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Groundwater flow equation, overview, derivation, and solution

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Abstract. Darcy's law is the basic law of flow, and it produces a partial differential equation is similar to the heat transfer equation when coupled with an equation of continuity that explains the conservation of fluid mass during flow through a porous media. This article, titled the groundwater flow equation, covers the derivation of the groundwater flow equations in both the steady and transient states. We look at some of the most common approaches and methods for developing analytical or numerical solutions. The flaws and limits of these solutions in reproducing the behavior of water flow on the aquifer are also discussed in the article.

1 Introduction

Excessive exploitation of fresh water has raised demand for groundwater, prompting many academics to concentrate their efforts on understanding the phenomena of groundwater flow, and as a result, new research has focused on groundwater flow modeling [1-8]. One of the basic answers for deterministic mathematical models of groundwater flow is the Theis equation. To utilize the groundwater equation while studying a leaking aquifer, Hantush included water leakage as a volumetric sink/source factor. [9]. Theis groundwater flow equation [9] was developed under certain circumstances that do not match real-world data. Theis groundwater equation implies that the media through which the flow occurs is homogenous and that the medium does not vary in distance or time. The main mathematical tool used to derive the Theis groundwater flow equation, on the other hand, is the well-known Darcy's law, which was also obtained experimentally using sand; this simply implies that there is no form of heterogeneity, or layers, which is in contradiction with what is observed in the real field observation, and thus a new flow equation is required. The characteristics of the newly suggested fractional-order derivative allow it to simulate the flow of water in various levels or scales inside a geological structure known as an aquifer.
As a result, we reformulate the well-known Darcy law to create a novel groundwater flow equation inside a restricted aquifer to accurately reproduce the movement of water through porous media in various levels of the aquifer [9]. The difference between the rate input and outflow from an annular cylinder is the change in the volume of water inside the annular region, which is one of the most significant facts in the model of flow within a restricted aquifer.

Groundwater flow, as represented by the variational order derivative, is a very complicated phenomenon that does not lend itself well to the study of analytical models.

Through the discretization of space and time, numerical techniques provide approximate solutions to the governing equation. The system's changeable internal characteristics, borders, and stresses are approximated within the discredited problem domain. Deterministic, distributed-parameter numerical models may relax the strict idealized conditions of analytical or lumped-parameter models, making simulations more realistic and flexible. There has been a lot of research on finite difference methods for constant-order time or space fractional diffusion equations [10, 11]. Chen et al. developed an implicit difference approximation method for constant-order temporal fractional diffusion equations [12]. Weighted average finite difference techniques were presented by Yuste et al. [13]. For fractional diffusion equations, Podlubny et al. developed the matrix method [14], while Hanert provided a flexible numerical system for the discretization of the space-time fractional diffusion problem [15]. Zhuang et al. recently looked at numerical schemes for the VO space fractional advection-dispersion equation [16], while Lin et al. looked at the explicit scheme for the VO nonlinear space fractional diffusion equation [17].

2 Derivation of Groundwater Equation

We begin by deducting partial differential equations that occur in describing flows in porous media phenomena. The general groundwater flow equation is deducted from Darcy's law and the continuity equation. The law of conservation of mass for transient flow shows that the net rate of change of density is exactly opposite to the net rate of change of volume itself (V), in other words, the net rate of penetration of a fluid in a control volume is exactly equal to the net rate of change of storage of the mass of fluid in the same control volume.

$$Inflow\ NetRate = Inflow - Outflow = Storage\ Change\ Rate \quad (1)$$

This is equivalent:

$$Inflow\ NetRate = -div(\rho\vec{v}) = -(\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}) \quad (2)$$

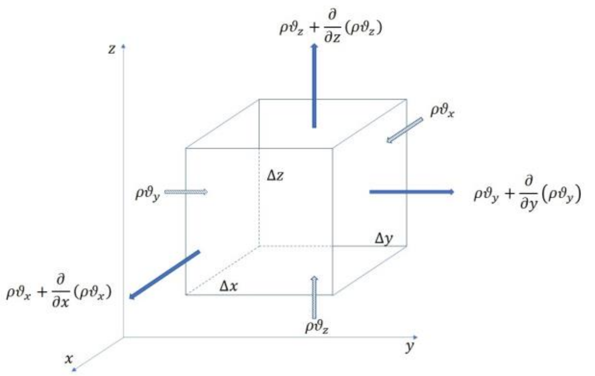

Figure 1: Groundwater control volume

It's worth noting that the change in storage inside the control volume is zero in steady-state flow. In a transitory flow, the storage change should be greater than zero. As a result, equation 2 will become:

$$-div(\rho\vec{v}) = -\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial(\rho\eta)}{\partial t} \quad (3)$$

Where η is the porosity of porous medium. The time rate of fluid mass change per unit volume of the control volume is referred to as $\frac{\partial(\rho\eta)}{\partial t}$ (the term has dimensions M/L^3T)). It is a saturated porous media, according to us. The $\frac{\partial(\rho\eta)}{\partial t}$ the word will become when utilizing the chain rule.

$$\frac{\partial(\rho\eta)}{\partial h}\frac{\partial h}{\partial t} = \frac{\partial(\rho\eta)}{\partial t} \quad (4)$$

The change rate in fluid storage in the control volume is linked to the change rate in the hydraulic head when there is a transient saturated flow. As a result, the $\frac{\partial(\rho\eta)}{\partial t}$ term becomes:

$$\eta\frac{\partial\rho}{\partial h} + \rho\frac{\partial\eta}{\partial h} = \frac{\partial(\rho\eta)}{\partial t} \quad (5)$$

The $\rho\frac{\partial\eta}{\partial h}$ term in equation 5 is the water mass generated by the porous media's compression or expansion, whereas the $\eta\frac{\partial\rho}{\partial h}$ term is the water mass produced by the fluid's compression or expansion. If the porosity rises $\frac{\partial\eta}{\partial h} > 0$ or the fluid density increases $(\frac{\partial\rho}{\partial h} > 0)$., water may enter the control volume in the saturated scenario.
we pose now α as the porous media compressibility and β as the fluid compressibility, and σ_e as a change in effective stress (Compression or expansion of the porous media).

For the saturated case:

$$d\sigma_e = -\rho g d\Phi \quad (6)$$

Were Φ is pressure head. since $d\Phi = (h - z) = dh - dz$ then:

$$d\sigma_e = -\rho g dh \quad (7)$$

Now we can define the compressibility of porous media α

$$\alpha = -\frac{dV_f}{V}\frac{1}{d\sigma_e} = \frac{d\eta}{d\sigma_e} \quad (8)$$

The fluid volume is V, and the control volume is it. Let's write equations 7 and 8 together.:

$$\frac{d\eta}{dh} = \alpha\rho g \quad (9)$$

We can define the fluid compressibility β as:

$$\beta = \frac{dV_f}{V_f}\frac{1}{dp} \quad (10)$$

We note p the pressure of the fluid. the expression of the change in pressure is given by:

$$dp = \rho g d\Phi = \rho g dh \quad (11)$$

And with $dV_f/V_f = d\rho/\rho$, we can write equation 10 as:

$$\beta = \frac{d\rho}{\rho}\frac{1}{\rho g dh} \quad (12)$$

or

$$\frac{d\rho}{dh} = \rho^2 g\beta \quad (13)$$

After substituting equations 9 and equation 13 into equation 4, we've got:

$$\frac{\partial}{\partial t}(\rho n) = (\rho\frac{\partial n}{\partial h} + n\frac{\partial\rho}{\partial h})\frac{\partial h}{\partial t} = (\rho^2 g\alpha + n\rho^2 g\beta)\frac{\partial h}{\partial t} \quad (14)$$

Now we can define the specific storage S_s as:

$$S_s = \rho g(\alpha + n\beta) \quad (15)$$

The specific storage S_s dimensions are L^{-1}; this word refers to the amount of water that an aquifer unit volume releases from storage for a unit decrease in hydraulic head.
We get the following by putting equation 15 into equation 14:

$$\frac{\partial}{\partial t}(\rho n) = \rho S_s\frac{\partial h}{\partial t} \quad (16)$$

After substituting equation 16 into equation 3, we've got:

$$-\frac{\partial}{\partial x}(\rho v_x)-\frac{\partial}{\partial y}(\rho v_y)-\frac{\partial}{\partial z}(\rho v_z)=\rho S_s\frac{\partial h}{\partial t} \quad (17)$$

We suppose that the density ρ it's a constant, equation 17 becomes:

$$\rho(-\frac{\partial}{\partial x}v_x-\frac{\partial}{\partial y}v_y-\frac{\partial}{\partial z}v_z)=\rho S_s\frac{\partial h}{\partial t} \quad (18)$$

We simplify equation 18 by eliminating ρ from both sides of the equation, we have compensated in v according to Darcy law ($v_x=K_x\frac{\partial h}{\partial x}$, $v_y=K_y\frac{\partial h}{\partial y}$,$v_z=K_z\frac{\partial h}{\partial z}$).

$$\frac{\partial}{\partial x}(K_x\frac{\partial h}{\partial x})+\frac{\partial}{\partial y}(K_y\frac{\partial h}{\partial y})+\frac{\partial}{\partial z}(K_z\frac{\partial h}{\partial z})=S_s\frac{\partial h}{\partial t} \quad (19)$$

Equation 19 represents the transient saturated-flow equation, when K_x,K_y,K_z are homogeneous, they will be constants, and equation 19 can be written: :

$$K_x\frac{\partial^2 h}{\partial x^2}+K_y\frac{\partial^2 h}{\partial y^2}+K_z\frac{\partial^2 h}{\partial z^2}=S_s\frac{\partial h}{\partial t} \quad (20)$$

If the porous media is also isotropic$K_x=K_y=K_z=K$, equation 20 is written:

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=\frac{S_s}{K}\frac{\partial h}{\partial t} \quad (21)$$

In the case of a confined aquifer, equation 21 is written:

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t} \quad (22)$$

$Where\, b\ is\ constant\ thickness, S=S_s b\ and T=Kb.$

3 Groundwater flow equation solution

Most of the studies interested in accurate solutions to the groundwater equation have adopted the common methods for solving similar differential equations, three common methods, the first one that depends on the separation of variables or known as Fourier, the second method used is the Botzman method, and there is another common third method which is the beta beta-derivative and beta-Sumudu transform method.

Through the discretization of space and time, numerical techniques solution provide approximate solutions to the governing equation. There has been a lot of research on finite difference methods for space-time fractional diffusion equations. Zhuang et al. [16], finite difference schemes (FDM) are suitable for simple geometries. They are adapted to the problems of Dirichlet but they hardly take into account the boundary conditions of the Von Neumann type (conditions on the derivatives). Finite element diagrams (FEM) are particularly suitable for complex geometries and are very effective. They are conventionally used in professional multiphysics calculation software. For our needs, they are complex to implement and are expensive in terms of computation time and memory space.

4 Limitations of solution

Numerical modeling is first relied upon in order to simplify the phenomenon, it was represented in order to facilitate the process of testing various hypotheses related to the development of the groundwater situation. Groundwater systems can only be reproduced to a certain degree of precision. Actual groundwater flow systems are much more complicated than conceptual models can usually depict. Because the precision of predictive simulations is difficult to determine, it's best to factor in a degree of uncertainty when making forecasts. Darcy's law is a simple mathematical statement that succinctly describes many known characteristics that groundwater moving in aquifers displays using the groundwater flow equation.

The flow will occur from high pressure to low pressure if there is no pressure gradient. There is a tendency with sophisticated model-generated visuals to overlook the inevitable ambiguities in portraying the actual system with a simpler conceptual model. The majority of mathematical models simulate the conceptual system pretty well.

5 Conclusion

Darcy's law is a simple mathematical statement that concisely explains several known properties of groundwater flowing through aquifers using the groundwater flow equation. With sophisticated model-generated graphics, it's easy to ignore the ambiguities that come with presenting the real system with a simpler conceptual model. Groundwater flow systems in real life are much more complex than conceptual models can represent. If there is no pressure gradient, the flow will occur from high to low pressure.

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