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The advanced defuzzification methods of the convex α −cut fuzzy sets

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Abstract. The main features of the fuzzy sets and their corresponding membership functions were presented in terms of the fuzzification process and further by the de-fuzzification operation. The convexity of the α −(alfa) cuts of the fuzzy sets is used in the decomposition of the fuzzy sets. The alfa cuts of the fuzzy sets were defined precisely in terms of the pair of functions and their lowest upper and greatest lower bounds. The convex combination of the intervals of the sub-regions of the fuzzy sets and their membership function were considered as the points of the de-fuzzified values of the fuzzy sets. The methods of the de-fuzzification to the crisp sets were presented by the formulas to find the defuzzification regions and de-fuzzified values. The compositional concepts of the inference as the expansion of the extension principle were introduced to formalize further the fuzzy reasoning by the set of fuzzy rules based on the approximate reasoning.

1 Fuzzy membership function and the defuzzification to the crisp set

Let U be a set called as a universe and $F \subset U$ with the corresponding elements such as $x \in F$, x is either affiliated with or not affiliated with F . We can define a characteristic function for $x \in F$ by the set of the ordered pairs $(x, 0), x \notin F$ or

$(x, 1), x \in F$, correspondingly to the standards of the classical set.

The characteristic function is the function from F to $\{0,1\}$:

$\phi_F(x) = \begin{cases} 1, x \in F \\ 0, x \notin F \end{cases}$, where $\{0,1\}$ is called the evaluation set. The value of 0 is associated with non-membership and the value 1 is associated with the membership.

In contrast to the conventional classical set theory, we can define a fuzzy set where the membership value ascribes a degree to a specified element affiliated with the set.

Definition. If the evaluation set is the closed interval of the real numbers such that $[0,1] \in \mathbb{R}$,then F is defined as the fuzzy set and the function $\phi_F(x)$ is the grade of membership of $x \in F$: $\phi_F: F \longrightarrow [0,1] \in \mathbb{R}$.

The fuzzy set F is the set of ordered pairs such as $F \coloneqq \{x, \phi_F(x)\}, x \in U$.

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The membership function of the fuzzy set $F \subseteq X$ is denoted as $\phi_F(x)$ and defined as it is in [13, 14,15] which can be interpreted by the following: nearer the value of the grade of membership to *1* then x is more affiliated to F :

$\phi_F(x): F \longrightarrow [0,1] \in \mathbb{R}$.

If the membership function $\phi_F(x): F \longrightarrow \Phi[0,1] \in \mathbb{R}$, where $\Phi[0,1] \in \mathbb{R}$ is a family of the closed intervals of the real numbers then the membership function represents the interval-valued fuzzy sets of F.

The main features of the membership function comprise of the core, support, and boundaries.

The normalized fuzzy set is the set where the membership function

$\phi_F(x) = 1$.

If $\phi_F(x) < 1$, then such set is a sub-normal fuzzy set.

Definition. For any elements of the universe the inequality $x \in X$, $x_1 < x < x_2$ upholds that $\phi_F(x) \geq \min [\phi_F(x_1), \phi_F(x_2)]$ and such fuzzy set is called the convex fuzzy set.

The membership function of the convex fuzzy set is strictly increasing and decreasing while the values of x are increasing [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

If the criterion of the convexity of the fuzzy sets is violated, then such fuzzy sets are non-convex.

The figure 1 presents the main features such as support, core, boundaries and the shape of the convex fuzzy set and non-convex fuzzy set.

Definition. The crossover point of the fuzzy set is the point where $\phi_F(x) = 0.5$.

The height of the fuzzy set is the point where the maximum value of the membership function reached:

$$hght_F(x) = max\phi_F(x)$$

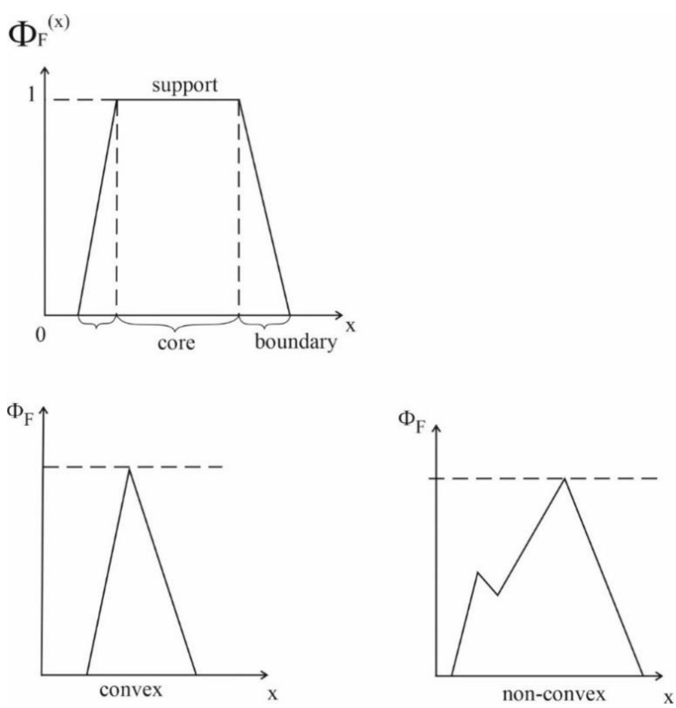

Fig. 1. the main features such as support, core, boundaries and the shape of the convex fuzzy set and non-convex fuzzy set.

The compelling idea of the defuzzification is engaging to the principle to find the point where the fuzzy set is converted to the crisp set.

The defuzzification to the crisp set occurs at the point or region of the domain of the fuzzy set where the fuzzy set is transformed into the crisp set.

One of the methods of the defuzzification is the α–cut defuzzification method which is known as the decomposition method.

2 α – cut decomposition method

Definition. The elements of the fuzzy set where the membership function approaches the value to the certain degree of α is called the α–cut of the fuzzy set [13, 14]:

$F^{\alpha}(x) = \{(x, \phi_F(x)), \phi_F(x) \geq \alpha, x \in X, \alpha \in [0,1]\}$.

The Figure 2 illustrates the α–cut of the fuzzy set.

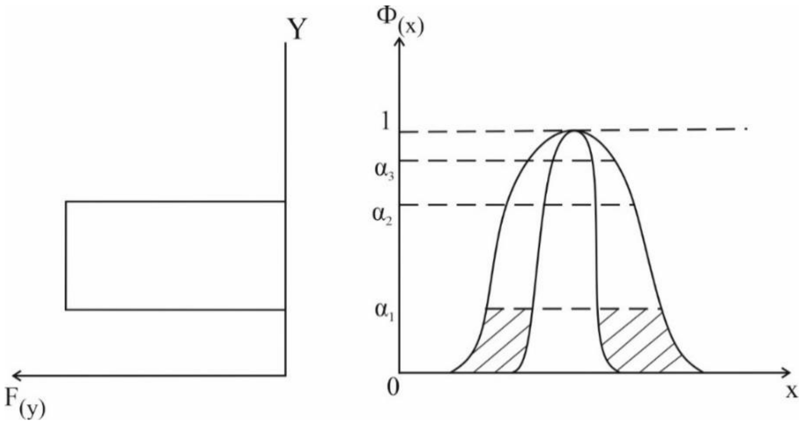

Fig. 2. The α–cut of the fuzzy set.

Definition. The fuzzy set is called the power fuzzy set if the fuzzy set represents the union of the α–cuts:

$$F^{\alpha} = \cup F_{\alpha} = F_{\alpha_1} \cup F_{\alpha_2} \cup \dots \cup F_{\alpha_n} \text{ and } F_{\alpha}(x) = \alpha F^{\alpha}(x).$$

The following theorem represents the defuzzification principle based on the α–cuts.

Theorem 1. If the union of the α–cuts is

$\cup_{\alpha \in [0,1]} F_{\alpha} = F_{\alpha}(x)$ then $F_{\alpha}(x) = \alpha F^{\alpha}(x)$ is the power fuzzy set.

Proof. Let us denote the fuzzy set $F_{\alpha_i}(x) = \{(x_i, \alpha_i)\} \div \frac{\alpha_i}{x_i},\ i = 1,2, \dots, n.$

Then, according to the definition of the α–cuts we can represent the fuzzy sets as stated above:

$$F_{\alpha_1}(x) = \frac{\alpha_1}{x_1} + \frac{\alpha_1}{x_2} + \dots + \frac{\alpha_1}{x_n},$$

$$F_{\alpha_2}(x) = \frac{0}{x_1} + \frac{\alpha_2}{x_2} + \dots + \frac{\alpha_2}{x_n},$$

$$F_{\alpha_n}(x) = \frac{0}{x_1} + \frac{0}{x_2} + \dots + \frac{0}{x_{n-1}} + \frac{\alpha_n}{x_n}.$$

Here 0 is the empty set and $\alpha_1, \alpha_2, \dots, \alpha_n$ are
α_i–cuts, $\alpha_i \in [0,1], i = 1,2, \dots, n$.
Hence, we can represent the cuts as the union:

$$F_{\alpha_1} \cup F_{\alpha_2} \cup \dots \cup F_{\alpha_n} = F_\alpha\left(F_{\alpha_1} \cup F_{\alpha_2} \dots \cup F_{\alpha_n}\right) = \alpha F^\alpha(x) \blacksquare$$

Based on the decomposition theorem we can state the following two corollaries.

Corollary 1. The cardinality of the power fuzzy set is $|F| = \sum_{x \in F} \phi_F(x)$ and the relative cardinality is $\|F\| = \frac{|F|}{|x|}$.

Corollary 2. (The inclusion principle): The fuzzy sets $F_1,\ F_2 \subset F(x), F_1 \subseteq F_2$ are inclusive if for $\forall x \in X, \phi_{F_1} \le \phi_{F_2}$.

In a case of the strict inequality the inclusion is strict, too.

The convexity of the fuzzy set can be extended to the α–cuts of the fuzzy set.

Definition: The fuzzy set is convex if for $\forall x_1, x_2$ and small $\lambda > 0$ the membership function is determined by the minimum operator:

$\phi_{F_\alpha}(\lambda x_1 + (1-\lambda)x_2) \ge \min\,(\phi_{F_\alpha}(x_1), \phi_{F_\alpha}(x_2))$.

The fuzzy set is convex if all its cuts are convex in terms of the convexity.

We can re-define the cuts of the fuzzy sets by the pair of the fuzzy set with reference to their lowest upper bound and greatest lower bound defined as it is [13,14, 15, 16, 17, 18, 19].

Definition. α–cut of the membership function $\phi_F(x) \ge \alpha, \alpha \in [0,1]$ is given by the pair of the functions $\left(I(\alpha), S(\alpha)\right)$ as it is:

$$I(\alpha) = \begin{cases} inf\,\phi_{F_\alpha}(x), & \alpha > 0 \\ \inf Supp\phi_{F_\alpha}(x), & \alpha = 0 \end{cases} \text{ and } S(\alpha) = \begin{cases} sup\phi_{F_\alpha}(x), & \alpha > 0 \\ supSupp\phi_{F_\alpha}(x), & \alpha = 0 \end{cases}$$

Next theorem states the convexity principle of the cuts of the fuzzy sets.

Theorem 2. The α–cut fuzzy set $F_\alpha(x) = \alpha F^\alpha(x) = \cup_{(0,1]} F_\alpha,\ F_\alpha(x) = \{\left(x, \phi_F(x)\right), \phi_F(x) \ge \alpha, \forall\alpha \in (0,1] \in \mathbb{R}\}$, where $F_\alpha(x) \ne \emptyset,\ \phi_F(x): F_\alpha \to [0,1] \in \mathbb{R}$, is the convex fuzzy set if the inequality upholds:

$$\alpha(\lambda x_1 + (1-\lambda)x_2) \ge \min\left(\alpha(x_1), \alpha(x_2)\right), x_1, x_2 \in F_\alpha \subseteq X,\ \alpha(x): F_\alpha(x) \longrightarrow [0,1] \in \mathbb{R}.$$

Proof. If we choose the element $\dot{x} \in F_\alpha$, then there is existing the neighborhood around this point such as $N(\dot{x})$.

Suppose we choose one more point such as $\ddot{x} \ne \dot{x}, \ddot{x} \in F_\alpha, \phi_{F_\alpha}(\ddot{x}) \le \phi_{F_\alpha}(\dot{x})$.

Using the definition of the convexity we can obtain the following inequality:

$$\alpha(\lambda\ddot{x} + (1-\lambda)\dot{x}) \ge \min\left(\alpha(\dot{x}), \alpha(\ddot{x})\right) = \alpha\left((\dot{x}), (\ddot{x})\right) = \alpha(\ddot{x}).$$

The point $\lambda\ddot{x} + (1-\lambda)\dot{x} \in F_\alpha,\ \ddot{x} \in F_\alpha \cap N(\dot{x})$. Hence, we have arrived at the statement which contradicts the convexity at the point $\dot{x}$.

3 The fuzzy $\alpha-$ cuts in terms of the union, intersection and complement of the fuzzy sets

The operator OR $F_1 \vee F_2$ for the cuts of the union are: $(F_1 \cup F_2)^{>\alpha} = F_1{}^{>\alpha} \cup F_2{}^{>\alpha} = \{x, \max(\phi_{F_1}, \phi_{F_2})\}, x \in X$

The operator AND $F_1 \wedge F_2$ for the cuts of the intersection are: $(F_1 \cap F_2)^{>\alpha} = F_1{}^{>\alpha} \cap F_2{}^{>\alpha} = \{x, \min(\phi_{F_1}, \phi_{F_2})\}, x \in X$

The operator NOT $F^c = \neg F$ for the cuts of the complement are: $(F^c)^{>\alpha} = F^{\leq 1-\alpha} = \{\phi_F(x) \leq 1 - \alpha\}, x \in X.$

The Cartesian product of the fuzzy sets $F_1 \times F_2$, $F_1 \subseteq X_1, F_2 \subseteq X_2$ is a fuzzy set based on the domain $X_1 \times X_2$ with the corresponding membership function as

$$\phi_{F_1 \times F_2}(x_1, x_2) = \min\left(\phi_{F_1}(x_1), \phi(x_2)\right)$$

The Cartesian co-product $F_1 + F_2$ is the fuzzy set with the corresponding membership function as

$$\phi_{F_1 + F_2}(x_1, x_2) = \max\left(\phi_{F_1}(x_1), \phi_{F_2}(x_2)\right. .$$

4 The aggregation of the fuzzy sets by the union and intersection

The function t norm aggregates two fuzzy membership functions as it is by the standard intersection rule:

$$t\colon [0,1] \times [0,1] \to [0,1]. \text{ Here } t(\phi_F(x), 1) = \phi_F(x)$$

$$\phi_{F_1 \cap F_2}(x) = t\left(\phi_{F_1}(x), \phi_{F_2}(x)\right) = \min\left(\phi_{F_1}(x), \phi_{F_2}(x)\right)$$

$t-$norm is the binary two valued function which meets the characteristics of the associative, the commutative, monotonically increasing, symmetric functions such as

$t(\phi_{F_1}, \phi_{F_2}) = t(\phi_{F_2}, \phi_{F_1})$ symmetry

$t\left(\phi_{F_1}(x), t(\phi_{F_2}(x), \phi_{F_3}(x))\right) = t\left(t\left(\phi_{F_1}(x), \phi_{F_2}(x)\right), \phi_{F_3}(x)\right)$ associativity

If $\phi_{F_1}(x) \leq \phi_{F_2}(x)$ and $\phi_{F_3}(x) \leq \phi_{F_4}(x)$, then

$t\left(\phi_{F_1}(x), \phi_{F_3}(x)\right) \leq t(\phi_{F_2}(x), \phi_{F_4}(x))$

The function $s-$co-norm is the binary two-valued function such as

$s\colon [0,1] \times [0,1] \to [0,1]$. Here $s(\phi_F(x), 0) = \phi_F(x)$

$$\phi_{F_1 \cup F_2}(x) = s\left(\phi_{F_1}(x), \phi_{F_2}(x)\right) = \max\left(\phi_{F_1}(x), \phi_{F_2}(x)\right)$$

Similarly, the characteristics of the associative, the commutative, monotonicity of the functions are applicable to the $s-$co-norms such as

$$s(\phi_{F_1}, \phi_{F_2}) = s(\phi_{F_2}, \phi_{F_1}) \text{ symmetry}$$

$$s\left(\phi_{F_1}(x), s\left(\phi_{F_2}(x), \phi_{F_3}(x)\right)\right) = s\left(s\left(\phi_{F_1}(x), \phi_{F_2}(x)\right), \phi_{F_3}(x)\right) \quad \text{associativity}$$

If

$$\phi_{F_1}(x) \le \phi_{F_2}(x) \text{ and } \phi_{F_3}(x) \le \phi_{F_4}(x) ,$$

then

$$s\left(\phi_{F_1}(x), \phi_{F_3}(x)\right) \le s(\phi_{F_2}(x), \phi_{F_4}(x))$$

There are existed analogous t −norm and s −co-norm operators such as minimum t_{min} , algebraic product t_{ap} , bounded product t_{bp} ,drastic product t_{dp} and, concurrently, maximum s_{max} ,algebraic sum s_{as} , bounded sum s_{bs} ,drastic sum s_{ds} as it is:

for $\phi_{F_1}(x) = \alpha, \phi_{F_2}(x) = \beta$ there are the following operators involved such as

$$t_{min}(\alpha, \beta) = min(\alpha, \beta) = \alpha \wedge \beta$$

$$t_{ap}(\alpha, \beta) = \alpha \times \beta$$

$$t_{bp}(\alpha, \beta) = 0 \vee (\alpha + \beta - 1)$$

$$t_{dp}(\alpha, \beta) = \begin{cases} \alpha \because \beta = 1 \\ \beta \because \alpha = 1 \\ 0 \because \alpha < 1, \beta < 1 \end{cases}$$

The $t -$ norm operators are related to each other by these inequalities:

$$t_{min} \ge t_{ap} \ge t_{bp} \ge t_{dp}$$

For the co-norms the following operators involved such as

$$s_{max}(\alpha, \beta) = max(\alpha, \beta) = \alpha \vee \beta$$

$$s_{as}(\alpha, \beta) = \alpha + \beta - \alpha \times \beta$$

$$s_{bs}(\alpha, \beta) = 1 \wedge (\alpha + \beta)$$

$$s_{ds}(\alpha, \beta) = \begin{cases} \alpha \because \beta = 0 \\ \beta \because \alpha = 0 \\ 1 \because \alpha > 0, \beta > 0 \end{cases}$$

There are associated with these operators the following inequalities:

$$s_{ds} \ge s_{bs} \ge s_{as} \ge s_{max}$$

There are the order relations as the inequalities with the consideration to their reciprocate values:

$$s_{ds} \ge s_{bs} \ge s_{as} \ge s_{max} \ge t_{min} \ge t_{ap} \ge t_{bp} \ge t_{dp}$$

The triangular norms and co-norms are uniquely connected to each other by the following principle such as

$$s_t\left(\phi_{F_1}(x),\phi_{F_2}(x)\right) = 1 - t(1-\phi_{F_1}(x), 1-\phi_{F_2}(x)),$$

$$t_s\left(\phi_{F_1}(x),\phi_{F_2}(x)\right) = 1 - s(1-\phi_{F_1}(x), 1-\phi_{F_2}(x))$$

Theorem 3. If F_1, F_2 are fuzzy sets of the universe X and their complements are

$$F'_1 = X \setminus F_1,\ F'_2 = X \setminus F_1 \text{, then } (F_1 \cup_t F_2)' = F'_1 \cap_t F'_2 .$$

Proof. Let $x \in (F_1 \cup_t F_2)'$. Then $x \notin F_1 \cup_t F_2$. Hence $x \notin F_1, x \notin F_2$.
From this we may conclude that $x \in F'_1, x \in F'_2$. Therefore,
$x \in (F'_1 \cap_t F'_2) \subseteq (F_1 \cup_t F_2)'$. Thereafter, $F'_1 \cap_t F'_2 \subseteq (F_1 \cup_t F_2)'$ ■

Morgan's Law for the sets is contented by this theorem to connect the union and intersection of the compliments of the fuzzy sets as it is:

$$(F_1 \cap_t F_2)' = F'_1 \cup_t F'_2,$$

$$(F_1 \cup_t F_2)' = F'_1 \cap_t F'_2$$

The triangular norm and co-norm connected by the union and intersection to generate the consequent membership function as it is:

$$\phi_{F_1 \cap_t F_2}(x) = s_t(\phi_{F_1}(x),\phi_{F_2}(x)) \text{ for } F_1 \cup_t F_2 \subseteq F_1 \cap_t F_2 .$$

If we keep, $\phi_{F_1}(x) = \alpha, \phi_{F_2}(x) = \beta$ then the fuzzy compliment's operator is the continuous function c such as :
$c: [0,1] \to [0,1]$ by satisfying to the boundary, monotonicity , involution conditions:

$$c(0) = 1,\ c(1) = 0yj$$

$$c(\alpha) \geq c(\beta), \qquad \alpha \leq \beta$$

$$c\big(c(\alpha)\big) = \alpha$$

As it is stated the norms and co-norms are having two-fold nature then the Morgan's theorem can be reinstated in terms of the compliments to connect the norm and co-norm operators with their compliments.

Theorem 4. The $t-$ norm is uniquely assigned to $s-$ co-norm by the compliment operator c:

$$(\mathscr{F} \cap_t \mathcal{G})^c = \mathscr{F}^c \cup_t \mathcal{G}^c$$

$$(\mathscr{F} \cup_t \mathcal{G})^c = \mathscr{F}^c \cap_t \mathcal{G}^c$$

where $\mathscr{F}, \mathcal{G}$ are fuzzy sets associated with the membership functions $\phi_{\mathscr{F}}(x), \phi_{\mathcal{G}}(x)$

Proof. By the definition of the intersection, we can determine the compliment of the intersection of the fuzzy sets of $\mathscr{F}$ and $\mathcal{G}$

$c \coloneqq \mathscr{F} \cap_t \mathcal{G}$ with the corresponding membership function

$$\phi_c(x) \coloneqq t\left(\phi_{\mathscr{F}}(x), \phi_{\mathcal{G}}(x)\right), \forall x \in X$$

Since there are involved the properties of the symmetry and associativity of the norms then the following upholds:

$$\mathcal{F} \cap_t \mathcal{G} \subseteq \mathcal{F},\ \ \mathcal{F} \cap_t \mathcal{G} \subseteq \mathcal{G} \Rightarrow \mathcal{F} \cap_t \mathcal{G} \subseteq \mathcal{F} \cap \mathcal{G}$$

Since every intersection generates the dual union then we can state:

$$\mathcal{F} \cap_t \mathcal{G} \coloneqq (\mathcal{F}^c \cap_t \mathcal{G}^c)^c$$

Therefore, the relations follow with respect to norms :

$$(\mathcal{F} \cap_t \mathcal{G})^c = \mathcal{F}^c \cup_t \mathcal{G}^c$$

$$(\mathcal{F} \cup_t \mathcal{G})^c = \mathcal{F}^c \cap_t \mathcal{G}^c$$

There is the following theorem which states that for the convex fuzzy sets there is existing the defuzzied value of the argument of the aggregated crisp function [13, 14].

Theorem 5. Let $F_\alpha(x) \neq \emptyset$ be a α–cut fuzzy set. If the point x belongs to the convex fuzzy set, then $\check{x} \in X$ is the upper bound of the membership function and the point $\check{x} = x^*$ is the de-fuzzified value of $\phi_{F_\alpha}(x^*) = sup\phi(\check{x})$.

Proof. Suppose there is existing $\hat{x} \in X, \phi_{F_\alpha}(\hat{x}) < \phi_{F_\alpha}(\check{x})$. The convexity criterion of the fuzzy set leads to the inequality: $\lambda\hat{x} + (1 - \lambda)\check{x} \in F_\alpha(x), \lambda \in [0,1]$.

Correspondingly, followed by the convexity at the point we can use the inequality such as: $\phi_{F_\alpha}(\check{x}) < \phi_{F_\alpha}(\lambda\hat{x} + (1 - \lambda)\check{x}), \lambda \in (0,1)$. Since the function is convex and $\phi_{F_\alpha}(\hat{x}) < \phi_{F_\alpha}(\check{x})$, then we have that $\phi_{F_\alpha}(\check{x}) > \phi_{F_\alpha}(\lambda\hat{x} + (1 - \lambda)\check{x}), \lambda \in (0,1)$. Hence, we have gotten the contradiction and $\phi_{F_\alpha}(x^*) = sup\phi(\check{x})$ and x^* is the de-fuzzified value.

There is following next theorem, which states that the de-fuzzified value can be found at the lowest upper bound of the membership function of the α–cut fuzzy sets.

Theorem 6. For the α–cut convex fuzzy set let consider the interval $(\delta, \varepsilon) \in [x_1, x_2]$, $\delta < \varepsilon$, and $\phi_{F_\alpha}(\delta) < \phi_{F_\alpha}(\varepsilon)$, such that $\phi_{F_\alpha}(z) \leq \phi_{F_\alpha}(\delta)$ for $z \in [x_1, \delta)$. Then

(1) $\phi_{F_\alpha}(z) = inf\phi_{F_\alpha}(x)$ at $z \in [x_1, \delta)$.

(2) If $\phi_{F_\alpha}(\delta) \geq \phi_{F_\alpha}(\varepsilon)$, then $\phi_{F_\alpha}(z) \leq \phi_{F_\alpha}(\varepsilon)$ and $\phi_{F_\alpha}(z) = sup\phi_{F_\alpha}(x)$ for $z \in (\varepsilon, x_2]$.

Proof. Let us consider the interval $z \in [x_1, \delta)$. Next, let use the contradiction to suppose that $\phi_{F_\alpha}(z) > \phi_{F_\alpha}(\delta)$. Because of the convexity at δ we obtain that $\phi_{F_\alpha}(\delta) < \max\{\phi_{F_\alpha}(z), \phi_{F_\alpha}(\varepsilon)\} = \phi_{F_\alpha}(\varepsilon)$ and this result contradicts that $\phi_{F_\alpha}(z) < \phi_{F_\alpha}(\delta)$. Thereafter $\phi_{F_\alpha}(z) = sup\phi_{F_\alpha}(x)$.

The part 2 is being proven by utilizing the same contradiction.

5 Defuzzification to the crisp sets and methods to find the defuzzification regions

The defuzzification to the crisp set occurs at the region where we convert the fuzzy set into the crisp set. One of the methods of the defuzzification is the **α–cut method**. Here for the given fuzzy set $F(x)$ we are going to define the crisp set $F_\alpha(x) = \{x: \phi_{F_\alpha}(x) \geq \alpha, 0 < \alpha < 1$ [13, 14,15].

E.g. 1. The fuzzy set is given as the matrix: $F(x) = \begin{pmatrix} 1 & 0.95 & 0.7 \\ 0.4 & 0.9 & 0.3 \\ 0.5 & 0.2 & 0 \end{pmatrix}$.

For the $\alpha = 0.95$ the crisp set is $F_\alpha(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

For the $\alpha = 0.3$ the crisp set is $F_\alpha(x) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

There are inference methods with antecedents and consequent are considered in Fuzzy Logic there. Among them the graphical approaches by Mamdani, Sugeno and etc. [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29,30].

The composition rule of inferences by Zadeh [9] is the expanded extension principle when the membership function is not a bijective one. In such scenario the extension principle is the only particular case of the rule of the inference.

If we assume that the $F -$ is the fuzzy relation at mapping $X \times Y$, $F_X -$is the fuzzy set on X , then the deriving fuzzy set $F_Y -$is the cylindrical shape as the extension $f(F_x)$ with the base of F_x . In terms of the membership functions there is

$\phi_{f(F_x)}(x, y) = \phi_{F_x}(x),\ x \in X, y \in Y$

Based on the intersection we arrive to

$$\phi_{f(F_x)\cap F}(x, y) = min\ \{\phi_{f(F_x)}(x, y), \phi_F(x, y)\} = min\{\phi_{F_x}(x), \phi_F(x, y)\}$$

Since there is the cylindrical projection in the form of the intersection such as $f(F_x) \cap F$ here , then we yield to the following:

$$\phi_{F_y}(y) = max_x\ min\{\phi_{F_x}(x), \phi_F(x, y)\} = \vee_x\{\phi_{F_x}(x) \wedge \phi_F(x, y)\}$$

This rule of inference helps to acknowledge the acceptance of the fuzzy set reasoning based on the If-Then rules.

The α–cut of the fuzzy sets as the decomposition of the fuzzy sets leads to the aggregated crisp sets defined by the Sugeno systems, which says:

If two antecedents $x_1 \in F_{\alpha_1}, x_2 \in F_{\alpha_2}$ generates $y = \phi_\alpha(x_1, x_2)$, then $y = \phi_\alpha(x_1, x_2)$ is the consequent crisp function.

The aggregated crisp function can be obtained by utilization of the various methods such as [6, 7, 8, 9, 10, 11, 12]:

The weighted average defuzzification method here exists by the following rules.

Rule 1. If $x_1 \in F_{\alpha_1}, x_2 \in F_{\alpha_2}$, then $y_1 = \beta x_1 + \gamma x_2 + \delta$.

Rule 2. $x_1 \in F_{\alpha_3}$, $x_2 \in F_{\alpha_4}$, then $y_2 = \zeta x_1 + \varepsilon x_2 + \epsilon$, $\beta, \gamma, \delta, \varepsilon, \epsilon, \zeta \in [0,1]$.

The following formula provided by utilization of the Sugeno systems offers the formula to find the consequent crisp function as it is: $y^* = \frac{w_{y_1}y_1 + w_{y_2}y_2}{w_{y_1} + w_{y_2}}$, where w_{y_1}, w_{y_2} – the value of the assigned weight to the consequents.

The Fig. 3 illustrates the Sugeno approach to find the aggregated crisp function.

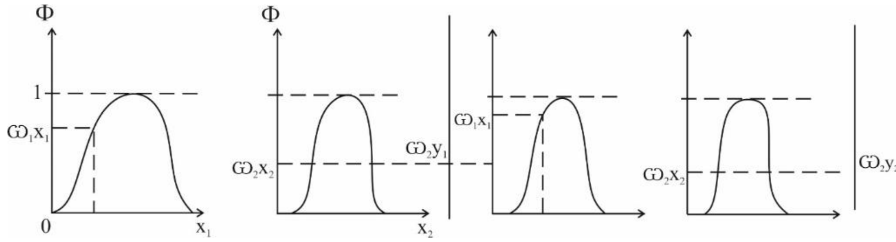

Fig. 3. The Sugeno approach to find the aggregated crisp function.

The value of $x^*, y^* = f(x^*)$ refers to the defuzzification to the crisp sets to find the argument of the crisp sets.

There are existing various methods to obtain the de-fuzzified value of the argument.

(1) Center of the summation method: $x^* = \frac{\sum_{i=1}^{n} \phi_{x_i}(x) x_i}{\sum_{i=1}^{n} \phi_i(x)}$.

(2) Center of the largest area method: $x^* = \frac{\int \phi(x) x dx}{\int \phi(x) dx}$.

(3) First or last Maximum method: the first maximum is given at $x^* = inf_{x \in F} \phi(x) = hght\phi(x), x \in F \subseteq X$; the last maximum is given at $x^* = \sup\ \phi(x) = hght\phi(x), x \in F \subseteq X$.

Let us consider the rule system with two antecedents and one consequence.

Mamdani systems: If x_1 is F_1 and x_2 is F_2, then y is F_y. Here two inputs as the antecedents generates one output and all sets F_1, F_2, F_y – fuzzy sets.

That means the output is the fuzzy set, which requires the further application of one of the defuzzification methods such as α–cut methods.

In the case of the multiple inputs: If $x_1 \longrightarrow F_1^k$ and $x_2 \longrightarrow F_2^k$, then $y^k \longrightarrow F_y^k, k = 1,2, \dots, n$.

For the case of the ***2-input Mamdani systems*** there are existed two methods:

(1) Max-Min inference method.

(2) Max-product inference methods.

It is worth to note that wherever the antecedents are connected by the topological operator AND there is utilized the minimum operator of the membership function. Whenever there is used the topological operator OR, then there is used the maximum of the membership function.

(1) Max-Min inference method. There is considered 2 rule system with two antecedents and one consequent:

Rule 1. If $x_1 \longrightarrow F_1^1$ AND $x_2 \longrightarrow F_2^1$, then $y_1 \longrightarrow F_y^1$. Here there is used MIN operator.

Rule 2. If $x_1 \longrightarrow F_1^2$ OR $x_2 \longrightarrow F_2^2$, then $y_2 \longrightarrow F_y^2$. Here there is used MAX operator.

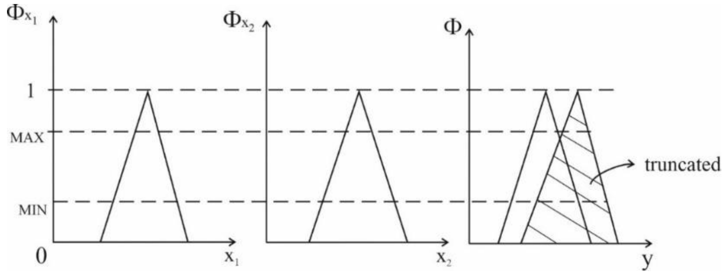

Fig. 4. The Max-Min inference method, where the shaded region is the truncated membership fuzzy function, which was aggregated by Max and Min.

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The Fig. 4 illustrates the Max-Min inference method, where the shaded region is the truncated membership fuzzy function, which was aggregated by Max and Min. The shaded region represents the region, which is supposed to be further defuzzied to obtain the crisp set. This can be done by α–cut method.

(2) Max-product inference method. The difference between MAX-MIN and Max-product methods is in the obtained defuzzied region, which is in a case of Max-Min method is called the truncated region while in a case of Max-product method it is called the scaled region.

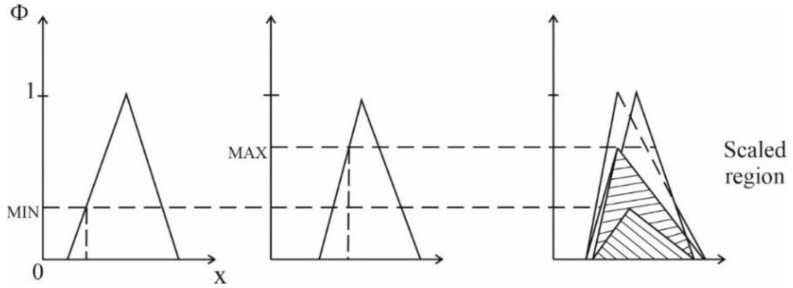

Fig. 5. How to obtain the scaled shaded region for the defuzzied values.

The rule 1. There is used MIN operator for AND in Max product method.

The rule 2. There is used Max operator for OR in Max product method.

The Fig. 5 illustrates how to obtain the scaled shaded region for the defuzzied values. The graphs from two rules integrated in each other to obtain the scaled region which is shaded.

Centroid method for the defuzzification to crisp set

The centroid method is alternatively called the method of the center of gravity to find the defuzzification values. There is the formula is given to find the de-fuzzified value of the crisp set:

$$x^* = \frac{\int F(x)xdx}{\int F(x)dx}.$$

There is given Fig .6 to illustrate the centroid method to find the de-fuzzified value of x.

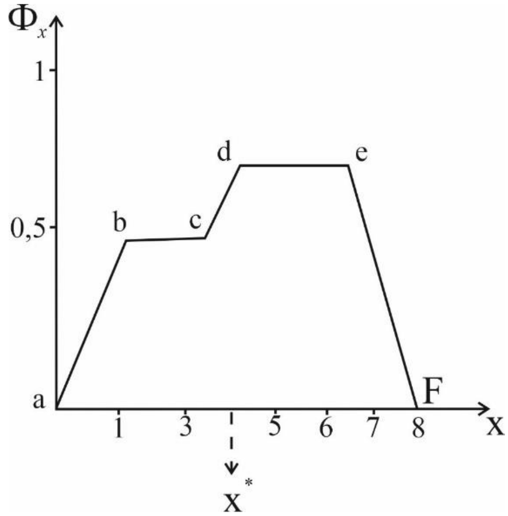

Fig. 6. Illustrate the centroid method.

The equation of the lines from a to b, b to c , c to d ,d to e and e to f is given by using the point-slope formula for the line, or the slope formula for the line, e.g. such as: $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$.

The general equation of the line is $y=mx+c$ in slope-intercept form. If we apply the integration process to find the de-fuzzified value, then we must evaluate the following integrals over the certain specified regions:

$$x^*=\frac{\int_0^1 y_{ab}xdx+\int_1^{3.5} y_{bc}xdx+\int_{3.5}^4 y_{cd}xdx+\int_4^6 y_{de}xdx+\int_6^8 y_{ef}xdx}{\int_0^1 y_{ab}dx+\int_1^{3.5} y_{bc}dx+\int_{3.5}^4 y_{cd}dx+\int_4^6 y_{de}dx+\int_6^8 y_{ef}dx}=A.$$

So, after evaluating the integrals we obtain the de-fuzzified value for $x^*=A\subseteq\mathbb{R}$ by using the centroid method.

Next method is called

Maximum principle or height method

This method is applicable if the fuzzy sets and their union have the maximum height: $hght\big(F(x)\big)=max\phi_F(x^*)\geq\phi_F(x),\forall x\in X.$

Center of the Largest Area defuzzification method

This particular method is used wherever the fuzzy set has more than one subregions and the center of the gravity can be used to calculate the de-fuzzified value: $x^*=\frac{\int\phi_{LA}(x)xdx}{\int\phi_{LA}(x)dx}$, where LA sub-notation is used to denote the region with the largest area and x is the center of the gravity of the largest region.

Method of the First or Last Maxima to find the de-fuzzified value

For this method the de-fuzzified value is found at the greatest lower bound:
$x^*=inf_{x\in X}\{\phi_F(x)=hght(F(x)).$

Center of the Sums method to find the de-fuzzified value

The de-fuzzified value can be found by using the formula: $x^*=\frac{\sum_{i=1}^{n}x\phi_F(x)}{\sum_{i=1}^{n}\phi_F(x)}$.

The weighted average method to find the de-fuzzified value

This method is applicable to the symmetric functions. There is used the following formula to evaluate the de-fuzzified value: $x^*=\frac{\sum x\phi_F(x)}{\sum\phi_F(x)}$.

6 Conclusion

The fuzzification concepts such as the fuzzy sets and their corresponding membership functions with further descriptors such as the core, support plane, boundaries, heights, sub-normal and normalized fuzzy sets were introduced to be mostly relevant to the fuzzification and farther defuzzification operations required to apply in processing of the soft solutions.

The compelling idea to find the de-fuzzified value of the soft solution has led the route of the research to redefine the alfa-cuts of the fuzzy sets as the pair of functions substantiated on their lower and upper bounds.

The decomposition of the fuzzy power sets was outlined in terms of the decomposition theorem as the main tool in the de-fuzzification process.

The farther steps to find the de-fuzzified values of the soft solutions were presented on the basis of the alfa-cuts of the convex fuzzy sets. There was proven the next theorem which supports the convexity principle of the alfa-cuts of the fuzzy sets.

Next development was shown in the form of the theorem which was introduced to help to find the de-fuzzified value of the convex alfa-cuts of the fuzzy sets. This theorem stated that the de-fuzzified values of the soft solution is located in the region of the upper and lower bounds of the membership function.

Further, there was introduced one more theorem which was the extended version of the previous one to prove that the de-fuzzified value(s) belongs to the upper bounds of the sub-regions of the alfa-cuts.

The defuzzification formulas and defuzzification operations to the crisp sets were shown by utilizing the inference methods with multiple antecedents and consequents rules and formulas.

References

1. R. Aliev, B. Fazlollahi, *Soft Computing and Its Applications in Business and Economics* (Springer, Verlag-Berlin Heidelberg, 2004)
2. G.J. Klir, B. Juan, *Fuzzy sets and Fuzzy Logic. Theory and Applications* (NJ, Prentice Hall, 1995)
3. G.J. Klir, B. Juan, *Fuzzy sets Theory. Foundations and Applications* (Prentice Hall, NJ, 1997)
4. R.A. Aliev, F.T. Aliev, M.D. Babaev, *Fuzzy process control and knowledge engineering* (Koln, Verlag, TUV Rheinland, 1991)
5. R.A. Aliev, R.R. Aliev, *Soft Computing and its applications* (World Scientific, 2001)
6. L.A. Zadeh, J. Information Sciences **3**, 117-200 (1971)
7. L.A. Zadeh, Fuzzy sets and Systems **1**, 3-22 (1978)
8. L.A. Zadeh, J. Information and Controls **8**, 338-353 (1965)
9. L.A. Zadeh, J. IEEE Trans Systems, Man. and Cybernetics **3**, 28-144 (1973)
10. L.A. Zadeh, J. Information Sciences **3** (1971)
11. L.A. Zadeh, IEEE Software **11(6)**, 48 (1971)
12. L.A. Zadeh, J. IEEE Transactions on fuzzy systems **2(4)** (1971)
13. D.D. Gadjiev, Chebyshevski sbornik **1(73)**, 21 (2019)
14. D.D. Gadjiev, I. Kochetkov, A. Rustanov, IOP Conf. Series **403**, 012187 (2019) DOI: 10.1088/1755-1315/403/1/012187
15. D.D. Gadjiev, et al, ICCCT 2021, Series Algorithms for Intelligent Systems, 237-245 (2021) DOI https://doi.org/10.1007/978-981-16-3246-4
16. M. Bugeja, *Non-Linear Swing-Up and Stabilizing Control of an Inverted Pendulum System* (EUROCON Ljublana, Slovenia, 2003)
17. N. Muskinja, B. Tovornik, IEEE Transactional on Industrial Electronics **53(2)** (2006)
18. A.K. Stimac, *Standup and Stabilization of the Inverted Pendulum* (Massachusetts Institute of Technology, 1999)
19. W. Zhong, H. Rock, IEEE Conference on Control Applications (2001)
20. J. Krishen, V.M. Becerra, IEEE International Symposium on Intelligent Control Germany, 2701-2706 (2006)
21. K. Craig, S. Awtar, Mechatronics **12**, 357-370 (2001)
22. A. Mirza, Journal of AMSE **55(3,4)** (2000)
23. S. Nundrakwang, T. Benjanarasuth, J. Ngamwiwit, N. Komine, ICCAS 2005 KINTEX, Gyeonggi-Do, Korea (2005)
24. M.E. Magana, F. Holzapfel, IEEE Transactions on Education **41(2)**, 165-170 (1998)

25. S. Yasunobu, M. Mori, Proceedings of the Sixth IEEE International Conference on Fuzzy Systems **3**, 1621-1625 (1997)
26. C.W. Ji, F. Lei, K. Kin, IEEE International Conference on Intelligent Processing Systems, ICIPS, 185-189 (1997)
27. Y. Becerikli, B.K. Celik, Mathematical and Computer Modeling **46(1-2)**, 24-37 (2007)
28. K.M. Passino, S. Yurkovich, *Fuzzy Control* (Addison Wesley Longman, Menlo Park, 1998)
29. K. Selcuk, Z. Bingul, C. Oysu, Journal of intelligent and Fuzzy Systems (2010) DOI:10.1007/978-3-540-85563-7_85
30. A. Gani, Z. Bas, H. Acikgoz, O.F. Kececioglu, International Journal of Engineering research and technology (IJERT) **3(9)** (2014) ISSN: 2278-0181