Forchheimer–Bénard Instability of the Non-Newtonian Fluid

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Abstract. The present paper examines the effect of vertical throughflow on the onset of convective instability in a horizontal porous layer filled with a non-Newtonian power-law fluid (PL). The permeable boundary layers are exposed to two different uniform constant temperature conditions, The Oberbeck-Boussinesq hypothesis is considered with the Darcy-Forchheimer model. A fourth-order eigenvalue problem is stemmed from the performance of the linear stability analysis, and the critical values are obtained using the shooting method combined with the Runge-Kutta method. The non-Newtonian Darcy-Rayleigh number (R), the Péclet number (Pe), the Forchheimer number (G), and the power-law index (n) are the parameters whose value play a crucial role in the onset of instability. The finding shows more stabilizing effects arise in pseudoplastic fluid than dilatant one at Peclet number Pe << 1 where the inverse behaviour takes place at large Peclet number even with the existence of the drag number or without it.

Keywords: thermal instability, porous media, non-Newtonian fluids, Darcy–Forchheimer model, form drag, Linear stability analysis.

1 Introduction

The Horton-Rogers-Lapwood problem (HRL) or convective instability has significant implications in various natural and engineering processes, such as geothermal energy extraction, oil recovery, soil water movement, and underground carbon dioxide CO2 sequestration [1-3]. Understanding and controlling this type of instability has a crucial impact on optimizing and avoiding undesirable outcomes.

HRL problem highlights the phenomenon where the flowing of the fluid becomes unstable and undergoes convective motion, leading to the formation of patterns and fluctuations. This problem is in general driven by the coupling between fluid flow and heat transfer. i.e., when a fluid goes through a system and exchanges heat with the solid matrix due to conduction, it can experience a temperature difference across the medium, which in turn yields a density variation that manifests the buoyancy forces. The majority of the books and papers dedicated to the problem belong to Ingham & Pop [4-5], Nield & Bejan [6], Vafai [7], Lagziri et al [12,14], and Rees et al [15].

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Recently, an upsurge of attention has been drawn to the thermo-convective instability of fluids with rheological behavior and this refers to the complexity introduced by the non-Newtonian fluid in comparison with Newtonian one. In other words, stability criteria and the onset of instability may be different for non-Newtonian fluids compared to Newtonian fluids. Therefore, the formation of patterns and flow structures in non-Newtonian fluids can be more diverse and intricate due to the variable viscosity and shear-dependent behaviour [8-9].

On the other hand, Darcy’s law is usually adopted to model power-law fluid (PL) in porous media, and a number of studies have considered this subject. For instance, Barletta et al. have examined the impact of vertical pressure gradient in HRL problem with a working fluid-type PL model [10]. Celli et al. have dealt with the case where the Prats problem is present and saturated with PL fluid [11]. For high flow rates, the inertia effect takes place which renders the Darcy model inapplicable. Several works have tackled this case such as Barletta et al. [13] who have investigated the onset of convection of a PL-saturating fluid in Forchheimer-Benard configuration with taking into account the viscous dissipation behaviour.

The importance of understanding the mechanisms of convective instability in non-Darcy flow generates a large body of scientific and technical literature. In this context, the current paper investigates the effect of vertical pressure gradient on the Forchheimer-Benard convection using a working fluid type of PL model. The horizontal boundary layers are assumed to be permeable and isothermal. The governing equations for the problem description are going through a normal modes method in order to achieve the eigenvalues problem. This latter is solved numerically by coupling in some way the method of Runge-Kutta and Shooting schemes.

### 2 Governing equations

The condition of the convection cell is examined in a horizontal porous channel with a thickness \( D \) confined between two permeable parallel plates and saturated with a non-Newtonian PL model. The effective viscosity of this latter behaves in the following equation:

\[
\tau = \eta(\dot{\gamma})^n. \tag{1}
\]

Here, \( \tau \) is the shear stress, \( \eta \) is the consistency coefficient, \( \dot{\gamma} \) is the shear rate and \( n \) is the PL index whose value defines if the fluid is pseudoplastic, dilatant, or Newtonian. The present problem is governed by the continuity equations, the energy equation, and the Darcy-Forchheimer equation with the validation of the Oberbeck-Boussinesq approximation [8-9]:

\[
\nabla \cdot \mathbf{u}^* = 0, \tag{2a}
\]

\[
\frac{\eta}{K} + |\mathbf{u}^*|^{n-1}\mathbf{u}^* + \frac{C_f \rho}{\sqrt{K}}|\mathbf{u}^*|\mathbf{u}^* = -\nabla^* P^* + \rho g(T^* - T_0)\mathbf{e}_z, \tag{2b}
\]

\[
\sigma^* \frac{\partial T^*}{\partial t} + \mathbf{u}^* \cdot \nabla^* T^* = \alpha \nabla^* T^*. \tag{2c}
\]

The uniform throughflow is applied perpendicularly to the permeable boundaries with a velocity \( w_0 \). Where the bottom layer is heated with temperature \( T_1 \) and the top one is kept at \( T_2 \) such that \( T_1 > T_2 \) see Fig. (1). Therefore, the boundary conditions are formulated as follows,

\[
z^* = 0: w^* = w_0, T^* = T_0 + \Delta T, \tag{2d}
\]

\[
z^* = D: w^* = w_0, T^* = T_0. \tag{2e}
\]
In these equations, \( \mathbf{u}^* \) represents the velocity vector, \((x^*, y^*, z^*)\) are the Cartesian coordinates, \( C_F \) is the dimensional drag coefficient (Forchheimer coefficient), \( \nabla P \) is the pressure gradient, \( \rho \) is the constant density, \( t^* \) is the time, \( \alpha \) is the thermal diffusivity, \( K \) is the modified permeability, \( \beta \) is the thermal expansion coefficient, \( T \) is the temperature, \( T_0 \) is a reference temperature, \( K \) is the permeability of the porous structure, \( \mathbf{e}_z \) is the unit vector in the vertical direction, and \( g \) is the gravitational acceleration that acts in the opposite direction to the \( z \)-axis (see Fig. (1)).

### 2.1 Dimensionless Method

The scaling variables and operators used in Eqs. (2) are represented below.

\[
(u^*, v^*, w^*) = \frac{a}{D} (u, v, w), \quad T^* = \Delta T T + T_0, \quad t^* = \frac{D}{\alpha} t, \quad (x^*, y^*, z^*) = D (x, y, z). \tag{3}
\]

In this case, the dimensionless parameters are written as,

\[
P_e = \frac{W_0 D}{\alpha}, \quad G = \frac{\rho C_F K D^{n-2}}{\sqrt{\eta a^{n-2}}}, \quad R = \frac{\rho \beta g K D \Delta T D^n}{\eta a^n}. \tag{4}
\]

Where \( P_e \) presents the Péclet number, \( G \) the form drag number, while the non-Newtonian Darcy-Rayleigh number is noted by \( R \). Consequently, the non-dimensional expression of Eqs. (1) can be determined as follows:

\[
\begin{align*}
\nabla \cdot \mathbf{u} &= 0, \tag{5a} \\
\nabla \times (|\mathbf{u}|^{n-1} \mathbf{u} + G |\mathbf{u}| \mathbf{u}) &= R \nabla \times (T \mathbf{e}_y), \tag{5b} \\
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \nabla^2 T, \tag{5c} \\
z = 0, w = P_e, T = 1, \tag{5d} \\
z = 1, w = P_e, T = 0. \tag{5e}
\end{align*}
\]

### 2.2 Basic state

A uniform parallel throughflow with a velocity \( \mathbf{u}_B \) directed along the \( z \)-axis is written as,

\[
\mathbf{u}_B = (0, 0, P_e), \tag{6a}
\]
\[ T_B(z) = \frac{e^{Pe-e^{Pez}}}{e^{Pe-1}}. \]  

(6b)

Where the solution of the basic temperature is tackled through Eq. (5c). The subscript “B” is used to represent the “basic flow”.

### 3 Linear stability analysis

The initial solution is perturbed by invoking small disturbances into Eqs. (6) in the given form.

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}_B + \varepsilon \hat{\mathbf{U}}, \\
T &= T_B + \varepsilon \hat{T}.
\end{align*}
\]

(7a) (7b)

Assuming \( \varepsilon \ll 1 \) implies the ignorance of all terms \( (\varepsilon^2) \) in the resulting equations attained by substituting Eqs. (7) into Eqs. (5), namely

\[
\begin{align*}
\left( \frac{\partial^2 \hat{\mathbf{W}}}{\partial x^2} + \frac{\partial^2 \hat{\mathbf{W}}}{\partial y^2} \right) (n + G|P_e|^2-n) + \frac{\partial^2 \hat{\mathbf{W}}}{\partial z^2} (1 + G|P_e|^2-n) &= \frac{R}{|P_e|^{n-1}} \left( \frac{\partial^2 \hat{T}}{\partial x^2} + \frac{\partial^2 \hat{T}}{\partial y^2} \right), \\
\frac{\partial \hat{T}}{\partial t} + P_e \frac{\partial \hat{T}}{\partial z} + \hat{W} e^{Pe} = \nabla^2 \hat{T}, \\
z &= 0, \hat{W} = 0, \hat{T} = 0, \\
z &= 1, \hat{W} = 0, \hat{T} = 0.
\end{align*}
\]

(8a) (8b) (8c) (8d)

The perturbed temperature and velocity components take the form of normal modes.

\[
\{ \hat{\mathbf{W}}(x, y, z, t), \hat{T}(x, y, z, t) \} = \{ f(z), h(z) \} e^{i(a_0 x + a_1 y + \omega t)}. 
\]

(9)

By substituting formulations of Eq. (9) into Eqs. (8), the resulting eigenvalue problem for neutrally stable modes can be noted as follows.

\[
\begin{align*}
f''(1 + G|P_e|^2-n) - a^2 f(n + G|P_e|^2-n) &= -ah \frac{R}{|P_e|^{n-1}}, \\
h'' - P_e h' + a^2 h + i\omega h + e^{Pe(1-z)} f &= 0, \\
z &= 0, f = 0, h = 0, \\
z &= 1, f = 0, h = 0.
\end{align*}
\]

(10a) (10b) (10c) (10d)

Furthermore, the symbol \( a \) denotes the horizontal wave number, and \( \omega \) refers to the dimensionless complex frequency. Since convection is stationary the real part \( R(\omega) = 0 \).

### 4 Numerical Method

Numerical solutions serve to solve Eqs. (10) characterized by non-constant coefficients with dependence on \( z \). The Runge–Kutta method employed to deal with ordinary differential equations is based on the option of the adaptive step size. To manage the use of this method, we transform Eqs. (10) into an initial value problem written as,

\[
\begin{align*}
z &= 0, f = 0, f' = 1, h = 0, h' = y_1 + i y_2.
\end{align*}
\]

(11)
The unknown constants $\gamma_1$ and $\gamma_2$ have real values and can be determined together with a pair $(R, \omega)$ through the validation of the boundary conditions $z = 1$. This operation required a shooting method to be applied for every given value of $(a, P_e, G, n)$ in order to obtain the eigenvalues. The pair $R_c$ and $a_c$ characterizing the absolute minimum of the marginal curves and can be extracted directly by taking the derivative of Eqs. (10) with respect to the $a$. Mathematica software 10.3 is being used to compute both methods.

5 Result and Discussion

![Plots of neutral stability curves for fixed $G = 0$ and different values $P_e$.](image)

Fig. 2. Plots of neutral stability curves for fixed $G = 0$ and different values $P_e$. 
Fig. 3. Plots of neutral stability curves for fixed $G = 0.08$ and different $P_e$ values.

The Figure. (2)-(3) are relative to the neutral stability for fixed $G = 0$ and $G = 0.08$ with a given Péclet parameter $P_e = 0.4, 0.8, 1, 3$ for different PL index values $n = 0.4, 0.6, 1, 1.6, 2$. The absence of $G$ in Fig. (2) displays that the Péclet number acts as a stabilizing in the case of pseudoplastic (with $n < 1$) fluids, whereas it behaves as a destabilizing for dilatant ones (with $n > 1$). Otherwise, the presence of the form drag in Fig. (3) shows that the stability of non-Newtonian fluids is greatly amplified, especially when a strong flow rate is involved, on the contrary, $G$ does not have a large effect on the case of Newtonian fluid (with $n = 1$). The plots of $R_c$ and $a_c$ for fixed values of $G = 0$ and $G = 0.08$ versus $n$ and different Péclet $P_e = 0.4, 0.8, 1, 3$ are displayed in Fig. (4)-(5). For $G = 0$, $R_c$ is a nonmonotonic function of $Pe$ for pseudoplastic behaviour while it becomes monotonic for dilatant one. In the case of $G = 0.08$ the reverse scenario appears for a large value of $P_e$ when a pseudoplastic
case is taken into consideration. The critical values of $a_c$ drawn in the right-hand frames of Fig. (4)-(5) exhibit no dependence on $G$ for Newtonian and dilatant fluids, otherwise, the curves of $a_c$ for pseudoplastic case can move monotonically upward when the value of $G$ deviate from zero. Briefly,

**Fig. 4.** Plots of critical values curves ($R_c, a_c$) for fixed values of $G = 0$

**Fig. 5.** Plots of critical values curves ($R_c, a_c$) for fixed values of $G = 0.08$.

### 5 Conclusion

The study carried out in this paper aims to examine the influence of form drag number $G$, on the initiation of thermal instability in a porous system saturated with a PL fluid. A vertical uniform flow in a horizontal porous layer with impermeable isothermal boundaries is considered. The normal mode approach is invoked in order to perform the stability analysis. Mathematica 10.3 is the software used to compute the eigenvalue problem by employing the
Runge-Kutta solver and the shooting method. The numerical finding revealed the following outcomes:

- For pseudoplastic fluids, a small Péclet number has a stabilizing effect, while for dilatant ones, it has a destabilizing effect. The inverse behaviour can be noticed for a large Peclet number.
- The stability of the pseudoplastic fluid is greatly amplified in the presence of form drag, especially when a strong flow rate is involved.
- As form drag effects increase, the critical Rayleigh number experiences significant growth, while the wavenumber experiences only a small variation.

References