Finding the temperature distribution in a rectangular plate with variable thermal conductivity in one coordinate

Aleksandr Kanareykin¹,*

¹Sergo Ordzhonikidze Russian State University for Geological Prospecting, 23, Miklouho-Maclay str., 117997 Moscow, Russia

Abstract. The work is devoted to the issues of stationary heat transfer. The article presents a solution for the distribution of the temperature field in a rectangular plate. At the same time, the law of change of thermal conductivity along one of the coordinates is known. Heat exchange at the two opposite ends of the plate occurs under boundary conditions of the third kind, there is no heat exchange at the other two. The solution is found by decomposition into a functional series. As a result, an analytical expression of the plate temperature distribution is obtained in the form of a Fourier series containing modified zero-order Bessel functions.

1 Introduction

The increase in energy efficiency and compactness of heat exchangers is closely related to the intensification of heat exchange processes [1]. Many works are devoted to this. Two-dimensional problems of thermal conductivity can be singled out separately [2-4]. Similar problems usually arise when describing heat transfer processes in thin plates. Such processes, for example, are found in refrigeration units. At the same time, there are various methods for solving such problems. Analytical methods exist to solve them, but it is not possible to obtain the solution of some inhomogeneous and nonlinear problems of thermal conductivity by analytical methods. The solution of such problems is carried out using numerical methods. [5-7].

The paper considers a case with a linear change in thermal conductivity along one coordinate with given boundary conditions.

2 Main Part

Consider a homogeneous plate with the specified dimensions (fig. 1). At the same time, its thickness is significantly less than the length and width. In this case, heat exchange at the two opposite ends of the plate occurs at α = const, there is no heat exchange at the other two.

* Corresponding author: kanareykins@mail.ru
In the case of a change in thermal conductivity along one of the coordinates, the thermal conductivity equation will take the form

$$\frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \lambda \frac{\partial^2 T}{\partial y^2} = 0$$

(1)

In this case, the nonlinear equation (1) must satisfy the following boundary conditions: there is no heat exchange at the two opposite ends

$$\frac{\partial T(x,0)}{\partial y} = 0$$

(2)

$$\frac{\partial T(x,b)}{\partial y} = 0$$

(3)

and on the rest, heat exchange is carried out due to convection, obeying Newton's law, and heat is also supplied to them, which is a function of the y coordinate

$$-\lambda \frac{\partial T(0,y)}{\partial x} + \alpha_1 T(0,y) = q_1(y)$$

(4)

$$-\lambda \frac{\partial T(a,y)}{\partial x} + \alpha_2 T(a,y) = q_2(y)$$

(5)

where $\alpha_1$ and $\alpha_2$ are the given heat transfer coefficients, $q_1$ and $q_2$ are the heat flux densities.
Let the coefficient of thermal conductivity depend linearly on the x coordinate as follows

\[ \lambda = \lambda_0 + \lambda_1 x \]  

(6)

Then the functional dependence of the thermal conductivity can be represented as

\[ \lambda = \lambda_1 \rho \]  

(7)

In this case, the thermal conductivity equation (1) will be rewritten as follows

\[ \frac{\partial}{\partial \rho} \left( \rho \frac{\partial T}{\partial \rho} \right) + \rho \frac{\partial^2 T}{\partial y^2} = 0 \]  

(8)

and the boundary conditions (2, 3) will take the following form

\[ - \lambda_1 x_0 \frac{\partial T(x_0, y)}{\partial \rho} + \alpha_1 T(x_0, y) = q_1(y) \]  

(9)

\[ - \lambda_1 (x_0 + a) \frac{\partial T(x_0 + a, y)}{\partial x} + \alpha_2 T(x_0 + a, y) = q_2(y) \]  

(10)

We are looking for a solution to problem (8) in the form of a functional series

\[ T = \sum_{n=0}^{\infty} T_n(\rho) \cos \frac{n \pi y}{b} \]  

(11)

Substitute this solution into equation (8). After separating the variables, we get the following expression

\[ \frac{d^2 T_n}{d \rho^2} + \frac{1}{\rho} \frac{d T_n}{d \rho} - \frac{n^2 \pi^2}{b^2} T_n = 0, n = 0,1,2,\ldots, \infty \]  

(12)

Its solution is

\[ T_n = A_n I_0 \left( \frac{n \pi \rho}{b} \right) + B_n K_0 \left( \frac{n \pi \rho}{b} \right), n = 0,1,2,\ldots, \infty \]  

(13)

For \( n = 0 \), equation (12) will take the form
\[
\frac{d^2 T}{d\rho^2} + \frac{1}{\rho} \frac{dT}{d\rho} = 0
\]

(14)

Its solution is

\[
T_0 = A_0 + B_0 \ln \rho
\]

(15)

To obtain boundary conditions, we decompose the functions \(q_1(y)\) and \(q_2(y)\) into a Fourier series

\[
q_1(y) = \sum_{n=0}^{\infty} q_{1n} \cos \frac{n\pi y}{b}
\]

(16)

\[
q_2(y) = \sum_{n=0}^{\infty} q_{2n} \cos \frac{n\pi y}{b}
\]

(17)

In turn, the Fourier coefficients \(q_{1n}\) and \(q_{2n}\) are determined as follows

\[
q_{1n} = \frac{2}{b} \int_{0}^{b} q_1(y) \cos \frac{n\pi y}{b} dy
\]

(18)

\[
q_{2n} = \frac{2}{b} \int_{0}^{b} q_2(y) \cos \frac{n\pi y}{b} dy
\]

(19)

For \(n = 0\) we have

\[
q_{10} = \frac{1}{b} \int_{0}^{b} q_1(y) dy
\]

(20)

\[
q_{20} = \frac{1}{b} \int_{0}^{b} q_2(y) dy
\]

(21)

In this case, the boundary conditions for the \(T_n\) functions will be written as follows

\[
-\lambda_1 x_0 \frac{dT_n(x_0)}{d\rho} + \alpha_1 T_n(x_0) = q_{1n}
\]

(22)
\[- \lambda_1 (x_0 + a) \frac{dT_n(x_0 + a)}{d\rho} + \alpha_2 T(x_0 + a) = q_{2n} \]  

(23)

First, we define the coefficients \( A_0 \) and \( B_0 \). Why solve a system of two equations

\[- \lambda_1 B_0 + \alpha_1 (A_0 + B_0 \ln x_0) = q_{10} \]  

(24)

\[- \lambda_1 B_0 + \alpha_2 [A_0 + B_0 \ln(x_0 + a)] = q_{20} \]  

(25)

where from

\[ A_0 = \frac{\alpha_2 \ln(x_0 + a) - \lambda_1 \frac{\alpha_2}{\alpha_1} q_{10} + (\lambda_1 - \alpha_1 \ln x_0)q_{20}}{\alpha_1 \alpha_2 \ln \left(1 + \frac{a}{x_0}\right)} \]  

(26)

\[ B_0 = \frac{\alpha_1 q_{20} - \alpha_2 q_{10}}{\alpha_1 \alpha_2 \ln \left(1 + \frac{a}{x_0}\right)} \]  

(27)

To determine the unknown coefficients \( A_n \) and \( B_n \), it will be necessary to solve the following system of equations

\[ \begin{bmatrix} \alpha_1 I_0 \left( \frac{n \pi x_0}{b} \right) - \lambda_1 \frac{n \pi x_0}{b} I_1 \left( \frac{n \pi x_0}{b} \right) \end{bmatrix} A_n + \begin{bmatrix} \alpha_1 K_0 \left( \frac{n \pi \rho}{b} \right) + \lambda_1 \frac{n \pi x_0}{b} K_1 \left( \frac{n \pi \rho}{b} \right) \end{bmatrix} B_n = q_{10} \]  

(28)

\[ \begin{bmatrix} \alpha_1 I_0 \left( \frac{n \pi (x_0 + a)}{b} \right) + \lambda_1 \frac{n \pi x_0}{b} I_1 \left( \frac{n \pi (x_0 + a)}{b} \right) \end{bmatrix} + \]

\[ + \begin{bmatrix} \alpha_1 K_0 \left( \frac{n \pi (x_0 + a)}{b} \right) - \lambda_1 \frac{n \pi (x_0 + a)}{b} K_1 \left( \frac{n \pi (x_0 + a)}{b} \right) \end{bmatrix} B_n = q_{2n} \]  

(29)

The solution of the resulting system is found by Kramer’s formulas

\[ A_n = \frac{\Delta_1}{\Delta} \]  

(30)
\[ B_n = \frac{\Delta_2}{\Delta} \]  

(31)

where

\[
\Delta = \left[ I_0\left(\frac{n\pi x_0}{b}\right)K_0\left(\frac{n\pi(x_0 + a)}{b}\right) - K_0\left(\frac{n\pi x_0}{b}\right)I_1\left(\frac{n\pi(x_0 + a)}{b}\right)\right] \alpha_1^2 - \\
- \left\{ \frac{n\pi x_0}{b} \left[ I_1\left(\frac{n\pi x_0}{b}\right)K_0\left(\frac{n\pi(x_0 + a)}{b}\right) + K_0\left(\frac{n\pi x_0}{b}\right)I_1\left(\frac{n\pi(x_0 + a)}{b}\right)\right] + \right. \\
+ \frac{n\pi(x_0 + a)}{b}\left[ I_0\left(\frac{n\pi x_0}{b}\right)K_1\left(\frac{n\pi(x_0 + a)}{b}\right) + K_0\left(\frac{n\pi x_0}{b}\right)I_1\left(\frac{n\pi(x_0 + a)}{b}\right)\right]\right\} \alpha_1 \lambda_1 + \\
+ \left[ I_1\left(\frac{n\pi x_0}{b}\right)K_1\left(\frac{n\pi(x_0 + a)}{b}\right) - K_1\left(\frac{n\pi x_0}{b}\right)I_1\left(\frac{n\pi(x_0 + a)}{b}\right)\right] \lambda_1^2 \frac{n\pi x_0}{b} \frac{n\pi(x_0 + a)}{b} \\
\Delta_1 = \left[ \alpha_1 K_0\left(\frac{n\pi(x_0 + a)}{b}\right) - \lambda_1 \frac{n\pi(x_0 + a)}{b} K_1\left(\frac{n\pi(x_0 + a)}{b}\right)\right] q_{1n} - \\
- \left[ \alpha_1 K_0\left(\frac{n\pi x_0}{b}\right) + \lambda_1 \frac{n\pi x_0}{b} K_1\left(\frac{n\pi x_0}{b}\right)\right] q_{2n} \\
\Delta_2 = \left[ \alpha_1 I_0\left(\frac{n\pi(x_0 + a)}{b}\right) + \lambda_1 \frac{n\pi(x_0 + a)}{b} I_1\left(\frac{n\pi(x_0 + a)}{b}\right)\right] q_{1n} + \\
+ \left[ \alpha_1 I_0\left(\frac{n\pi x_0}{b}\right) - \lambda_1 \frac{n\pi x_0}{b} I_1\left(\frac{n\pi x_0}{b}\right)\right] q_{2n} \\
(32) 

3 Conclusions

In this paper, we solved the problem of finding a temperature field in a rectangular plate with a given linear change in thermal conductivity along one of the coordinates at adiabatically isolated opposite boundaries and under given boundary conditions of the third kind on the other two. An analytical expression was obtained for finding the temperature field of the plate in the form of a series containing modified zero-order Bessel functions.
References

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