Nonlinear Eigen frequencies of a functionally
graded porous nano-beam with respect to the
coulomb and Casimir forces

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Abstract. A new mathematical model of a porous functionally graded micro/nano-beam under the action of the Casimir force and the Coulomb force is constructed in this paper. The construction of the mathematical model takes into account the Euler-Bernoulli kinematic model. The dimensional-dependent parameter is taken into account by modified moment theory of elasticity. The variational, differential equations, boundary and initial conditions are derived from Ostrogradsky-Hamiltonian variational principle. The problem of non-linear natural oscillations of a beam under the action of force Casimir and Coulomb force is solved. The influence of the Casimir and Coulomb forces on the nonlinear eigenfrequencies of the micro/nano-beam is shown.

1 Introduction

New developments in nano-, micro-, and macroscale technology are mainly based on the development and production of various miniature inertial and external information sensors, micromotors, and transducers. The elements of nano-/micro-/macroscale technology are devices with electrical, optical, and micromechanical structures integrated into the volume or surface of a solid body. In most nano/microelectromechanical elements, there are two electrodes: a movable and a stationary one. A potential difference causes the mobile electrode to deflect. In addition to the Coulomb force, MEMS/NEMS beam elements are influenced by vacuum fluctuations, the Casimir force, which significantly affects the deflection of the beam from its original state and changes the natural frequency of the structural element. Quite a number of papers are now devoted to the phenomenon of retraction under the influence of the Casimir force [1-7]. In the work of Y. Tadi Beni [8], in which the static instability of nanostructures under the action of the Casimir force is investigated. Modified moment theory of elasticity, gradient theory of elasticity and surface theory of elasticity are used to describe the dimension dependent behaviour. The study of surface effect on static beam retraction is considered in [9 - 10]. The effect of temperature on the retraction effect under the Casimir force is considered in [11 -13]. The effect of marginal effect on retraction is considered in [14]. A limited number of articles have been devoted to nonlinear oscillations of micro/nanoelectromechanical beam structures [15 -17].

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under the action of Coulomb and Casimir forces. In the above papers, the authors consider homogeneous materials. Functionally graded porous beams under the action of Casimir and Coulomb forces were considered in [18]. The authors considered the effects of geometric nonlinearity, flexoelectricity, material distribution on the free vibrations of the beams and the retraction effect. However, the authors did not consider the influence of different types of porosity, the ratio of fractions of metallic and ceramic phases, the value of porosity of the material on the natural non-linear vibrations of the beam under the influence of Casimir and Coulomb forces. This paper eliminates the aforementioned gap.

2 Mathematical model

To construct a new mathematical model for the statics and dynamics of porous functionally graded Euler-Bernoulli nanoscale beams of length $a$, thickness $h$ and width $b$ is presented as a three-dimensional (3D) domain $\Omega$: $\Omega = \{x \in [0; a]; y \in [-b/2; b/2]; z \in [-h/2; h/2]\}, 0 \leq t \leq \infty$ of the space $\mathbb{R}^3$ in Cartesian rectangular coordinate system (Fig. 1). The origin of coordinates is located in the centre of the beam on its midline, $x$-axis, $y$-axis are parallel to the sides of the beam, $z$-axis is directed downwards. Next, we consider a girder of width equal to $b = 1$.

![Fig.1. Schematic diagram.](image)

Let the material properties of the nano-beam, such as Young's modulus, Poisson's ratio and density, be determined using the following relations [19]:

$$E(z) = (E_c - E_m(z)) \cdot \left(\frac{1}{2} + \frac{z}{h}\right)^{k} + E_m(z) - (E_c + E_m(z)) \cdot \varphi(z),$$

$$v(z) = (v_c - v_m(z)) \cdot \left(\frac{1}{2} + \frac{z}{h}\right)^{k} + v_m(z) - (v_c + v_m(z)) \cdot \varphi(z),$$

$$\rho(z) = (\rho_c - \rho_m(z)) \cdot \left(\frac{1}{2} + \frac{z}{h}\right)^{k} + \rho_m(z) - (\rho_c + \rho_m(z)) \cdot \varphi(z),$$

<table>
<thead>
<tr>
<th>Table 1. Types of porosity.</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-PFGM \quad \varphi(z) = \frac{r}{2}</td>
</tr>
</tbody>
</table>

![Image of table 1](image)
where $\Gamma$ is a porosity index (Table 1), $E_c$, $E_m$, $\nu_c$, $\nu_m$, $\rho_c$, $\rho_m$ - Young's modulus, Poisson's ratio and density associated with the ceramic and metal phases of the FGM. If $\Gamma = 0$, there are no pores. Note that in the proposed mathematical model Young's modulus, density and Poisson's ratio are considered as a function.

The non-zero components of the symmetric tensor of total deformation taking into account the Bernoulli-Hayler hypothesis will take the form:

$$e_{xx} = -x \frac{\partial^2 w}{\partial x^2}$$

here $w$ is deflection.

To account for dimension-dependent effects, the modified moment theory of elasticity [20] is taken into account. The components of the symmetric curvature gradient tensor will be:

$$\chi_{xy} = \frac{1}{2} \left( \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right)$$

Determining ratios for beam material are:

$$\sigma_{xx} = \frac{E(z)}{1-\nu(z)} e_{xx}, m_{xy} = \frac{E(z) l^2}{1+\nu(z)} \chi_{xy},$$

here $\sigma_{xx}$ is a Cauchy tensor component, $m_{xy}$ is a component of the symmetrical moment tensor of higher order, $l$ - is an additional independent material parameter of length.

The equations of motion of the nanobar element with consideration of electrostatic influences and Casimir force, the boundary and initial conditions are obtained from the Ostrogradsky-Hamilton energy principle:

$$\int_0^l (\delta K - \delta U + \delta W) \, dt = 0$$

Here $K$ is kinetic energy, $U$ is potential energy, $W$ - the work of external forces. Given the micropolar theory, the potential energy $U$ in an elastic body, is written in the form $U = \frac{1}{2} \int_O (\sigma_{ij} e_{ij} + m_{ij} \chi_{ij}) \, d\Omega$.

Kinetic energy $K = \frac{1}{2} \int_O \rho \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] \, d\Omega$. Variation in the work of external forces $\delta W = \delta W_{\text{coul}} + \delta W_{\text{elas}}$, $\delta W_{\text{coul}} = \int_0^a F_{\text{coul}}(\delta w) \, dx$ - variation in the work of the electrostatic force (Coulomb force), $\delta W_{\text{elas}} = \int_0^a F_{\text{elas}}(\delta w) \, dx$. $F_{\text{coul}} = \frac{e_c \varepsilon_0 V^2}{2(\delta_0 - w)^2}$ - Coulomb force, $F_{\text{elas}} = \frac{E_c \varepsilon_0 \varepsilon_0 V^2}{2(\delta_0 - w)^2}$ - Casimir's force, $\varepsilon_c$ - dielectric constant, $\varepsilon_0$ - electric constant, $V$ - potential difference between beam and electrode, $\hbar$ - Planck constant, $c$ - the speed of light in a vacuum.

$$\int_0^l \left( \frac{E(x) x^2}{1-\nu(x)} + \frac{E(x) l^2}{2(1+\nu(x))} \right) \, dx \frac{\partial^4 w}{\partial x^4} - \frac{\hbar c \pi}{240(\hbar_0 - w(x))^4} \frac{e_c \varepsilon_0 V^2}{2(\hbar_0 - w)^2} + \int_0^a \rho(z) \, dz \frac{\partial^2 w}{\partial t^2} = 0$$

The boundary conditions are rigid pinch (both ends $x = 0, x = a$ are fixed):

$$w(0, t) = w(a, t) = \frac{\partial w(0, t)}{\partial x} = \frac{\partial w(a, t)}{\partial x} = 0$$
The following initial conditions applied are:

\[ w(x, 0) = H \left( 1 - \cos \left( \frac{2\pi x}{a} \right) \right) \frac{\partial w(x, 0)}{\partial t} = 0 \]  

(10)

The following dimensionless parameters are introduced for the numerical analysis:

\[ \bar{w} = \frac{w}{h_0}, \quad \bar{u} = \frac{ua}{k_0}, \quad \bar{x} = \frac{x}{a}, \quad \bar{z} = \frac{z}{h_0}, \quad \bar{t} = \frac{t}{\sqrt{\rho_0}}, \quad \bar{\rho}(z) = \frac{\rho(z)}{\rho_0}, \quad \bar{E}(z) = \frac{E(z)}{E_0}, \quad \gamma = \frac{\gamma}{\alpha}; \quad \lambda = \frac{\lambda}{h_0} \]

\[ K_1 = \frac{e_v e_0 V^2 a^2}{2h_0 E_0}, \quad D_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\bar{E}(z) x^2}{(1+\nu(z))} \right) dz, \quad D_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\bar{E}(z) y^2}{(1+\nu(z))} \right) dz, \quad D_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\rho}(z) dz. \]

\[ K_2 = \frac{K_0}{(1-\bar{w})^4}, \quad q_e = \frac{K_1}{(1-\bar{w})^2} \]

here \( \gamma \) is a dimensionally dependent parameter, \( E_0 \) and \( \rho_0 \) - Young's modulus and density of ceramics.

The equation in dimensionless form will be:

\[ (D_0 + D_1) \frac{\partial^4 w}{\partial x^4} - q_k - q_e + D_2 \frac{\partial^2 w}{\partial t^2} = 0 \]  

(11)

Boundary conditions - rigid pinch point:

\[ w(0, t) = w(1, t) = \frac{\partial w(0, t)}{\partial x} = \frac{\partial w(1, t)}{\partial x} = 0 \]  

(12)

Initial conditions:

\[ w(x, 0) = H \left( 1 - \cos \left( 2\pi x \right) \right) \frac{\partial w(x, 0)}{\partial t} = 0 \]  

(13)

In dimensionless form, the dash in the equation is omitted.

3 Materials and methods

The partial differential equation (11-13) is non-linear, so it needs to be solved numerically. In order to obtain reliable results the solution of this equation was carried out by two different methods of nature: 1. Finite difference method with approximation \( O(\delta^2) \) by spatial coordinate and fourth-order Runge-Kutta method for numerical integration by temporal coordinate. 2. The Bubnov-Galerkin method.

3.1 Finite Difference Method

When solving the resulting system of equations, the finite difference method was used. For this purpose a uniform grid with the number of nodes \( n \) was imposed on the girder area. Partial derivatives on a spatial coordinate are replaced by central finite difference approximations. After the introduced replacement the system of equations is transformed to the system of ordinary differential equations.
The system is then reduced to the Cauchy problem with respect to the evolutionary variable, which is solved by the Runge-Kutta method of 4th order accuracy by the method of replacement of variables.

The convergence of the finite difference method is achieved for the number of partition points \( n = 40 \). Comparison of dependencies of the ratio of the square of the non-linear natural frequency to the square of the linear natural frequency \( \mu = \frac{\omega_n^2}{\omega_0^2} \) as a function of the amplitude of the initial deflection \( H \), obtained by the finite difference method for different values of \( n \), is shown in Figure 2. In this case the full function of the Casimir force and the Coulomb force is used.

### 3.2 The Bubnov-Galerkin method

When solving the eigenvalue problem using the Galerkin method, the deflection function was represented as: \( w_n = A(t)(1 - \cos(2n \pi x)) \). The function of the Casimir force and the Coulomb force are decomposed into a McLaren series:

\[
q_k = K_0 \left( 1 + 4w + 10w^2 + 20w^3 + 35w^4 + 56w^5 + 84w^6 + 120w^7 + 165w^8 + 220w^9 + \ldots \right),
\]

\[
q_\epsilon = K_1 \left( 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 + 7w^6 + 8w^7 + 9w^8 + 10w^9 + \ldots \right).
\]

We limit ourselves to the function \( q_k \) by 14 row members, and \( q_\epsilon \) - by eight terms of the series. A justification for this choice will be given when describing the convergence of the methods used.

Applying the Bubnov-Galerkin procedure we obtain a non-linear differential equation

\[
\begin{align*}
-1371172.8515625K_0A^{13}(t) & - 577670.0439453125K_0A^{12}(t) - \\
-240310.73828125K_0A^{11}(t) & - 985124.7665625K_0A^{10}(t) - \\
-39693.671875K_0A^9(t) & - 15668.5546875K_0A^8(t) - 6032.8125K_0A^7(t) - \\
-2252.25K_0A^6(t) & - 805.5K_0A^5(t) - 275.625K_0A^4(t) - 87.5K_0A^3(t) - \\
-25K_0A^2(t) & - 402.1875K_1A^2(t) - 187.6875K_1A^1(t) - 86.625K_1A^0(t) - \\
-39.375K_1A^4(t) & - 17.5K_1A^3(t) - \frac{60}{8}K_1A^2(t) - \frac{2}{3D_2} + \frac{d^2A(t)}{dt^2} = 0
\end{align*}
\]

where the square of the natural linear frequency of the beam including the linear part of the Casimir force and the Coulomb force is:

\[
\omega_0^2 = \frac{2}{3D_2} (8\pi^4n'(D0 + D1) - 6K_0 - 3K_1)
\]
Equation (15) does not have an exact solution; to solve it, we will use one of the approximate methods, the Bubnov-Galerkin method. For this purpose, the time-dependent deflection function will be represented as:

$$A(t) = H \cos(\omega t)$$

Let us apply the Bubnov-Galerkin procedure over a quarter of a period

$$\int_{\pi/2}^{\pi/2} \cos(\omega t) dt = 0$$

To analyse the amplitude-frequency relationship, express the ratio of the squared non-linear frequency to the linear natural frequency $\mu = \frac{\omega^2}{\omega_0^2}$ nano-beams under the action of the Casimir force and the Coulomb force.

$$\mu(H) = \frac{\omega^2}{\omega_0^2} = 1 - \frac{K_n}{\pi \omega_0^2 d_z} \left\{ \frac{50H}{3} + \frac{262.5\pi H^2}{16} + \frac{147H^3}{32} + \frac{4027.5\pi H^4}{32000} + \frac{36036H^5}{35} + \frac{211155\pi H^6}{256} + \frac{1408 \cdot 12252240H^7}{43008} + \frac{693 \cdot 232792560\pi H^8}{512 \cdot 64512} + \frac{3328 \cdot 232792560H^9}{693 \cdot 30720} + \frac{21021 \cdot 5354228880\pi H^{10}}{2048 \cdot 2027520} + \frac{5120 \cdot 26771144400H^{11}}{429 \cdot 1622016} \right\} - \frac{5H}{32} + \frac{105\pi H^2}{32} + \frac{21H^3}{256} + \frac{3465\pi H^4}{85.8H^4} + \frac{225225\pi H^5}{4096}$$

Preliminarily, a conclusion was made about the necessary number of terms in the expansion of the Casimir force function and the Coulomb force function into the McLaren series (Fig. 3). In all graphs in the Coulomb force decomposition into series $m = 8$. It was concluded that for a geometrically linear problem, the number of terms in the McLaren series expansion should be greater than eleven for the Casimir force and eight terms for the Coulomb force. Fig. 4 shows a comparison of results obtained by methods different in nature: finite difference method with approximation $O(\delta^2)$ and the Bubnov-Galerkin method.
In order to validate the results obtained, the solution was compared with the in-situ and numerical experiments described in [15 - 17] (Table 2).

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</thead>
<tbody>
<tr>
<td>210</td>
<td>27.638</td>
<td>27.496</td>
<td>27.575</td>
<td>27.95</td>
</tr>
</tbody>
</table>

4 Results and discussion

The influence of the parameter $k$ to change the dependency $\mu(H)$ was studied. The parameter $k$ is responsible for the ratio of the fraction of the ceramic phase to the metal phase. The higher the value of the parameter $k$, the more metal is contained in the material. The parameters used for the numerical experiment are shown in Table 3.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Base quantity</th>
<th>Base unit</th>
<th>Symbol</th>
<th>Base quantity</th>
<th>Base unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_m$</td>
<td>modulus of elasticity of pure aluminum (Al)</td>
<td>70000 MPa</td>
<td>$E_c$</td>
<td>modulus of elasticity of ceramics</td>
<td>410000 MPa</td>
</tr>
<tr>
<td>$\nu_m$</td>
<td>poisson's ratio of pure aluminum (Al)</td>
<td>0.35</td>
<td>$\nu_c$</td>
<td>poisson's ratio of ceramics</td>
<td>0.26</td>
</tr>
<tr>
<td>$\rho_m$</td>
<td>density of pure aluminum (Al)</td>
<td>2400 kg/m$^3$</td>
<td>$\rho_c$</td>
<td>density of ceramics</td>
<td>3100 kg/m$^3$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>dimensionless parameter</td>
<td>50</td>
<td>$\gamma$</td>
<td>dimensionless parameter</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Graphs $\mu(H)$ depending on $k$ are shown in Fig. 5. Material with a coarse porosity $\Gamma = 0.3$ was taken for analysis. The influence of the nonlinear Casimir and Coulomb function has a stronger effect on the pure aluminium beam than on the ceramic beam. By varying the ceramic-to-metal ratios in the material, the desired frequency response of the nanobar can be achieved. With a combination of large pores $\Gamma = 0.3; 0.4$ and metal to ceramic ratio $k = 5$
The frequency response of the beam will change more dramatically with changes in vibration amplitude than that of an aluminium beam. We consider the effect of the pore size in a functionally graded material. Fig. 6 shows the change in linear to non-linear natural frequency $\mu = \frac{\omega^2}{\omega_0^2}$ depending on the amplitude $H$ at different pore sizes $\Gamma = 0.1; 0.2; 0.3; 0.4$. Changing the pore size as well as the proportions of ceramic and metallic phases can both improve and degrade the performance of the material. With a fine porosity and a high ceramic phase content, the beam element will approach ceramic in its frequency response and will have a higher load-bearing capacity than brittle ceramics in terms of its physical characteristics. Let's consider how the amplitude-frequency characteristics change depending on the pore thickness distribution. Fig. 7 with the same parameters $\Gamma = 0.1$ and $k = 2$ show the dependencies $\mu(H)$ for a material with homogeneous porosity U-PFGM, material X- PFGM - large pores at the edge, small pores in the middle, material O- PFGM - small pores at the edge of the beam, large pores in the middle of the beam. When comparing the dependencies $\mu(H)$, it can be concluded that in terms of its frequency response, a beam made of material with X- PFGM porosity is closer to a ceramic beam, while a beam made of material with homogeneous porosity is closer to a metal beam.

![Fig. 5. Effect of the k-factor on the amplitude-frequency characteristics of a nano-beam with homogeneous porosity](image1)
![Fig. 6. Effect of G-factor on the amplitude-frequency characteristics of a nano-beam with homogeneous porosity](image2)
![Fig. 7. Influence of porosity type on the amplitude-frequency characteristics of the nano-beam](image3)

### 5 Conclusion

The mathematical model of a porous functional gradient micro/nano beam in an electrostatic field with consideration of the Casimir force is constructed. The mathematical model of vibration takes into account scale effects by means of modified moment theory of elasticity. The mathematical model is verified by comparing its results with experimental data and results of other authors. The problem of nonlinear eigenvibrations of a nanotube was solved. The effect of porosity type, pore size, and ratio of volume fractions of ceramic and metallic phases on the natural vibration frequency of a rigidly clamped functionally graded porous micro-/nano-beam is studied. The results may be useful in the design of MEMS/NEMS beams based on metal-ceramic materials.

### 6 Acknowledgements

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References