete elliptic integral of the second kind

$$\int_0^{\pi}\sqrt{ch^2a_0-\cos^2\beta}\cos^2\beta d\beta = \int_{-\pi/2.}^{\pi/2} ch.a_0\sqrt{1-\frac{\sin^2 x}{ch^2a_0}}\sin^2 x d(-x) =$$

$$= 2ch.a_0E\left[\frac{sh^2a_0}{3}F\left(\frac{1}{ch.a_0},\frac{\pi}{2}\right)+\left(\frac{2-ch^2a_0}{3}\right)E\left(\frac{1}{ch.a_0},\frac{\pi}{2}\right)\right] \qquad (26)$$

where F is a complete elliptic integral of the first kind.

The coefficients a_{2n} are determined by the formula

$$a_{2n} = \frac{2}{\pi}\int_0^{\pi}\sqrt{ch^2\alpha_0-\cos^2\beta}\,(cos^{2n}\beta - C_n^2\cos^{2n-2}\beta\sin^2\beta + ...)d\beta =$$

$$= \frac{4ch.a_0}{\pi}\int_0^{\pi/2}\sqrt{1-\frac{\sin^2 x}{ch^2a_0}}(\sin^{2n}x - C_n^2\sin^{2n-2}x\cos^2 x + ...)dx =$$

$$=\frac{2ch.a_0}{\pi}\left[B\left(\frac{2n+1}{2},\frac{1}{2}\right)F_1\left(\frac{2n+1}{2},-\frac{1}{2},n+1,\frac{1}{ch.a_0}\right)-\right.$$

$$\left.-C_{2n}^2 B\left(\frac{2n-1}{2},\frac{3}{2}\right)F_1\left(\frac{2n-1}{2},-\frac{1}{2},n+1,\frac{1}{ch.a_0}\right)+\dots\right] \quad (27)$$

where B(y,z) is the beta function, F_1(y,z,m;k) is a hypergeometric function, which is a special function represented by a hypergeometric series that includes many other special functions as specific or limiting cases

$$T(\alpha,\beta)=\sum_{n=0}^{\infty}\left(A_{2n}ch2n\alpha\cos 2n\beta + B_{2n+1}sh(2n+1)\alpha\sin 2n\beta\right)-$$

$$-\frac{q_v c^2}{4\lambda}\left(sh^2\alpha+\cos^2\beta\right) \quad (28)$$

in which A_{2n} is found from the ratio (28)

$$A_{2n}=\left\{\frac{q_u}{2\lambda}c^2 sh2a_0\pi + 4B_{2n+1}ch(2n+1)a_0 + Bi\frac{q_u}{2\lambda}c^2 ch.a_0\left[sh^2 a_0 E\left(\frac{1}{ch.a_0},\frac{\pi}{2}\right)+\right.\right.$$

$$\left.\left.+\frac{sh^2 a_0}{3}F\left(\frac{1}{ch.a_0},\frac{\pi}{2}\right)+\left(\frac{2-ch^2 a_0}{3}\right)E\left(\frac{1}{ch.a_0},\frac{\pi}{2}\right)\right]\right\}:$$

$$: Bi\pi ch2n\alpha_0 a_{2n},\ n=0,1,2\dots \quad (29)$$

and B_{2n+1} from the ratio (15).

We will analyze the resulting solution. In the case of a surface without adiabatic isolation, for reasons of symmetry, we should put Bn =0. Then we get the solution given in [2].

$$T(\alpha,\beta)=\sum_{n=0}^{\infty}A_{2n}ch2n\alpha\cos 2n\beta-\frac{q_v c^2}{4\lambda}\left(sh^2\alpha+\cos^2\beta\right) \quad (30)$$

3 Conclusions

In this paper, we present a solution for the distribution of the temperature field in a body with an elliptical cross-section under boundary conditions of the third kind. The solution is obtained for the case of adiabatic isolation of half of the wall in the system of elliptic coordinates in the form of a trigonometric series containing hypergeometric functions. The obtained result is compared with the case of the absence of adiabatic isolation and compared with other results of the study.

References

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