Kinematics study for a spatial manipulator

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Abstract. For industrial robot design and control problems, it is necessary to determine both the positions of its parts relative to the stationary coordinate system, absolute positions of the parts, and their relative positions, generalised coordinates. The first problem is called the forward problem, and the second is the inverse problem for manipulator positions. The purpose of the paper is to solve the direct problem on manipulator positions, i.e., to determine and study positions of manipulator sections with known generalised coordinates. In the paper a spatial manipulator with four degrees of freedom is considered. Kinematic characteristics of the last link of the manipulator - grabber are defined. The kinematic characteristics are the coordinates of current position, velocity, and acceleration of the grip. Kinematic characteristics are found by applying the vector matrix method, based on the application of transition matrices from one reference system to another and rotation vector matrices. The vector matrix method belongs to the universal methods and is designed for use in mathematical computer simulation systems. Keywords: Vector matrix method, spatial manipulator, kinematic characteristics, mechanics of industrial robots.

1 Introduction

A manipulator is an executive mechanism of an industrial robot equipped with actuators and a working organ, by means of which the working functions are performed \[1, 2\]. The ability to reproduce motions is achieved by giving the manipulator several degrees of freedom, along which a controlled movement is carried out in order to obtain a given movement of the working body - grip.

The motion of a spatial manipulator with four degrees of freedom (Fig. 1) is given by indicating the time-dependence of the four generalised coordinates \(s(t), f(t), \phi_{z1}(t), \theta_{z2}(t)\). In a gripper (grabber) \(C\) of a solid body is attached to the arm \(T\), on which the point \(M\) is marked. The task is to investigate the kinematics of a spatial manipulator: determine point \(M\) positions of the body \(T\) for any point in time, determine an array of velocity and acceleration values of the point \(M\) of the body \(T\) for any time moment in global coordinates, and calculate the matrix of angular velocity and angular acceleration of the body \(T\) \[3, 4\].

Several coordinate systems are used to set the positions of the manipulator links: global coordinate system and local coordinate systems: \(x_1y_1z_1, x_2y_2z_2, x_3y_3z_3\).

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2 Methods

The kinematic characteristics of the spatial manipulator can be found by applying the vector matrix method. Let us assume the following input data. The incremental movements of AO and BD are given by functions \( s(t) = 10 + 5 \sin(2\pi t) \) (cm) and \( f(t) = 15 + 20 \cos^2(\pi t) \) (cm) respectively. Rotation of the links is given by the rotation angles versus time around the axes \( z_1 \) and \( z_2 \) (Fig. 1): \( \phi_z(t) = (\pi / 4) \sin(3\pi t) \) (rad) and \( \theta_{z2}(t) = (\pi / 3) + (\pi / 4) \cos(\pi t) \) (rad). Point \( M \) coordinates in the local reference frame associated with the body \( T \) is: \( x_M = 8 \text{cm}, y_M = -2 \text{cm}, z_M = 4 \text{cm} \).

2.1 Determining coordinates of a single point \( M \) for the body \( T \), placed in the arm grip

Let us determine the position of the point \( M \) of the body \( T \) in the global coordinate system [5, 6].

The global and local point coordinate matrices are depicted through the radius-vector matrices \( \vec{r} \) with an appropriate ordinal index.

Equation of motion for a point \( M \) in global coordinates will be described by a matrix:

\[
r_{M0} = r_{A0} + H_z(\phi(t))^T \cdot (r_{D1}(t) + H_{z2}(\theta(t))^T \cdot (r_{C2}(t) + r_{M3}(t))) .
\]

(1)

Here:

\[
r_{A0}(t) = \begin{pmatrix} 0 \\ 0 \\ s(t) \end{pmatrix}, r_{D2}(t) = \begin{pmatrix} AB \\ 0 \\ f(t) \end{pmatrix}, r_{C2}(t) = \begin{pmatrix} DC \\ 0 \\ 0 \end{pmatrix}, r_{M3}(t) = \begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix}.
\]

Rotation matrices \( H_z(\phi(t)) \) and \( H_{z2}(\theta(t)) \) around the axes \( z_1 \) and \( z_2 \) respectively:
\[ H_z(\phi(t)) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_z(\theta(t)) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (2)

Dependency plots of the global coordinates of a given point \( M \) of time in the range from 0 to \( t_1 = 500 \) s, and the trajectory of the point \( M \) in global coordinates are shown in Figure 2 and Figure 3:

![Fig. 2. Dependency graphs for global point coordinates \( M \) on time.](image1)

![Fig. 3. Spatial trajectory of point \( M \) at global coordinates.](image2)

\((X, Y, Z)\)

**2.2 Determining a single point \( M \) velocity of the body \( T \), placed in the arm grip**

To obtain a point velocity matrix \( M \) in global coordinates, it is possible to differentiate equation (1) by time, or calculate an array of point velocity values \( M \) by numerical differentiation in a sufficiently small step \( \Delta t \) \([7, 8]\).

Array of point \( M \) velocity matrices in a global reference frame is:
\[ v_{M0k} = \frac{T_{M0(k+1)} - T_{M0k}}{\Delta t}, \]  

(3)

here \( r_{M0k} \) - an array of point \( M \) coordinate matrices in a global reference frame \( (k = 0,1,\ldots,n) \). The process of changing the velocity projections of a point \( M \) over time is shown in Figure 4.

Fig. 4. Diagrams of changes in projection velocity of a point \( M \) from time in the global coordinate system.

Modulus of velocity of the point \( M \) is defined by the expression:

\[ v_{M0k} = \sqrt{v_{M0k}^T \cdot v_{M0k}}. \]  

(4)

The process of changing the velocity modulus of a point over time is illustrated in Figure 5.

Fig. 5. Diagram of the change in velocity modulus of a point \( M \) for the body \( T \) in time.
Array of point M acceleration matrices is obtained using the numerical rate differentiation procedure:

$$a_{M0k} = \frac{v_{M0(k+1)} - v_{M0k}}{\Delta t},$$

(5)

The process of changing the acceleration projections of point M over time is shown in Figure 6.

The modulus of acceleration of point M is determined by the expression:

$$a_{M0k} = \sqrt{|a_{M0k}^T \cdot a_{M0k}|}. \quad (6)$$

The process of changing the acceleration modulus of point M over time is shown in Figure 7.

Let us define the angular velocity matrix of the body $T$. The body performs two rotations: around the axis $z_1 (z_0)$ at an angular velocity $\phi_{z_1}(t)$ and around the axis $z_2$ at angular velocity $\theta_{z_2}(t)$. Absolute angular velocity matrix of the body $T$ can be found as a sum:
\[ \omega_0(t) = \omega_{\phi}(t) + \omega_{\theta}(t), \]  
\text{(7)}

Here \( \omega_{\phi}(t) = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi}(t) \end{pmatrix} \) - matrix of transfer angular velocity of a body \( T \),

\[ \omega_{\theta}(t) = H_z(\phi(t))^T \cdot H_{z2}(\theta(t))^T \cdot \omega_{\theta2}(t) \] - matrix of the relative angular velocity of the body \( T \) on the global coordinate system,

\[ \omega_{\theta1}(t) = H_{z2}(\theta(t))^T \cdot \omega_{\theta2}(t) \] - matrix of the relative angular velocity of the body \( T \) in the first local coordinate system;

\[ \omega_{\theta2}(t) = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}(t) \end{pmatrix} \] - matrix of the relative angular velocity of the body \( T \) in the second local coordinate system [9, 10].

Absolute angular velocity matrix of the body \( T \) takes the form:

\[ \omega_0(t) = \omega_{\phi}(t) + H_z(\phi(t))^T \cdot H_{z2}(\theta(t))^T \cdot \omega_{\theta2}(t). \]  
\text{(8)}

The process of changing the projections of the angular velocity of the body over time is shown in Figure 8. The figure shows that the rotation occurs only around the axis \( z_0 \) \((z_1)\). Rotations around the axes \( x_0, y_0 \) is not available.

**Fig. 8.** The process of changing the angular velocity projections of the bodies \( T \) in time.

The modulus of angular velocity is determined by the formula (9):

\[ \omega_0(t) = \sqrt{\omega_0^T(t) \cdot \omega_0(t)} \]  
\text{(9)}

The process of changing the angular velocity modulus of the body over time is shown in Figure 9.
2.3 Determining the angular acceleration of a body $T$, placed in the arm grip

Calculate the angular acceleration of the body $T$ by numerically differentiating the angular velocity matrix:

$$\varepsilon_{M0}(u) = \frac{\omega(u+1) - \omega(u)}{\Delta t}$$  \hspace{1cm} (10)

The process of changing the projections of the angular acceleration of a body $T$ in time is shown in Figure 10. The figure shows that the angular accelerations around the axes $x_0, y_0$ are not available.

$$\varepsilon_{M0}(u) = \sqrt{[\varepsilon_{M0}(u)]^2 \cdot [\varepsilon_{M0}(u)]}$$  \hspace{1cm} (11)
The process of changing the modulus of angular acceleration of a body over time is shown in Figure 11.

![Diagram of changes in the modulus of angular acceleration of a body T in time.](image)

Fig. 11. Diagram of changes in the modulus of angular acceleration of a body T in time.

3 Conclusions

A direct problem of manipulator kinematics on the example of a manipulator with four degrees of freedom is solved in the paper by finding position, velocity and acceleration of a grip at any instant of time with known generalised coordinates. The solution of the direct problem is done by vector-matrix method in Mathcad simulation environment.

References