Calculation of a fiber–reinforced concrete specimen based on the orthotropic model

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Abstract. The calculation of fiber-reinforced concrete is performed by the finite element method using the stresses approximations. The solution is based on the additional energy functional. The nodes equilibrium equations are obtained using the possible displacements principle. To take into account the nonlinearity of concrete deformations, an orthotropic model is used in the plane case. The solution of the nonlinear problem is performed by the method of variable elasticity parameters. The paper gives a calculation result of a tension specimen.

1 Introduction

The addition of various fibers to the concrete mixture to improve its mechanical characteristics began a long time ago. Since then, numerous studies have been carried out to demonstrate the success of reinforcing concrete with various types of fibers. In [1], a critical analysis of previous studies is presented with an emphasis on the optimization of fiber-reinforced concrete to increase its strength under various loading conditions. This review cites 227 publications. The article [2] explores the possibility of using high-strength concrete reinforced with steel fibers for the columns of a multi-storey building. The paper [3] considers the use of dispersed reinforcement of concrete with glass fibers. The effectiveness of concrete reinforcement with fiber has been proven, since its strength and deformation characteristics increase. Another research area is the use of a combination of plastic and steel fibers for dispersed reinforcement of concrete [4-7]. It is shown that self-compacting concrete reinforced with polyolefin fiber showed high performance in both fresh and solid state.

A significant number of studies are aimed at optimizing the parameters of fiber-reinforced concrete [8-11]. The shape, size and volume content of fibers affect the density, strength and workability of concrete. The paper [9] proposes a technique following which it is possible to obtain an optimized fiber-reinforced concrete with a dense structure, high fiber efficiency and excellent mechanical properties. For the fibers manufacture, it is possible to use products of secondary processing of polyethylene terephthalate [12-14]. As shown in [12], the use of PET fibers provides an increase in the tensile strength of concrete up to 66%, the compressive strength of the studied samples of fiber-reinforced concrete decreased by 3–25% within the range of variable factors.

In papers [15-18], the various fiber-reinforced concrete elements and structures are investigating. In [15], flat finite elements are considered for modelling the tension of fiber-
reinforced concrete samples. The fibers were modelled by rod finite elements working in tension-compression. A non-linear shear relationship between fiber and concrete is introduced into the calculation, which models the possibility of fiber slippage. Concrete is considered as a linearly elastic material with brittle tensile cracking.

In works [19-25] for the calculation of the beams, plates and plane problems of the theory of elasticity, the finite element method based on the stresses approximation was used. This method allows obtaining more accurate stresses at nodal points. Therefore, this article proposes to use this approach for the calculation of structures made of fiber-reinforced concrete.

2 Methods

To solve the problem of tension-compression specimen of fiber-reinforced concrete (Fig. 1), we use the finite element method based on the stress approximations [19-25]. The concrete matrix stress state is plane, and the randomly arranged fibers have a tension-compression state.

Fig. 1. Calculation scheme of fiber-reinforced concrete specimen.

The solution of the plane elasticity theory problem for fiber-reinforced concrete in stresses can be obtained on the additional energy functional basis:

$$\Pi = \frac{1}{2} \iint \sigma^T E^{-1} \sigma \, d\Omega + \frac{1}{2} \sum_{i=1}^{nf} k_{f,i} \sigma_{f,i}^2 + \int T^T \Delta \, dS \rightarrow \min. \quad (1)$$

$$\sigma = (\sigma_x, \sigma_y, \tau_{xy})^T$$ is concrete stress vector; $\Delta$ is vector of the specified displacements; $T$ is boundary force vector; $E$ is concrete stiffness matrix; $\Omega$ is subject area; $S$ is domain boundary; $\sigma_{f,i}$ is stress in $i$-th fiber; $k_{f,i} = \frac{E_{f,i}}{A_{f,i}}$ is stiffness coefficient of $i$-th fiber; $n_f$ is number of fibers; $A_f$ is fiber cross-sectional area; $l_f$ is fiber length; $E_{f,i}$ is fiber elasticity modulus.

When solving, we will assume that the forces (stresses) in each fiber are constant along the length, and the fibers are connected to concrete at nodal points. The concrete matrix is represented by a rectangular finite elements set. The stresses are approximated by piecewise constant functions (Fig. 2). On Fig. 2, the constant stresses associated with the node are highlighted in one color. In each of the finite element quarter, the stresses are equal to the stresses at the corresponding node. We denote the nodal stress vector as $\sigma_i = (\sigma_{x,i}, \sigma_{y,i}, \tau_{xy,i})^T$. 
Fig. 2. Piecewise constant approximations of stresses.

In accordance with the minimum additional energy principle, the stresses in concrete and forces in fibers must satisfy the equilibrium equations. To obtain the nodes equilibrium equations for the finite element mesh, we use the possible displacements principle.

Fig. 3. The possible node displacements.

For a rectangular finite element, we introduce a local coordinate system (Fig. 3) associated with its center and the base functions, which are expressed in normalized local coordinates in the following form:

$$N_i(x, y) = \frac{(1+\xi_i\xi)(1+\eta_i\eta)}{4}, \quad \xi = \frac{2x}{a}, \quad \eta = \frac{2y}{b}, \quad i = 1, 2, 3, 4. \tag{2}$$

The index $i$ denotes the local node number of the finite element; $x, y$ are coordinates along the X and Y axes, respectively; $\xi_i, \eta_i$ are local coordinates of a node $i$, which take the values 1 or -1. As possible displacements of each node, consider the displacements $\delta u_i$ and $\delta v_i$ along the global coordinate system axes:

$$\delta u_i = N_i(x, y), \quad \delta v_i = N_i(x, y). \tag{3}$$

The finite element deformations caused with possible displacement along the X axis are

$$\delta \varepsilon_x = \frac{\partial (\delta u_i)}{\partial x} = \frac{\xi_i(1+\eta_i\eta)}{2a}, \quad \delta \gamma_{xy} = \frac{\partial (\delta u_i)}{\partial y} = \frac{\eta_i(1+\xi_i\xi)}{2b}. \tag{4}$$

The possible work of the concrete internal forces is

$$\delta U^b_{ki} = \int_0^a \int_0^b t(\sigma_x \delta \varepsilon_x + \tau_{xy} \delta \gamma_{xy}) \, dx \, dy = \sum_{j=1}^4 \frac{bt}{8} \varepsilon_i \left(1 + \frac{1+\eta_i\eta_j}{2}\right) \sigma_{x,j} + \sum_{j=1}^4 \frac{at}{8} \eta_i \left(1 + \frac{1+\xi_i\xi_j}{2}\right) \tau_{xy,j}. \tag{5}$$
Fig. 4. Bonding fiber to concrete.

The fiber nodes are located in a rectangular concrete finite element (Fig. 4). A fiber deformation during possible displacement $\delta u_i$ are determined by its node’s relocations. The displacement of the initial or final fiber node $jj$ ($j=1$ or $j=2$) with coordinates $(x^f_j, y^f_j)$ along the global X axis is

$$\delta u^f_{jj} = N_i(x^f_j, y^f_j). \quad (6)$$

The deformations of the $jj$–th fiber, caused by the displacement of its node $j$, are determined by the following formula:

$$\delta \varepsilon^f_{jj,j} = \begin{cases} \frac{-N_i(x^f_j, y^f_j) \cos \alpha}{l_f}, & j = 1, \\
\frac{N_i(x^f_j, y^f_j) \cos \alpha}{l_f}, & j = 2 \end{cases}, \quad \cos \alpha = \frac{x^f_2-x^f_1}{l_f}. \quad (7)$$

The internal forces possible work of the fiber is determined by the sum of the fibers work, one or two nodes of which are in the considered finite concrete element $k$.

$$\delta U^f_{k,i} = \sum_{j=1}^{n_k^f} A_f l_f \sum_{j=1}^{2} \sigma_{f,ij} \delta \varepsilon^f_{jj,i}. \quad (8)$$

In accordance with the possible displacements principle, the sum of the internal forces works and the external forces potential for an equilibrium system is equal to zero.

$$\sum_{k \in \Omega_i} (\delta U^b_{k,i} + \delta U^f_{k,i}) + P_{x,i} = 0. \quad (9)$$

$\Omega_i$ is set of finite elements adjacent to node $i$; $P_{x,i}$ is the external forces potential, applied in the node and directed along the X axis. The equilibrium equation for possible displacement along the Y axis is formed in a similar way.

The nodes equilibrium equations for the entire system can be represented in the following matrix form:

$$L_b \sigma_b + L_f \sigma_f + P = 0. \quad (10)$$
\( \sigma_b \) is nodal stress vector for the entire system; \( \sigma_f \) is stress vector of all fibers. The matrix \( L_b \) has a band structure of non-zero elements, and the matrix \( L_f \) is block-diagonal, each fiber corresponds to an independent block. The additional energy (1), for the case of the specified displacements absence, can be represented in the following form:

\[
\Pi = \frac{1}{2} \sigma_b^T E_b^{-1} \sigma_b + \frac{1}{2} \sigma_f^T E_f^{-1} \sigma_f. \tag{11}
\]

\( E_b \) is stiffness matrix of concrete (material), which has a block-diagonal shape and is easily reversible. \( E_f \) is the fibers material stiffness matrix, which has a simple diagonal shape.

Using the Lagrange multipliers method, we add the equilibrium equations (10) to the functional (11).

\[
\Pi = \frac{1}{2} \sigma_b^T E_b^{-1} \sigma_b + \frac{1}{2} \sigma_f^T E_f^{-1} \sigma_f + w^T (L_b \sigma_b + L_f \sigma_f + P) \rightarrow \min. \tag{12}
\]

\( w \) is the nodal displacements vector of finite element mesh. Equating the derivatives of expression (12) to zero, we obtain the matrix equations system:

\[
\sigma_b = -E_b L_b^T w, \quad \sigma_f = -E_f L_f^T w, \quad L_b \sigma_b + L_f \sigma_f + P = 0. \tag{13}
\]

Substituting the stresses expressions into the third equation, we obtain:

\[
K w = P, \quad K = K_b + K_f, \quad K_b = L_b E_b L_b^T, \quad K_f = L_f E_f L_f^T. \tag{14}
\]

The matrix \( L_b \) has a band structure of nonzero elements, and the matrix \( E_b \) is block-diagonal (15). Therefore, when calculating the concrete stiffness matrix \( K_b \), an algorithm is used that takes into account the indicated features of the matrices. \( E_{b,i} \) is the stiffness matrix calculated for the area related to node \( i \) (Fig. 2).

\[
E_b = \begin{bmatrix}
E_{b,1} & & \\
& \ddots & \\
& & E_{b,n-1} \\
& & \\
E_{b,n}
\end{bmatrix}. \tag{15}
\]

Since the fibers are not connected to each other, the stiffness matrix of all fibers is formed as the sum of the individual fiber stiffness matrices. Note that the system of equations matrix (14) also has a band structure of nonzero elements, but its band width is greater than one of the standard finite element method in displacements.

To take into account the concrete deformations nonlinearity, we use an orthotropic model. In accordance with the orthotropic model, the ratios for the principal stresses and strains for concrete in a plane stress state, as a physically nonlinear material, are as follows:

\[
\tilde{\sigma} = \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} = \frac{1}{1 - \nu_{12} \nu_{21}} \begin{bmatrix}
E_1 & \nu_{12} E_1 & 0 \\


\nu_{21} E_2 & E_2 & 0 \\
0 & 0 & (1 - \nu_{12} \nu_{21}) G_{12}
\end{bmatrix} \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}. \tag{16}
\]
$E_1, E_2$ are the secant strain moduli in tension–compression in the principal axes direction; $\nu_{12}, \nu_{21}$ are transverse strain coefficients; $G_{12}$ is secant shear modulus. The matrix components are connected by $\nu_{12}E_1 = \nu_{21}E_2$. To determine the strain moduli, "equivalent" or effective strains are introduced:

$$\begin{pmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \end{pmatrix} = \frac{1}{1-\nu_{12}\nu_{21}} \begin{pmatrix} 1 & \nu_{12} \\ \nu_{21} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \frac{1}{E_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \quad (17)$$

Thus, the deformation law in each main direction is taken to be the same as the law of concrete deformation for a uniaxial stress-strain state. We write relation (16) in the following matrix form:

$$\bar{\sigma} = \bar{E}\bar{\varepsilon}. \quad (18)$$

Then, the relationship between stresses and strains along the X and Y coordinate axes will have the following form:

$$\sigma = E\varepsilon, \quad E = C^T\bar{E}C. \quad (19)$$

$$C = \begin{pmatrix} \cos^2 \alpha & \sin \alpha \cos \alpha \\ \sin^2 \alpha & \cos \alpha \sin \alpha \end{pmatrix}, \quad \alpha = \frac{1}{2} \arctan \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \quad (20)$$

In SP 63.13330.2018 (Russia), the analytical dependence of curvilinear diagrams of concrete deformation is given. Dependence allows you to calculate the deformation and secant modulus of elasticity depending on the stress level:

$$\varepsilon_i^0 = \frac{\sigma_i}{E_i}, \quad E_i = E_b\eta_i. \quad (21)$$

$$\eta_i = \bar{\eta} \pm (\eta_0 - \bar{\eta})\sqrt{1 - \omega_1\xi - \omega_2\xi^2}, \quad \xi = \frac{\sigma_i}{R_b}, \quad \bar{\eta} = \frac{R_b}{E_b\varepsilon_u}. \quad (22)$$

The "+" sign is used for the ascending branch of the chart, and the "-" sign is used for the descending branch. When compressing concrete, the required parameters are determined by formulas (23) – (25). For the ascending branch of the chart:

$$\eta_0 = 1, \quad \omega_1 = 2 - 2.5\bar{\eta}, \quad \omega_2 = 1 - \omega_1. \quad (23)$$

For the descending branch we have

$$\eta_0 = 2.05\bar{\eta}, \quad \omega_1 = 1.95\bar{\eta} - 0.138, \quad \omega_2 = 1 - \omega_1. \quad (24)$$

The diagram top point is determined by the formula depending on the concrete class $B$:

$$\varepsilon_u = \frac{B}{E_b} \lambda \frac{1 + 0.75\lambda B/60 + 0.2\lambda B}{0.12 + B/60 + 0.2 B}. \quad (25)$$
For heavy concrete $\lambda = 1$. When concrete is stretched, formulas (26) are used.

$$\xi = \frac{\sigma_i}{R_{bt}}, \quad R_b = R_{bt}, \quad \tilde{\eta} = (0.6 + 0.15R_{btn}/2.5). \quad (26)$$

When using diagrams with a descending branch, one stress level can correspond to two different strain values. Therefore, when solving a nonlinear problem by the method of variable elasticity parameters, it is necessary to obtain an expression that determines the magnitude of the stress depending on the deformation. Expressing from equation (22) the parameter $\xi$, we obtain the following expression for a stress:

$$\sigma_i = \frac{R_{b(bt)}(b + \sqrt{b^2 - 4ac})}{2a},$$

$$a = \left(\frac{R_{b(bt)}}{E_b\varepsilon_i}\right)^2 - \omega_2(\eta_0 - \tilde{\eta})^2, \quad b = \frac{2R_{b(bt)}}{E_b\varepsilon_i} \tilde{\eta} - \omega_1(\eta_0 - \tilde{\eta})^2, \quad c = \tilde{\eta}^2 - (\eta_0 - \tilde{\eta})^2. \quad (27)$$

In (27), deformations, as well as the calculated resistances of concrete, are substituted by positive ones (in modulus). The secant modulus of elasticity is

$$E_i = \frac{\sigma_i}{\varepsilon_i}. \quad (28)$$

Diagrams for concrete class $B30$, built according to formulas (27), are shown in Fig. 5.

![Fig. 5. The stress – strain diagrams for concrete class $B30$.](image)

The transverse deformations coefficients are determined by the following formulas, which ensure the symmetry of the concrete stiffness matrix:

$$\nu_{12} = \frac{E_v\nu_v}{E_1}, \quad \nu_{21} = \frac{E_v\nu_v}{E_2}. \quad (29)$$

The shear modulus for an anisotropic material is

$$G_{12} = \frac{E_1E_2}{E_1+E_2+2\nu_{12}E_1}. \quad (30)$$

As an estimate of the concrete strength in a plane stress state, we use the criterion:
\[ \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 + (R_b - R_{bt})(\sigma_1 + \sigma_2) \leq R_b R_{bt}. \]  

(31)

\( R_b, R_{bt} \) are ultimate strength of concrete under uniaxial compression and tension. This criterion gives an overestimated strength in biaxial compression. In this paper, it is supposed to consider the tension and compression problems of fiber-reinforced concrete samples in one direction, so the use of this criterion is quite acceptable. For fibers, any tension-compression diagram can be used, depending on their material. The nonlinear system of equations (14) is solved by the method of variable elasticity parameters.

### 3 Results and Discussions

The calculated scheme of the investigated specimen of fiber-reinforced concrete (Fig. 1) has the parameters presented in Table 1. The scheme was divided into 20 finite elements along the length, and into 10 along the width. The number of randomly distributed fibers is 300.

**Table 1. Parameters of the calculated scheme.**

<table>
<thead>
<tr>
<th>Length</th>
<th>Width</th>
<th>Thickness</th>
<th>Concrete</th>
<th>( E_b )</th>
<th>( v_b )</th>
<th>( R_b )</th>
<th>( R_{bt} )</th>
<th>( E_f )</th>
<th>( R_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50 cm.</td>
<td>25 cm.</td>
<td>6 mm.</td>
<td>B30</td>
<td>32500 MPa</td>
<td>0.2</td>
<td>22 MPa</td>
<td>1.75 MPa</td>
<td>210000 MPa</td>
<td>350 MPa</td>
</tr>
</tbody>
</table>

The specimen calculations were performed for the action of a tensile load of various intensity uniformly distributed at the end. For comparison, the concrete specimen without reinforcement was also calculated. The calculation results are shown in Fig. 6.

**Fig. 6.** The dependency load – displacement. The red line is fiber reinforced concrete, the blue line is concrete.

The results of calculations show that the destruction of fiber-reinforced concrete specimen occurs at the load of 10.5 kN/m (Fig. 7). The bearing capacity of the concrete specimen is 7% less. The fiber content is approximately 2.6% of the concrete volume. On Fig. 7 shows the fibers distribution over the specimen. The red fibers are stretched and the blue fibers are compressed.
Fig. 7. The fibers distribution over the specimen. The black thick lines show the main stresses directions in the nodes.

In the nodes where the destruction of the concrete occurred (24), the black thick lines show the main stresses directions. Note that the iterative process of solving the nonlinear equations system by the method of variable elastic parameters reaches the required solution accuracy of 1% in 4 iterations.

4 Conclusion

The calculation of fiber-reinforced concrete is performed by the finite element method using the stresses approximations. The solution is based on the additional energy functional. The nodes equilibrium equations are obtained using the possible displacements principle. The stresses are determined at the finite element grid nodes.

To take into account the nonlinearity of concrete deformations, an orthotropic model is used in the plane case. To obtain the necessary ratios for the orthotropic model, the current building regulations were used. The solution of the nonlinear problem is performed by the method of variable elasticity parameters, which shows fast convergence.

References


