

Solution of Thermal Conductivity Problem of a Finite Dimensions Plate with Two Heat Sources

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Abstract. Research in the basic sciences is a critical factor in the development of the civil engineering industry. Solving the problems of radiation-convective heat transfer from heated surfaces has always aroused interest from the point of view of science and practical engineering application of knowledge. However, analytical solutions to these problems are obtained for elementary cases, for example, for infinite plates heated uniformly, or the propagation of heat waves in them obeys certain laws. The solution of the coupled problem of radiation-convective transfer from the surface of these panels is complicated not only by the geometric shape, but also by the openness of the entire thermophysical system, which includes the transfer of thermal energy from the coolant (coolant for cooling systems) to the surface of the thermal panel, from the panel to the room air by convection, and radiation to surrounding bodies (enclosing structures, furniture, people). In turn, additional heat exchange by convection occurs between the air and the enclosing structures. This article considers the possibility of obtaining an analytical solution to the problem of temperature distribution on the surface of a plate with two heat sources. When deriving the formulas, the classical equations of thermodynamics (Newton-Richmann, Fourier's law, Helmholtz equation) were used. The general solution of the differential equation, in this case, is a linear combination of the Infeld and MacDonald functions. The research results can be applied to various areas of technical sciences: cooling of microprocessors, renewable sources of thermal energy, thermal and cooling panels for industrial production, automotive, marine shipbuilding, and of course heating and air conditioning systems for buildings and transport.

1 Introduction

Solving the problems of radiation-convective heat transfer from heated surfaces has always aroused interest from the point of view of science and practical engineering application of knowledge. However, analytical solutions of these problems are obtained for elementary cases, for example, for infinite plates, plates heated uniformly, or the propagation of heat waves in them obeys certain laws [1-10]. Thermal panels are widely used in hot water

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heating and cooling systems, and have a variety of rather complex geometric shapes (convectors, radiators, ceiling panels, floor panels). In this regard, thermal panels are quite complex objects from the point of view of mathematical description, and each of their designs is actually a special case. A review of modern works on this issue showed that to determine the thermal characteristics of thermal panels, the solution of analytical equations is rarely used, and, as a rule, exclusively for solving the problem of radiation heat transfer [11-14]. The method of numerical modeling [15-18], experimental studies on finished samples [19-21], and in some cases a combination of numerical modeling and experimental research [22] are used more often.

Thermal panels of the convector type (a set of plates), to which heat energy is supplied or extracted using a circulating heat / cooling medium, have become widespread in technology (see Fig. 1). Panels of this design use such technical means as air conditioners, chillers, fan coils, heating equipment, cooling systems of microprocessors. The complexity of the problem in this case seems to be that in its solution it is necessary to link two independent physicists: heat exchange inside the pipe, which depends on the hydraulic regime, and heat exchange of the pipe and plates on the outer surface of the panel.

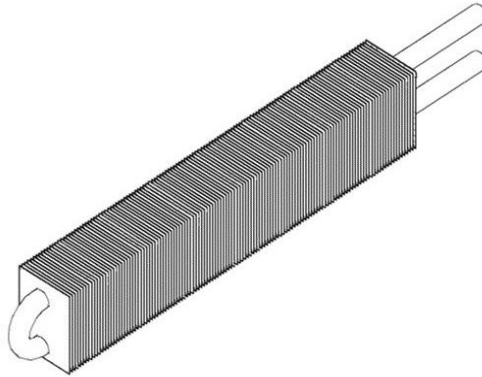


Fig. 1. Convector type thermal panel construction

To solve this class of "compound" or conjugate problems, it is necessary to consider several simpler problems, for example, heat transfer from a plate, which is crossed by two pipes at once (with a uniform temperature across the cross section), having different temperatures. After that, more complex tasks can be set, such as the distribution of air masses in the room and the distribution of air temperature, as well as temperatures on the surrounding surfaces.

2 Methods

Let us consider the problem of the temperature field in a rectangular plate with two heat sources under boundary conditions of the third kind. First, you need to solve the problem for a plate with one heat source. Because the plates in the convectors are made thin enough in relation to the other two geometric parameters – length a and thickness b , and the radius of the pipe R is also much less than the length and thickness, then

$$\begin{cases} h \ll a, b \\ R \ll a, b \end{cases}$$

In this case, the influence of edge effects and heat transfer from the side faces ah and bh of the plate will be negligible. This means that the selected problem can be

approximately considered as the problem of an infinite plate. According to the Newton-Richmann law, the heat flux from the plate surface is:

$$\vec{q} = \alpha u \vec{n}, \tag{1}$$

where u – temperature referenced from ambient temperature $u = T - T_{cp}$

On the other hand, according to Fourier's law, the heat flux is:

$$\vec{q} = -\mu \vec{\nabla} u \tag{2}$$

Let's write the law of conservation of energy:

$$c\rho \frac{\partial u}{\partial t} + \text{div} \vec{q} = Q, \tag{3}$$

where Q – amount of heat released per unit time per unit volume.

Taking into account (1) and (2), we obtain:

$$c\rho \frac{\partial u}{\partial t} = \mu \Delta u + Q \tag{4}$$

In the stationary case, taking into account convective and radiant heat transfer, due to the small thickness of the plate, the outflow of heat from its surfaces ab , can be taken into account by modifying Q :

$$\mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u + Q - \frac{2\alpha u}{h} - \frac{2\sigma\varepsilon \left((u + T_{cp})^4 - T_{cp}^4 \right)}{h} = 0, \tag{5}$$

where ε – plate surface emissivity.

Due to $u \ll T_{cp}$ linearize the term responsible for radiant heat transfer:

$$\mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u + Q - \frac{2\alpha u}{h} - \frac{2\sigma\varepsilon \cdot 3T_{cp}^3 u}{h} = 0 \tag{6}$$

Introducing the notation $\beta = \frac{2(\alpha + 3\sigma\varepsilon T_{cp}^3)}{\mu h}$, we obtain:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u - \beta u = -\frac{Q(x, y)}{\mu} \tag{7}$$

Equation (7) is the Helmholtz equation in Cartesian coordinates. In polar coordinates, it will take the form:

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) u - \beta u = -\frac{Q(r, \varphi)}{\mu} \tag{8}$$

We choose the center of the pipe as the pole of the polar coordinate system, then, due to axial symmetry, the temperature does not depend on the coordinate φ :

$$\frac{1}{r} (ru')' - \beta u = -\frac{Q(r)}{\mu} \tag{9}$$

The general solution of such a differential equation is a linear combination of the Infeld and Macdonald functions:

$$u = C_1 I_0(\sqrt{\beta}r) + C_2 K_0(\sqrt{\beta}r) \quad (10)$$

It is known that the Infeld function is singular at infinity $\lim_{r \rightarrow +\infty} I_0(\sqrt{\beta}r) = +\infty$; it is obvious that in our case the plate temperature at infinity should tend to the ambient temperature $u = 0$, hence $C_1 = 0$:

$$u = C_2 K_0(\sqrt{\beta}r) \quad (11)$$

Find the constant C_2 , in case the pipe temperature is set u_0 (since the heat transfer coefficient of the water - pipe wall system is very high, the pipe wall temperature is approximately equal to the water temperature):

$$C_2 = \frac{u_0}{K_0(\sqrt{\beta}R)}, \quad (12)$$

where R – outer radius of the pipe.

Then the solution in the case of a plate with a single heat source of a given temperature will take the form:

$$u = u_0 \frac{K_0(\sqrt{\beta}r)}{K_0(\sqrt{\beta}R)} \quad (13)$$

Consider the case with two heat sources with given temperatures u_{01} and u_{02} with radii R_1 and R_2 with distances between their centers l . By virtue of $R_1, R_2 \ll l$ we will look for an approximate solution in the form:

$$u = D_1 K_0(\sqrt{\beta}r_1) + D_2 K_0(\sqrt{\beta}r_2), \quad (14)$$

where r_1 and r_2 – distances to the centers of the first and second pipes, respectively.

For the temperatures of the external surfaces of the pipes, we obtain:

$$\begin{aligned} u_{01} &= D_1 K_0(\sqrt{\beta}R_1) + D_2 K_0(\sqrt{\beta}l) \\ u_{02} &= D_1 K_0(\sqrt{\beta}l) + D_2 K_0(\sqrt{\beta}R_2) \end{aligned} \quad (15)$$

Solving a system of linear algebraic equations with respect to unknowns D_1 and D_2 we get:

$$D_1 = \frac{u_{01} K_0(\sqrt{\beta}R_2) - u_{02} K_0(\sqrt{\beta}l)}{K_0(\sqrt{\beta}R_1) K_0(\sqrt{\beta}R_2) - [K_0(\sqrt{\beta}l)]^2} \quad (16)$$

$$D_2 = \frac{u_{02} K_0(\sqrt{\beta}R_1) - u_{01} K_0(\sqrt{\beta}l)}{K_0(\sqrt{\beta}R_1) K_0(\sqrt{\beta}R_2) - [K_0(\sqrt{\beta}l)]^2} \quad (17)$$

Then the temperature distribution will be described by:

$$u = \frac{[u_{01}K_0(\sqrt{\beta}R_2) - u_{02}K_0(\sqrt{\beta}l)]K_0(\sqrt{\beta}r_1) + [u_{02}K_0(\sqrt{\beta}R_1) - u_{01}K_0(\sqrt{\beta}l)]K_0(\sqrt{\beta}r_2)}{K_0(\sqrt{\beta}R_1)K_0(\sqrt{\beta}R_2) - [K_0(\sqrt{\beta}l)]^2} \quad (18)$$

For the same pipe radii, we get the formula:

$$u = \frac{K_0(\sqrt{\beta}R)[u_{01}K_0(\sqrt{\beta}r_1) + u_{02}K_0(\sqrt{\beta}r_2)] - (u_{01} + u_{02})K_0(\sqrt{\beta}l)}{[K_0(\sqrt{\beta}R)]^2 - [K_0(\sqrt{\beta}l)]^2} \quad (19)$$

For the same pipe radii and temperatures, the formula applies:

$$u = u_0 \frac{K_0(\sqrt{\beta}R)[K_0(\sqrt{\beta}r_1) + K_0(\sqrt{\beta}r_2)] - 2K_0(\sqrt{\beta}l)}{[K_0(\sqrt{\beta}R)]^2 - [K_0(\sqrt{\beta}l)]^2} \quad (20)$$

Since convectors are made with the same pipe diameters, then we will continue to calculate with respect to the formula (19). The formulas written above are valid at some distance from the boundary. We derive a formula that will describe the behavior of the temperature at the boundary:

$$u = u_1(r_1) + u_2(r_2) \quad (21)$$

$$\vec{\nabla}u = \vec{\nabla}u_1 + \vec{\nabla}u_2 = \frac{du_1}{dr_1} \vec{r}_1 + \frac{du_2}{dr_2} \vec{r}_2 \quad (22)$$

Counting the derivatives, we get:

$$\frac{du_1}{dr_1} \vec{r}_1 = - \frac{(u_{01}K_0(\sqrt{\beta}R)K_1(\sqrt{\beta}r_1)\sqrt{\beta})\vec{r}_1}{([K_0(\sqrt{\beta}R)]^2 - [K_0(\sqrt{\beta}l)]^2)r_1} \quad (23)$$

$$\frac{du_2}{dr_2} \vec{r}_2 = - \frac{(u_{02}K_0(\sqrt{\beta}R)K_1(\sqrt{\beta}r_2)\sqrt{\beta})\vec{r}_2}{([K_0(\sqrt{\beta}R)]^2 - [K_0(\sqrt{\beta}l)]^2)r_2} \quad (24)$$

When multiplying the formula (22) by $-\Pi$, let's find the heat flux density along the plate itself from the pipes to the edges:

$$\vec{q} = \left\{ u_{01}K_1(\sqrt{\beta}r_1)\frac{\vec{r}_1}{r_1} + u_{02}K_1(\sqrt{\beta}r_2)\frac{\vec{r}_2}{r_2} \right\} \frac{\Pi K_0(\sqrt{\beta}R)\sqrt{\beta}}{[K_0(\sqrt{\beta}R)]^2 - [K_0(\sqrt{\beta}l)]^2} \quad (25)$$

The density of the heat flow leaving the surfaces ab according to the formula (1):

$$\vec{q} = \alpha \vec{n} \frac{K_0(\sqrt{\beta}R)[u_{01}K_0(\sqrt{\beta}r_1) + u_{02}K_0(\sqrt{\beta}r_2)] - (u_{01} + u_{02})K_0(\sqrt{\beta}l)}{[K_0(\sqrt{\beta}R)]^2 - [K_0(\sqrt{\beta}l)]^2} \quad (26)$$

where \vec{n} – normal to surfaces ab .

Due to the fact that:

$$\begin{aligned} \frac{\partial u}{\partial n} &= (\vec{\nabla}u_1, \vec{n}) + (\vec{\nabla}u_2, \vec{n}) = |\vec{\nabla}u_1| \cos \varphi_1 + |\vec{\nabla}u_2| \cos \varphi_2 = - \left\{ \left| \frac{du_1}{dr_1} \right| \cos \varphi_1 + \left| \frac{du_2}{dr_2} \right| \cos \varphi_2 \right\} = \\ &= - \frac{\sqrt{\beta} K_0 (\sqrt{\beta} R)}{\left[K_0 (\sqrt{\beta} R) \right]^2 - \left[K_0 (\sqrt{\beta} l) \right]^2} \left\{ u_{01} K_1 (\sqrt{\beta} r_1) \cos \varphi_1 + u_{02} K_1 (\sqrt{\beta} r_2) \cos \varphi_2 \right\} \end{aligned} \quad (27)$$

At the same time:

$$\frac{\partial u}{\partial n} = - \frac{\alpha}{\text{III}} u \quad (28)$$

$$\frac{\sqrt{\beta} K_0 (\sqrt{\beta} R)}{\left[K_0 (\sqrt{\beta} R) \right]^2 - \left[K_0 (\sqrt{\beta} l) \right]^2} \left\{ u_{01} K_1 (\sqrt{\beta} r_1) \cos \varphi_1 + u_{02} K_1 (\sqrt{\beta} r_2) \cos \varphi_2 \right\} = \frac{\alpha}{\text{III}} u \quad (29)$$

Then the temperature value will take the form:

$$u = \frac{\sqrt{\beta} K_0 (\sqrt{\beta} R)}{\left[K_0 (\sqrt{\beta} R) \right]^2 - \left[K_0 (\sqrt{\beta} l) \right]^2} \left\{ u_{01} K_1 (\sqrt{\beta} r_1) \cos \varphi_1 + u_{02} K_1 (\sqrt{\beta} r_2) \cos \varphi_2 \right\} \frac{\text{III}}{\alpha} \quad (30)$$

Then the temperature at the boundary can be approximated using the arithmetic mean between formulas (19) and (30):

$$\begin{aligned} u_{sp} &= \frac{K_0 (\sqrt{\beta} R)}{2} \left\{ \frac{\left[u_{01} K_0 (\sqrt{\beta} r_1) + u_{02} K_0 (\sqrt{\beta} r_2) \right] - (u_{01} + u_{02}) K_0 (\sqrt{\beta} l)}{\left[K_0 (\sqrt{\beta} R) \right]^2 - \left[K_0 (\sqrt{\beta} l) \right]^2} + \right. \\ &\left. + \frac{\sqrt{\beta} \left[u_{01} K_1 (\sqrt{\beta} r_1) \cos \varphi_1 + u_{02} K_1 (\sqrt{\beta} r_2) \cos \varphi_2 \right] \frac{\text{III}}{\alpha}}{\left[K_0 (\sqrt{\beta} R) \right]^2 - \left[K_0 (\sqrt{\beta} l) \right]^2} \right\} \end{aligned} \quad (31)$$

3 Results

The analytical problem is solved for a plate with two sources of thermal energy, the temperature at the boundary, known in advance of 95 and 70 ° C. The ambient temperature is 18 ° C. The panel design is shown in figure 1. The calculation results for this plate are shown in the figure 2.

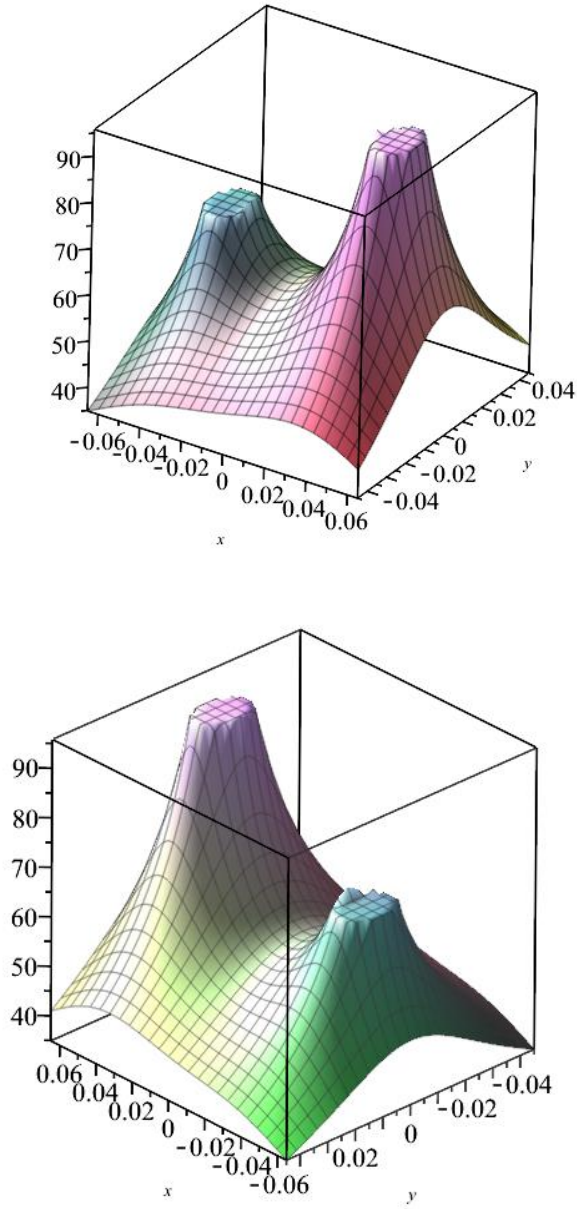


Fig. 2. Graphs of the temperature distribution in the plate obtained using the analytical solution

Solving this problem by numerical modeling earlier [17], the following temperature distribution was obtained (see Fig. 3).

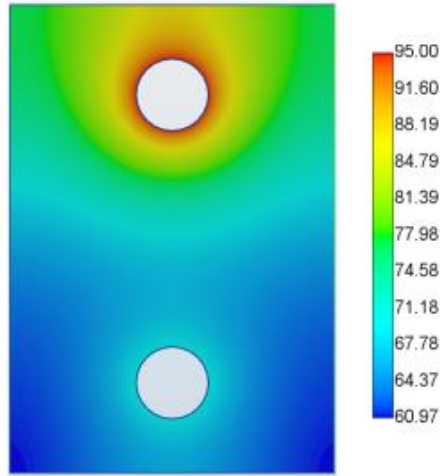


Fig. 3. Plot of the temperature distribution in the plate obtained by numerical solution

It can be seen that the results of solving the problem are somewhat different. However, the similarity of the solution is already visible. The analytical solution currently requires additional refinement, since it does not take into account the edge effects, considering the length, width of the plate and the distance between the pipes to be greater than the radius of both pipes and the characteristic distance of decreasing temperature $-\frac{1}{\sqrt{\beta}}$. It should be noted significant usability of the analytical solution of the problem, as there is no need to create models of different plates, and it is sufficient to set the initial parameters of the problem.

Also, in the future, you can trace the development of the front of the highest temperature from the upper source, and adjust the shape of the plate according to its shape. This will reduce the amount of material in places where the temperature becomes significantly less than at the source boundary.

4 Conclusions

According to the results of the study, the following conclusions can be drawn:

1. An analytical equation is obtained that allows us to determine the temperature distribution over the surface of a plate with two heat sources. This equation will allow us to determine with sufficient accuracy the optimal location of the heat sources, to ensure the maximum average temperature of the plate, and therefore the greatest efficiency in heat transfer to the heated air.
2. The uneven heating of the plate, observed in the temperature plots for these thermal panels, indicates the possibility of changing the shape of the plate, in order to reduce the material consumption of thermal panels, by excluding low-heat areas from them.

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