

# A survey of numerical schemes for transportation equation

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**Abstract.** The convection-diffusion equation is a fundamental equation that exists widely. The convection-diffusion equation consists of two processes: diffusion and convection. The convection-diffusion equation can also be called drift-diffusion equation. The convection – diffusion equation mainly characterizes natural phenomenon in which physical particles, energy are transferred in a system. The well-known linear transport equation is also one kind of convection-diffusion equation. The transport equation can describe the transport of a scalar field such as material feature, chemical reaction or temperature in an incompressible flow. In this paper, we discuss the famous numerical scheme, Lax- Friedrichs method, for the linear transport equation. The important ingredient of the design of the Lax- Friedrichs Method, namely the choice of the numerical fluxes will be discussed in detail. We give a detailed proof of the  $L^1$  stability of the Lax-Friedrichs scheme for the linear transport equation. We also address issues related to the implementation of this numerical scheme.

## 1 Introduction

It is of great significance to research methods of numerical implementation for hyperbolic equations. It has a practical application in several fields, such as fluid mechanics, environmental science, and electronic science [1-11]. Traditional numerical schemes on convection-diffusion equations are the finite difference method and finite volume method [1]. However, it is difficult to achieve high order precision using these traditional methods in dealing with the convection-diffusion equation due to numerical dissipation in the convection term [12-25]. In addition, in reality, we often encounter such a type of convective diffusion Process that convection dominates, and the initial conditions or boundary conditions change drastically. Spectral methods and finite element methods are closely related [1, 2, 26-29]; the main difference between these two numerical methods is that the basis functions used in spectral methods do not vanish on the whole domain, while the basis functions used in finite element methods have compact support[30-34]. For spectral method, there are many methods to find the spectral expansion coefficients  $\{v_k\}_{k=0}^N$ . The key idea of the spectral method is as follows. We first introduce the residual function for a series of the solution  $u_n(x, t)$ :

$$\mathcal{R}[u_n](x, t) \stackrel{\text{def}}{=} [\mathcal{L}u_n - f](x, t). \quad (1)$$

We evaluate the residual  $\mathcal{R}$  on the expansion(1), and the approximate solution quality is measured by the residual norm  $\|\mathcal{R}\|$ . Our goal is to reduce the residual in the domain  $\mathcal{U}$ . There are two representative techniques to find the expansion coefficients. One is the Galerkin method, and another one is the Collocation method.

### 1.1 Galerkin method

Given a set of basis functions  $\{\phi_k(x)\}_{k=0}^n$ , we have

$$u(x, t) \approx \sum_{k=0}^n v_k(t) \phi_k(x). \quad (2)$$

Then the residual function

$$\mathcal{R}[u_n](x, t) \approx \mathcal{L}(\sum_{k=0}^n v_k(t) \phi_k(x)) - f(x). \quad (3)$$

And we have to find a set of new basis functions  $\{\tilde{\phi}_i(x)\}_{i=0}^N$  be orthogonal to the residual  $\mathcal{R}[u_n]$ , i.e

$$\int_a^b \mathcal{R}[u_n] \tilde{\phi}_i(x) dx = 0 \quad \text{for } i = 1, \dots, N. \quad (4)$$

In this way we can compute

$$\int_a^b \tilde{\phi}_i \mathcal{L}(\sum_{k=0}^n v_k(t) \phi_k(x)) dx - \int_a^b \tilde{\phi}_i f(x) dx = 0 \quad \text{for } i = 1, \dots, N. \quad (5)$$

Thus, we get a set of N algebraic equations for  $\{v_k\}_{k=0}^N$ , when the function  $\mathcal{L}$  includes time derivatives, we have a set of ODEs of  $\{v_k\}_{k=0}^N$ , then the coefficients are found.

### 1.2 Collocation method

The main idea is to interpolate the solution at the collocation points approximate the derivatives of the solution by the derivatives of the interpolating polynomials. At a set of N+1 collocation points, the residual R is vanishes,

$$R(x_j) = 0, \quad j = 0, \dots, N. \quad (6)$$

A general formula for the interpolating polynomial  $I_N f$

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satisfying the above condition can be

$$(I_N f)(x) = \sum_{j=0}^N f(x_j) g_j(x), \quad (7)$$

where  $g_j(x)$  is a Nth-degree polynomial satisfying

$$g_j(x_k) = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}. \quad (8)$$

The polynomials  $\{g_j(x)\}_{j=0}^N$  are called cardinal functions, and  $I_N f(x)$  is a Nth-degree polynomial interpolating  $f$  satisfying

$$(I_N f)(x_k) = f(x_k), \quad k = 0, \dots, N.$$

And  $g_j(x)$  can be computed by the Lagrangian interpolation

$$g_j(x) = \frac{P(x)}{P'(x_j)(x-x_j)}, \quad (9)$$

where  $P(x) = \prod_{j=0}^N (x-x_j)$ . Then we can obtain the derivative of the interpolating polynomial  $I_N f$  by

$$(I_N f)'(x) = \sum_{j=0}^N f(x_j) g_j'(x), \quad (10)$$

In this way, we can easily compute the approximated values  $(I_N f)'(x_j)$ , and we can use matrix-vector to achieve the spectral collocation methods. Once we know the collocation points and the differentiation matrix, we can obtain a numerical approximations solution.

For linear problems, the two methods work equally well. However, for nonlinear problems, the collocation approach is a better choice, as it involves the products of numbers that are easy to handle.

In this paper, we mainly discuss the following hyperbolic equation,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{on } [0,1] \times (0, T). \quad (11)$$

We take partition of  $[0,1]$  into cells  $\{I_j\}_{j=1}^N$ .

Each finite volume  $I_j$  is defined in the following way,

$$I_j = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) = \left(x_i - \frac{x_i - x_{i-1}}{2}, x_i + \frac{x_{i+1} - x_i}{2}\right). \quad (12)$$

By integration on the equation  $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ , in each cell  $I_j$ , we have

$$\frac{d}{dt} \int_{I_j} u(x, t) dx = u\left(x_{i-\frac{1}{2}}, t\right) - u\left(x_{i+\frac{1}{2}}, t\right). \quad (13)$$

Through integration on time variable from  $t^n$  to  $t^{n+1}$ , we have

$$\int_{I_j} u(x, t^{n+1}) dx - \int_{I_j} u(x, t^n) dx = \int_{t^n}^{t^{n+1}} u\left(x_{i-\frac{1}{2}}, t\right) dt - \int_{t^n}^{t^{n+1}} u\left(x_{i+\frac{1}{2}}, t\right) dt. \quad (14)$$

Then, we have

$$\frac{1}{\Delta x} \int_{I_j} u(x, t^{n+1}) dx = \frac{1}{\Delta x} \int_{I_j} u(x, t^n) dx - \frac{1}{\Delta x} \left[ \int_{t^n}^{t^{n+1}} u\left(x_{i+\frac{1}{2}}, t\right) dt - \int_{t^n}^{t^{n+1}} u\left(x_{i-\frac{1}{2}}, t\right) dt \right], \quad (15)$$

we define  $u_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n) dx = \frac{1}{\Delta x} \int_{I_j} u(x, t^n) dx$ .

We get our numerical scheme in the form of

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left( \phi_{i+\frac{1}{2}}^n - \phi_{i-\frac{1}{2}}^n \right), \quad (16)$$

where  $\phi_{i\pm\frac{1}{2}}^n \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} u\left(x_{i\pm\frac{1}{2}}, t\right) dt$ .

### 1.3 Lax-Friedrichs Method

We take the following difference schemes into consideration

$$u_t \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}, \quad u_x \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}. \quad (17)$$

For the transport equation,  $u_t(x, t) + u_x(x, t) = 0$ , by substituting the difference formula we write above, we obtain that

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0. \quad (18)$$

Thus, we have  $u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x} (u_{i+1}^n - u_{i-1}^n)$ .

We introduce numerical flux function to rewrite the above scheme in conservative form,

$$\begin{aligned} f_{i-\frac{1}{2}}^n &= \frac{1}{2} [u_{i-1}^n + u_i^n], \\ f_{i+\frac{1}{2}}^n &= \frac{1}{2} [u_i^n + u_{i+1}^n]. \end{aligned} \quad (19)$$

Thus, we can see that this scheme is a conservative scheme. However, it is unstable unconditionally.

We replace  $u_i^n$  by  $\frac{1}{2}(u_{i+1}^n + u_{i-1}^n)$ . Then we get

$$u_i^{n+1} = \frac{1}{2} (u_{j-1}^n + u_{j+1}^n) - \frac{\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n). \quad (20)$$

This numerical scheme is so-called Lax-Friedrichs scheme. This scheme is stable.

If we introduce numerical flux as follows

$$\begin{aligned} \phi(u_j^n, u_{j+1}^n) &= \frac{u_j^n + \lambda u_{j+1}^n}{2} - \frac{\Delta x}{2\Delta t} (u_{j+1}^n - u_j^n) = \frac{u_j^n + u_{j+1}^n}{2} - \\ &\frac{\Delta x}{2\Delta t} (u_{j+1}^n - u_j^n), \end{aligned} \quad (21)$$

we can know that this scheme is conservative.

Theorem 1 For linear transportation equation, the stable condition of the Lax-Friedrichs scheme is  $|\frac{\Delta t}{2\Delta x}| \leq 1$ .

Proof. We suppose  $u_j^n = \widehat{u}^n e^{ikx_j}$  and substitute it into the Lax - Friedrichs scheme. We get

$$\widehat{u}^{n+1} e^{ikx_j} = \frac{1}{2} (\widehat{u}^n e^{ikx_{j-1}} + \widehat{u}^n e^{ikx_{j+1}}) - \frac{\Delta t}{2\Delta x} (\widehat{u}^n e^{ikx_{j+1}} - \widehat{u}^n e^{ikx_{j-1}}). \quad (22)$$

For simplicity, we use the symbol  $\mu$  to be  $\frac{\Delta t}{2\Delta x}$ .

Then, through computations, we get

$$\begin{aligned} \widehat{u}^{n+1} &= \widehat{u}^n \frac{1}{2} \left[ e^{-ik\Delta x} + e^{ik\Delta x} - \mu \left( e^{ik\Delta x} - e^{-ik\Delta x} \right) \right] \\ &= \widehat{u}^n \left[ \cos(k\Delta x) - i\mu \sin(k\Delta x) \right]. \end{aligned} \quad (23)$$

We define an amplification factor  $g(k, \Delta x, \Delta t)$  as

$$g(k, \Delta x, \Delta t) = \frac{\widehat{u}^{n+1}}{\widehat{u}^n} = \cos(k\Delta x) - i\mu \sin(k\Delta x). \quad (24)$$

Notice that stability condition for the scheme is  $|g(k, \Delta x, \Delta t)| \leq 1$ . Thus,

$$\begin{aligned} |g(k, \Delta x, \Delta t)| &= \left[ \cos^2(k\Delta x) + \mu^2 \sin^2(k\Delta x) \right]^{\frac{1}{2}} \\ &= 1 - \sin^2(k\Delta x) + \mu^2 \sin^2(k\Delta x) \Big]^{\frac{1}{2}} = \left[ 1 + (\mu^2 - 1) \sin^2(k\Delta x) \right]^{\frac{1}{2}} \leq 1. \end{aligned} \quad (25)$$

Thus,  $\mu^2 - 1 \leq 0 \Rightarrow \mu^2 \leq 1 \Rightarrow -1 \leq \mu \leq 1$ .

## 2 Local Truncation Error of the Lax – Friedrichs Scheme

In this section, we make calculation of the local truncation error  $\tau_i^n$  for the Lax-Friedrichs scheme. First, we substitute the exact solution in the Lax – Friedrichs scheme:

$$\tau_j^n = \frac{u(x_j, t^{n+1}) - \frac{1}{2} (u(x_{j-1}, t^n) + u(x_{j+1}, t^n))}{\Delta t} + \frac{u(x_{j+1}, t^n) - u(x_{j-1}, t^n)}{2\Delta x}. \quad (26)$$

Taking Taylor expansion with regarding to variables of time and space in  $u(x_j, t^{n+1})$ ,  $u(x_{j-1}, t^n)$ ,  $u(x_{j+1}, t^n)$ , we get the following expansion:

$$\begin{aligned} \tau_j^n &= \frac{u(x_j, t^{n+1}) - u(x_j, t^n) - \frac{\Delta x^2}{2} u_{xx}(x_j, t^n) + \mathcal{O}(\Delta x^4)}{\Delta t} \left( u_x(x_j, t^n) + \frac{\Delta x^2}{6} u_{xxx}(x_j, t^n) + \mathcal{O}(\Delta x^4) \right) \\ &= u_t(x_j, t^n) + \frac{\Delta t}{2} u_{tt}(x_j, t^n) + \mathcal{O}(\Delta t^2) - \frac{\Delta x^2}{2\Delta t} u_{xx}(x_j, t^n) + u_x(x_j, t^n) + \mathcal{O}(\Delta x^2). \end{aligned} \quad (27)$$

Further, we assume that  $u(x_j, t^n)$  is an exact solution. Thus,  $u_t(x_j, t^n) + u_x(x_j, t^n) = 0$ . We then can get that  $u_{tt}(x_j, t^n) = w_{xx}(x_j, t^n)$ . Combining these two equalities, we can obtain an expression of the truncation error:

$$\begin{aligned} \tau_j^n &= \frac{\Delta t}{2} u_{tt}(x_j, t^n) - \frac{\Delta x^2}{2\Delta t} u_{xx}(x_j, t^n) + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2) \\ &= \frac{\Delta x^2}{2\Delta t} (\mu^2 - 1) u_{xx}(x_j, t^n) + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2), \quad \forall j \in \mathbf{Z}, \forall n \geq 0 \end{aligned} \quad (28)$$

Obviously, the Lax - Friedrichs scheme is first order in space and time [33].

**Theorem** The Lax -friedrichs scheme is  $L^1$  stable for the equation.

**Proof.** Without loss of generality, we assume boundary conditions are periodic. Assume that  $u_j^n, v_j^n$  are two approximation solutions.

$$u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n),$$

$$v_j^{n+1} = \frac{1}{2} (v_{j+1}^n + v_{j-1}^n) - \frac{\Delta t}{2\Delta x} (v_{j+1}^n - v_{j-1}^n).$$

$$\begin{aligned} u_j^{n+1} - v_j^{n+1} &= \frac{1}{2} (u_{j+1}^n - v_{j+1}^n) - \frac{\Delta t}{2\Delta x} (u_{j+1}^n - v_{j+1}^n) + \frac{1}{2} (u_{j-1}^n - v_{j-1}^n) + \frac{\Delta t}{2\Delta x} (u_{j-1}^n - v_{j-1}^n), \\ u_j^{n+1} - v_j^{n+1} &= \frac{1}{2} (1 - \lambda_{j+1}) (u_{j+1}^n - v_{j+1}^n) + \frac{1}{2} (1 + \lambda_{j-1}) (u_{j-1}^n - v_{j-1}^n), \end{aligned} \quad (29)$$

where  $\lambda_j = \frac{\Delta t}{2\Delta x}$ .

$$\begin{aligned} |u^{n+1} - v^{n+1}|_{L^1} &= \sum_{j=1}^J |u_j^{n+1} - v_j^{n+1}| \\ &= \frac{1}{2} \sum_{j=1}^J \left| (1 - \lambda_{j+1}) (u_{j+1}^n - v_{j+1}^n) + (1 + \lambda_{j-1}) (u_{j-1}^n - v_{j-1}^n) \right| \\ &\leq \frac{1}{2} \sum_{j=1}^J \left( |1 - \lambda_{j+1}| |u_{j+1}^n - v_{j+1}^n| + |1 + \lambda_{j-1}| |u_{j-1}^n - v_{j-1}^n| \right) \\ &= \frac{1}{2} \sum_{j=1}^J \left( 1 - \lambda_{j+1} \right) |u_{j+1}^n - v_{j+1}^n| + \left( 1 + \lambda_{j-1} \right) |u_{j-1}^n - v_{j-1}^n| \left( \text{due to the fact that } |\lambda_j| < 1 \right) \\ &= \frac{1}{2} \sum_{j=1}^J \left( 1 - \lambda_j \right) |u_j^n - v_j^n| + \left( 1 + \lambda_j \right) |u_j^n - v_j^n| \end{aligned}$$

(because of periodicity)

$$= \frac{1}{2} \sum_{j=1}^J |u_j^n - v_j^n| = |u_j^n - v_j^n|_{L^1}. \quad (30)$$

We define the total variation of the solution of the hyperbolic equation as

$$TV(u(\cdot, t)) = \int \left| \frac{\partial u}{\partial x} \right| dx. \quad (31)$$

In the discrete case, we define it as

$$TV(u^n) = TV(u(\cdot, t^n)) = \sum_j |u_{j+1}^n - u_j^n|, \quad (32)$$

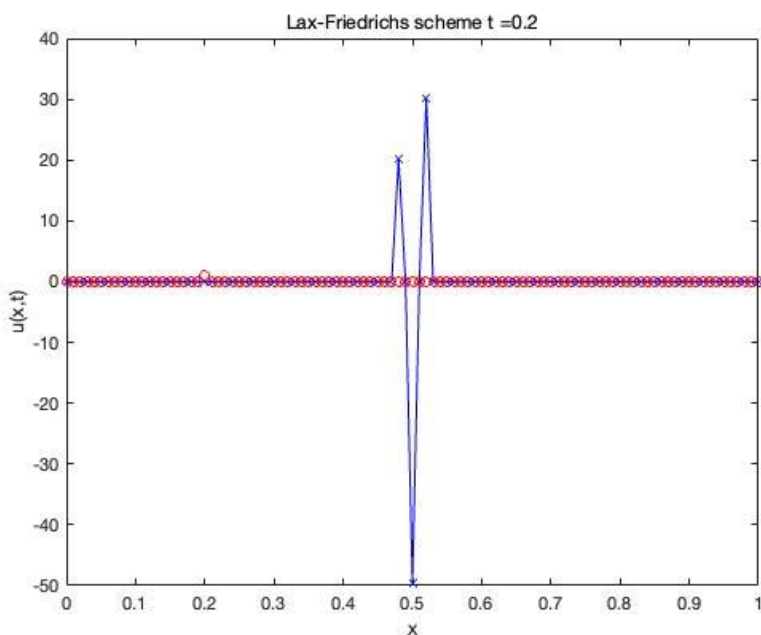
where  $u_j^n = u(x_j, t^n)$ .

We say a numerical scheme is a total variation diminishing (TVD) if it satisfies

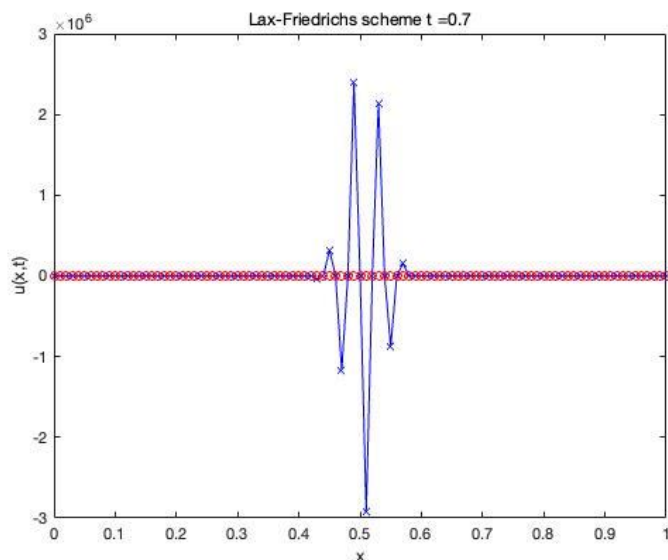
$$TV(u^{n+1}) \leq TV(u^n). \quad (33)$$

The Lax -Frechdrichs scheme is TVD for the transportation equation.

### 3 Numerical experiments



**Figure 1.** Lax-Friedrichs scheme t = 0.2



**Figure 2.** Lax-Friedrichs scheme t = 0.7

### 4 Conclusion

In this paper, we review the Lax-Friedrichs scheme for the linear transportation equation in detail. The Local Truncation Error of the Lax – Friedrichs scheme is obtained. The  $L^1$  stability of the Lax -friedrichs scheme for the equation is proved. All illuminated properties of the Lax-Friedrichs were validated numerically. The Lax-Friedrichs scheme is first-order accurate in both space and time. Plus, the Lax-Friedrichs is conservative. Hence, it is dissipative. For the future research, it remains to explore the effects of the Lax Friedrichs operator on non-uniform

grids. It also remains to investigate whether other schemes share the similar numerical properties of Lax-Friedrichs scheme.

### Appendix

```

%%%%%%%%%%
%%%%%%%%%%
%%%%%%%%%%
& matlab code of Lax – Frechdrichs scheme for
the transport equation
%
```

```

% u + u =0
% t x
%
% Domain: [0,1]
% Initial condition: Pulse
%
% u0(x)=up if x=xp=1/2; else u0(x)=0
%
% Transmissive boundary conditions
%
% Exact solution:
%u(x,t)=u0(x - t)
%
%%%%%%%%%%
%%%%%%%%%%
%%%%%%%%%%
clear all
clc
disp('Lax-Friedrichs scheme')
disp('-----')
disp('w + lambda w =0')
disp('t x')
disp('Domain: [a,b]')
disp('Transmissive boundary conditions')
disp('Initial condition: Pulse')
disp('w0(x)=wp if x=xp=(b+a)/2; else w0(x)=0')
disp('Exact solution')
disp('w(x,t)= w0(x-lambda*t)')
disp('-----')
a=0;
disp(['Lower end of the interval a = ', num2str(a)])
b=1;
disp(['Upper end of the interval b = ', num2str(b)])
lambda=1;
disp(['Velocity of propagation = ', num2str(lambda)])
m=100;
disp(['Number of nodes = ', num2str(m)])
deltax=(b-a)/(m);
disp(['deltax = ', num2str(deltax)])
x=[a:deltax:b];
deltat=0.1;
disp(['Time step = ', num2str(deltat)])
tmax=1;
disp(['Time end = ', num2str(tmax)])
mt=tmax/deltat;
% Courant number
cfl=lambda*deltat/deltax;
disp(['Courant number = ', num2str(cfl)])
up=1;
disp(['Value of the pulse wp=w0(xp) = ', num2str(up)])
disp('-----')
%
% Plot the initial condition at [a,b]
%
u0=zeros(1,m+1);
u0(m/2+1)=up;
figure(1)
plot(x,u0,'or')

```

```

xlabel('x'); ylabel('w(x,0)');
title('Initial condition');
%
% Initialization
%
ua=u0;
ue=u0;
%
%
for n=1:mt
%
uam(1:m)=ua(2:m+1);
uan(2:m+1)=ua(1:m);
%
un(2:m)=0.5*(uam(2:m)+uan(2:m))-...
cfl*0.5*(uam(2:m)-uan(2:m));
%
% Exact solution
%
% Exact position of the pulse
% at time t=n*deltat
%
xp=lambda*n*deltat;
%
for i=1:m+1
%
if(abs(x(i)-xp)<=deltax/2)
ue(i)=up;
else
ue(i)=0;
end
end
%
% Transmissive boundary conditions
%
un(1)=un(2);
un(m+1)=un(m);
%
figure(2)
plot(x,un,'xb',x,un,'b',x,ue,'or')
xlabel('x'); ylabel('u(x,t)');
title(['Lax-Friedrichs scheme t = ', num2str(n*deltat)]);
pause(1)
% Update
ua=un;
end

```

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