Longitudinal vibrations of rods

Lyalyusa Baragunova, Maryana Shogenova, and Lyana Kanukoyeva*

Kabardino-Balkarian State University named after Kh.M. Berbekov, 360004, Nalchik, Russia

Abstract. Elastic structures of buildings and machines, the dynamic behavior mathematical model of which is the problem of longitudinal vibrations of rods, are widespread in modern technology. In this regard, the study of issues related to the longitudinal vibrations of the rods is also an urgent problem currently. In terms of solving such problems, we considered the problem formulation of longitudinal (free and forced) vibration of rods, obtained the spectra of natural frequencies ω_n and own forms $\varphi_n(x)$ vibration, $u(x, t)$ – the function of cross-sections displacement in the longitudinal direction of the rod has been found.

1 Introduction

Let us consider all these points in more detail. Let us draw a design scheme (Fig. 1) and consider natural motions. It should be noted that they are described in detail in the following literature sources [1-5].

1.1 Natural motions

Fig. 1. Homogeneous rod and selected element dx

Let us consider a homogeneous rod (Fig. 1) with a total mass $m = \rho \cdot A$ (ρ - material density, A – cross-sectional area) vibrating in the longitudinal direction [6, 7]. In this case, we admit the use of the flat sections and the d'Alembert principle hypothesis. Longitudinal

* Corresponding author: shogenova_mar@mail.ru

sections' displacements are characterized by the function $u(x, t)$. The relative deformation is $\varepsilon = \frac{\partial u}{\partial x}$. Then, according to Hooke's law, it is possible to be written as:

$$N = \sigma A = EA\varepsilon = EA \frac{\partial u}{\partial x} \quad (1)$$

Let us consider the equilibrium of the selected element dx . The inertia force of (d'Alembert force) is equal to

$$dI = m \frac{\partial^2 u}{\partial t^2} dx$$

All forces applied to the selected element to the main axis of the rod are projected in the following way:

$$-N - m \frac{\partial^2 u}{\partial t^2} dx + N + \frac{\partial u}{\partial x} dx = 0$$

and taking into account (1), we have:

$$m \frac{\partial^2 u}{\partial t^2} - EA \frac{\partial^2 u}{\partial x^2} = 0.$$

After some transformations, we finally obtain the equation of transverse vibration of a homogeneous rod of constant cross section, which is a homogeneous differential equation in partial derivatives of hyperbolic type.

$$u_{tt} - a^2 u_{xx} = 0, \quad x \in (0, 1), \quad a^2 = E / \rho \quad (2)$$

Note that the equation (2) is identical to the classical equation of a single-span homogeneous string natural motion.

Further, the boundary conditions are added to the problem (2), which can be very diverse depending on the conditions for fixing the ends of the rod [8-10]. Let us consider a particular case of rigid attachment at both ends of the rod. In this case, the boundary conditions will have the form:

$$u(0, t) = 0, \quad u(1, t) = 0 \quad (3)$$

The solution to the problem (2), (3) is found using the method of variables separation as the product

$$u(x, t) = X(x) \cdot \omega t$$

Substitution by the method of variables separation in (2) gives:

$$X(x) = A \sin kx + B \cos kx, \quad k = \omega / a, \quad a = \sqrt{E / \rho}.$$

And substitution in boundary conditions (3) [$X(0) = 0, \quad X(1) = 0$.]

$$X(x) = k(A \cos kx + B \sin kx),$$

$B=0, \quad \cos k_1 = 0, \quad k_1 = \frac{\pi(2n-1)}{2l}, \quad n=1, 2, \dots, \quad \frac{\omega}{a}l = \frac{\pi(2n-1)}{2}.$

Next, we obtain the spectrum of natural frequencies of vibrations ω_n and own forms $\varphi_n(x)$

$$\omega_n = \frac{\pi a(2n-1)}{2l}, \quad n=1, 2, \dots \tag{4}$$

$$X(x) = A \sin kx, \quad \varphi(x) = \sin kx,$$

$$k_n = \frac{\pi(2n-1)}{2l}, \quad \varphi_n(x) = \sin \frac{\pi(2n-1)}{2l} x \tag{5}$$

For the input parameters $a=1, l=1$ the formula (4) gives the data for Table 1 of the first 4 natural frequencies

Table 1. Values of the first 4 natural motions at free vibrations

n	1	2	3	4
ω_n	1.571	4.712	7.854	10.9956

The corresponding eigenmodes were found using the MATLAB programming system. The graphs are shown in Fig. 2.

Fig. 2. Natural motions modes

2 Forced vibrations from dynamic disturbances

Under dynamic loading, the function of vibration amplitudes has the form

$$A(x) = \frac{F}{ES} \sin kx / k \cos kl.$$

For the stress amplitude, we have

$$A_{\sigma}(x)=\frac{F}{S}\frac{\cos kx}{\cos kl}.$$

This implies

$$A_{\sigma}(l)=\frac{F}{S},\qquad \omega=0\Rightarrow A_{\sigma}(x)=\frac{F}{S}.$$

Fig. 3. Amplitude graphs $H(x)$ from the dynamic disturbances

Table 2. Values of the first 8 natural motions for the forced vibrations from dynamic disturbances

n	1	2	3	4	5	6	7	8
ω_n	0.01	1	1.25	3.5	4.5	7.5	7.78	10.945

3 Forced vibrations from kinematic disturbances

For kinematic perturbations, the following formula is obtained

$$A(x)=b(\operatorname{tg} kl\sin kx+\cos kx).$$

It gives the amplitude of stresses and its properties

$$A_{\sigma}(x)=Ekb\left(\operatorname{tg} kl\cos kx-\sin kx\right),$$

$$A_{\sigma}(l)=0;\quad \omega=0\Rightarrow k=0\Rightarrow A_{\sigma}(x)=0.$$

In both cases, from $\cos kl=0\Rightarrow A_{\sigma}(x)=\infty$, since in this case the frequency of disturbances ω coincides with natural frequencies ω_n . This means that resonance leads to infinite values of the voltage amplitude.

Here are the graphs of the $H(x)$ amplitudes along the rod. The resonance frequency graphs are not shown, since they strive for ∞ .

Fig. 4. H (x) amplitude graphs from kinematic disturbances

Table 3. Values of the first 8 natural motions for the forced vibrations from kinematic disturbances

n	1	2	3	4	5	6	7	8
ω_n	0.01	1	1.25	3.5	4	5.5	6.8	9.2

4 Stresses in the cross-sections of the rod at natural motions (ωn).

In the process of vibrations, not only the deflections of the rod $u(x, t)$, but also normal stresses $\sigma(x, t)$ and longitudinal force $N(x, t)$ change.

Normal stresses according to Hooke's law will be

$$\sigma(x, t) = E\varepsilon(x, t) = Eu'(x, t),$$

The corresponding longitudinal forces

$$N(x, t) = S\sigma(x, t) = ESu'(x, t)$$

These formulas determine the stresses and longitudinal force during natural motions.

Based on the calculations results, it is possible to build the stress diagrams which will give an opportunity to carry out the calculations on strength.

Fig. 5. Spectra of natural frequencies ω_n and own forms $\varphi_n(x)$ vibration

Table 4. Values of natural motions spectra

n	1	2	3	4
ω_n	1.571	4.712	7.854	10.9956

5 Conclusion

The obtained spectra of natural frequencies ω_n and own forms $\varphi_n(x)$ vibration, the function of cross-sections displacement in the longitudinal direction of the rod $u(x, t)$ has been found.

References

1. N.S. Koshlyakov, E.B. Gliner., M.M. Smirnov, Partial differential equations of mathematical physics (Higher school, 1970).
2. V.V. Bolotin, Vibrations in technology. Reference book in 6 volumes. Volume 1. Oscillations of Linear Systems (Mechanical Engineering, Moscow, 1978).
3. M.M. Ilyin, K.S. Kolesnikov, Yu.S. Saratov, Oscillation theory (MSTU, Moscow, 2003).
4. Ya.G. Panovko, Strength, stability, vibrations. Handbook in three volumes. Volume 3 (Birger, Mechanical engineering, 1968).
5. Ya.G. Panovko, Introduction to the theory of mechanical vibrations (Nauka, Moscow, 1964).
6. A.N. Tikhonov, A.A. Samarsky, Equations of mathematical physics (Nauka, Moscow, 1977).
7. V.M. Starzhinsky, Applied methods of nonlinear oscillations, Chief editorial board of physical and mathematical literature (Moscow, 1977).
8. Kh.P. Kulterbaev, Foundations of the theory of vibrations. Fundamentals of theory, tasks for homework, examples of solutions (Kabardino-Balkarian State University, Nalchik, 2003).
9. Kh.P. Kulterbaev, I.M. Abdul Salam, I.M. Paizulaev, Bulletin of the Dagestan State Technical University. Technical science **45 (3)** (2018).

10. Z. Yong-Jun, H Shuan-Hai, S. Yi-Fan, Z. Xiao-Xing, Journal of Chang'an University (natural science edition) **2 (28)**, 49-52 (2008).