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Numerical solution of problem of equilibrium of parallelepiped in stresses

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Abstract. Usually, planar and spatial problems of the theory of elasticity in stresses are solved using the Airy or Maxwell stress function, respectively. Solving a plane problem usually comes down to solving a biharmonic equation. In the spatial case, it can be solved by the Castigliano variational method. This work is devoted to the mathematical and numerical solution of the spatial problem of elasticity theory directly concerning stresses. In this case, the boundary value problem consists of three equilibrium equations and three Beltrami-Michell equations with the corresponding boundary conditions. At the same time, equilibrium equations are considered as missing boundary conditions. Symmetric finite-difference equations are constructed, and the problems of equilibrium of a parallelepiped under the action of domed and uniformly distributed loads are solved numerically. The reliability of the results obtained is substantiated by comparing the numerical results with the well-known Filonenko-Borodich solution and with the results of solving the boundary value problem of the equilibrium of the parallelepiped with respect to displacements.

1 Introduction

The formulation of the boundary value problem of the theory of elasticity concerning stresses is an actual problem in solid mechanics. It is known that boundary value problems in stresses are based on the Saint-Venant conditions on the continuity of deformations and can be written in the form of Beltrami-Michell equations [1]. Then the boundary value problem in stresses consists of six Beltrami-Michell differential equations, three equilibrium equations, and three boundary conditions. In this case, the problem of insufficiency of boundary conditions is solved by considering the equilibrium equation on the boundary of a given domain [2]. In the three-dimensional case, the boundary value problem in stresses was first solved by Filonenko-Borodich [3] by a variational method based on the expansion of the stress tensor in a series concerning trigonometric functions and continued by others [5, 6]. Two-dimensional stress problems are usually solved by introducing the Airy's stress function satisfying the equilibrium equation, and reduced to solving a biharmonic equation [7, 8]. In [9, 10], Maxwell-type stress functions were used to solve the spatial elasticity problems in stresses, similar to the two-dimensional case.

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In the works of Pobedri [11-13], a new formulation of the boundary value problem in stresses is proposed, where the equilibrium equations are considered boundary conditions. In a particular case, the classical Beltrami-Michell equations follow from the new setting. The work [14] also studies the new Pobedri boundary value problem in stresses. The issues of the existence and uniqueness of the solution of boundary value problems in stresses and the equivalence of new and classical formulations are considered in the following work [15]. Dynamic boundary value problems concerning stresses are considered in the works of Konovalov [16].

This work is devoted to formulating and numerical solution of the spatial problem of elasticity theory directly concerning stresses. The problem of the equilibrium of a parallelepiped under the action of domed and uniformly distributed loads is considered. Discrete equations are constructed by the finite difference method and solved by the iterative method.

2 Formulation of the boundary value problem of the theory of elasticity in stresses

It is known that the boundary value problem of the theory of elasticity in stresses consists of the equilibrium equation, the Beltrami-Michell equation with the corresponding boundary conditions [1,2], i.e.

$$\sigma_{ij,j} + X_i = 0 \tag{1}$$

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu}\sigma_{,ij} = -(X_{i,j} + X_{j,i}) - \frac{\nu}{1-\nu}\delta_{ij}X_{k,k}, \quad \sigma = \sigma_{kk} \tag{2}$$

$$\sigma_{ij}n_j\Big|_{\Sigma_1} = S_i \tag{3}$$

where $\sigma_{ij} -$ is stress tensor, $\lambda,\ \mu -$ are elastic Lame constants, $\theta -$ is spherical part of strain tensor, $X_i -$ is body forces, $S_i -$ is surface load, $\Sigma -$ is surface , $V -$ is volume, $\delta_{ij} -$ is Kronecker symbol, $n_i -$ is components of the outer normal to the surface Σ, $\nu = \lambda/(\lambda+\mu)/2 -$ is Poisson's ratio, $\nabla^2 -$ is Laplace operator.

In the absence of body forces, the boundary value problem (1-3) has the form [11]:

$$\sigma_{ij,j} = 0 \tag{4}$$

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu}\sigma_{,ij} = 0, \quad \sigma = \sigma_{kk} \tag{5}$$

$$\sigma_{ij}n_j\Big|_{\Sigma_2} = S_i \tag{6}$$

The equations (4-6) system consists of six Beltrami-Michell equations and three equilibrium equations with appropriate boundary conditions. It can be seen from this system that the number of differential equations is nine, the number of boundary conditions is three, and the boundary value problem is not closed.

According to the studies of Pobedri [2, 11, 14], considering the equilibrium equation on the boundary of a given area as a boundary condition, i.e.

$$\sigma_{ij,j}\Big|_{\Sigma} = 0 \tag{7}$$

Combined with equations (4-6), this represents the classical boundary value problem of the theory of elasticity in stresses.

Note that consideration of the equilibrium equation on the boundary of the region V under consideration allows us to formulate the boundary value problem of the theory in a closed form concerning stresses. Multiplying the Beltrami-Michell equations by δ_{ij}, one can show the validity of the following harmonic equation

$$\nabla^2 (\sigma_x + \sigma_y + \sigma_z) = 0 \tag{8}$$

It is known that in the case of a plane problem, the strain compatibility equation, using Hooke's law and the equilibrium equation, can be reduced to the form

$$\nabla^2 (\sigma_x + \sigma_y) = 0, \tag{9}$$

which also follows from (7) in the case of plane deformation.

3 Boundary value problem in stresses for a parallelepiped

In [19, 20], it is shown that in the case of a plane problem, the boundary value problem in stresses can be composed of the third Beltrami-Michell equation and two equilibrium equations. Similarly to the plane problem of the theory of elasticity, the spatial problem in stresses can be formulated as a system of partial differential equations consisting of three equilibrium equations and three Beltrami-Michell equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{10}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \tag{11}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{12}$$

$$\frac{\partial^2 \sigma_{xy}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial y^2} + \frac{\partial^2 \sigma_{xy}}{\partial z^2} + \frac{1}{1+\nu}\left(\frac{\partial^2 \sigma_x}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial x \partial y} + \frac{\partial^2 \sigma_z}{\partial x \partial y}\right) = 0 \tag{13}$$

$$\frac{\partial^2 \sigma_{xz}}{\partial x^2} + \frac{\partial^2 \sigma_{xz}}{\partial y^2} + \frac{\partial^2 \sigma_{xz}}{\partial z^2} + \frac{1}{1+\nu}\left(\frac{\partial^2 \sigma_x}{\partial x \partial z} + \frac{\partial^2 \sigma_y}{\partial x \partial z} + \frac{\partial^2 \sigma_z}{\partial x \partial z}\right) = 0 \tag{14}$$

$$\frac{\partial^2 \sigma_{yz}}{\partial x^2} + \frac{\partial^2 \sigma_{yz}}{\partial y^2} + \frac{\partial^2 \sigma_{yz}}{\partial z^2} + \frac{1}{1+\nu}\left(\frac{\partial^2 \sigma_x}{\partial y \partial z} + \frac{\partial^2 \sigma_y}{\partial y \partial z} + \frac{\partial^2 \sigma_z}{\partial y \partial z}\right) = 0 \tag{15}$$

with appropriate boundary conditions

$$\begin{aligned}
&(\sigma_x n_1 + \sigma_{xy} n_2 + \sigma_{xz} n_3)\big|_{\Sigma_2} = S_1, \\
&(\sigma_{xy} n_1 + \sigma_y n_2 + \sigma_{yz} n_3)\big|_{\Sigma_2} = S_2, \\
&(\sigma_{xz} n_1 + \sigma_{yz} n_2 + \sigma_z n_3)\big|_{\Sigma_2} = S_3,
\end{aligned} \tag{16}$$

As the missing boundary conditions, according to the works of Pobedri [2], we consider the equilibrium equations on the boundary of a given area, i.e.

$$\begin{aligned}
&\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right)\Bigg|_{\Sigma} = 0, \\
&\left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z}\right)\Bigg|_{\Sigma} = 0 \\
&\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right)\Bigg|_{\Sigma} = 0
\end{aligned} \tag{17}$$

Thus, equation (10-17) is a spatial boundary value problem of the theory of elasticity in stresses.

Consider the well-known boundary value problem (10-17) on the equilibrium of a parallelepiped of size (2a,2b,2c) with the origin of the coordinate system at the center of the parallelepiped [3]. Let the parallelepiped be in equilibrium under the action of the load S applied on opposite faces perpendicular to the OZ axis; the remaining faces are free from loads. Let the parallelepiped be under the action of compressive forces from both sides along the OX axis; the other sides are free from loads (Fig. 1) i.e.

$$\begin{aligned}
&\sigma_z\big|_{z=\pm c} = S, \\
&\sigma_{xz}\big|_{z=\pm c} = 0, \quad \sigma_{yz}\big|_{z=\pm c} = 0
\end{aligned} \tag{18}$$

According to the boundary conditions (16), the corresponding normal and shear stresses are set on the remaining faces. For example,

for $x=a$:

$$\sigma_x = 0,\ \sigma_{xy} = 0,\ \sigma_{xz} = 0 \tag{19}$$

Three additional conditions are required on each face of the parallelepiped. They can be found in additional conditions (17).

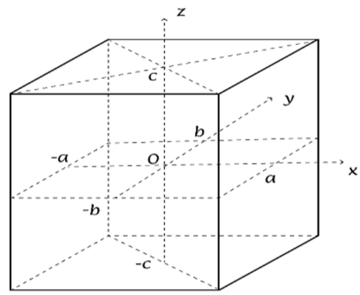

Fig. 1. Parallelepiped with faces and shear stresses

For example, on the face x=a, the condition for σ_y can be obtained from $(17)_2$, which, taking into account (19), has the form

$$\frac{\partial \sigma_y}{\partial y}\Big|_{x=a} = \frac{\partial \sigma_{yz}}{\partial z}\Big|_{x=a} \tag{20}$$

Similarly, from $(17)_3$, one can find the condition for σ_z i.e.

$$\frac{\partial \sigma_z}{\partial z}\Big|_{x=a} = \frac{\partial \sigma_{yz}}{\partial y}\Big|_{x=a} \tag{21}$$

The condition for σ_{yz} is chosen from the following:

$$\sigma_{yz}\Big|_{x=a} = \sigma_{yz}\Big|_{x=a-h} \tag{22}$$

Since the problem under consideration is solved by the finite difference method, at small grid steps the value of $\sigma_{yz}\Big|_{x=a}$ can be approximately equal to the value of $\sigma_{yz}\Big|_{x=a-h_1}$ from the previous neighboring layer. Similarly, boundary and additional boundary conditions can be defined for the remaining faces.

Thus, it is easy to see from relations (18-22) that there are six boundary conditions on each face of the parallelepiped.

4 Finite-difference equations for the spatial problem of the theory of elasticity in stresses

Let the boundary value problem (10-17) be considered in a parallelepiped with edge lengths l_1 = 2a, l_2 = 2b, and l_3 = 2c (Fig. 1). By dividing the lengths of the sides of the parallelepiped by N_r, one can find the grid step $h_r = l_r / N_r, r = \overline{1,3}$ along the coordinate axes. Then the coordinates of the nodal points have the form $x_i = ih_1$ $(i = \overline{0, n_1})$

, $y_j = jh_2 \ (j = \overline{0, n_2})$, $z_k = kh_3 \ (\mathrm{k} = \overline{0, n_3})$ [6, 3]. By replacing the derivatives in equations (10-15) with the corresponding finite-difference relations, we can find that

$$\frac{\sigma^{x}_{i+1,j,k} - \sigma^{x}_{i,j,k}}{h_1} + \frac{\sigma^{xy}_{i,j+1,k} - \sigma^{xy}_{i,j-1,k}}{2h_2} + \frac{\sigma^{xz}_{i,j,k+1} - \sigma^{xz}_{i,j,k-1}}{2h_3} = 0, \tag{23}$$

$$\frac{\sigma^{y}_{i,j+1,k} - \sigma^{y}_{i,j,k}}{h_2} + \frac{\sigma^{xy}_{i+1,j,k} - \sigma^{xy}_{i-1,j,k}}{2h_1} + \frac{\sigma^{yz}_{i,j,k+1} - \sigma^{yz}_{i,j,k-1}}{2h_3} = 0 \tag{24}$$

$$\frac{\sigma^{z}_{i,j,k+1} - \sigma^{z}_{i,j,k}}{h_3} + \frac{\sigma^{xz}_{i+1,j,k} - \sigma^{xz}_{i-1,j,k}}{2h_1} + \frac{\sigma^{yz}_{i,j+1,k} - \sigma^{yz}_{i,j-1,k}}{2h_2} = 0 \tag{25}$$

$$\frac{\sigma^{xy}_{i+1,j,k} - 2\sigma^{xy}_{ijk} + \sigma^{xy}_{i-1,j,k}}{h_1^2} + \frac{\sigma^{xy}_{i,j+1,k} - 2\sigma^{xy}_{ijk} + \sigma^{xy}_{i,j-1,k}}{h_2^2} + \frac{\sigma^{xy}_{i,j,k+1} - 2\sigma^{xy}_{ijk} + \sigma^{xy}_{i,j,k-1}}{h_3^2} + \frac{1}{1+v}\{\frac{\sigma^{x}_{i+1,j+1,k} - \sigma^{x}_{i-1,j+1,k} - \sigma^{x}_{i+1,j-1,k} + \sigma^{x}_{i-1,j-1,k}}{4h_1h_2} + \frac{\sigma^{y}_{i+1,j+1,k} - \sigma^{y}_{i-1,j+1,k} - \sigma^{y}_{i+1,j-1,k} + \sigma^{y}_{i-1,j-1,k}}{4h_1h_2} + \frac{\sigma^{z}_{i+1,j+1,k} - \sigma^{z}_{i-1,j+1,k} - \sigma^{z}_{i+1,j-1,k} + \sigma^{z}_{i-1,j-1,k}}{4h_1h_2}\} = 0 \tag{26}$$

$$\frac{\sigma^{xz}_{i+1,j,k} - 2\sigma^{xz}_{ijk} + \sigma^{xz}_{i-1,j,k}}{h_1^2} + \frac{\sigma^{xz}_{i,j+1,k} - 2\sigma^{xz}_{ijk} + \sigma^{xz}_{i,j-1,k}}{h_2^2} + \frac{\sigma^{xz}_{i,j,k+1} - 2\sigma^{xz}_{ijk} + \sigma^{xz}_{i,j,k-1}}{h_3^2} + \frac{1}{1+v}\{\frac{\sigma^{x}_{i+1,j+1,k} - \sigma^{x}_{i-1,j+1,k} - \sigma^{x}_{i+1,j-1,k} + \sigma^{x}_{i-1,j-1,k}}{4h_1h_3} + \frac{\sigma^{y}_{i+1,j+1,k} - \sigma^{y}_{i-1,j+1,k} - \sigma^{y}_{i+1,j-1,k} + \sigma^{y}_{i-1,j-1,k}}{4h_1h_3} + \frac{\sigma^{z}_{i+1,j+1,k} - \sigma^{z}_{i-1,j+1,k} - \sigma^{z}_{i+1,j-1,k} + \sigma^{z}_{i-1,j-1,k}}{4h_1h_3}\} = 0 \tag{27}$$

$$\frac{\sigma^{yz}_{i+1,j,k} - 2\sigma^{yz}_{ijk} + \sigma^{yz}_{i-1,j,k}}{h_1^2} + \frac{\sigma^{yz}_{i,j+1,k} - 2\sigma^{yz}_{ijk} + \sigma^{yz}_{i,j-1,k}}{h_2^2} + \frac{\sigma^{yz}_{i,j,k+1} - 2\sigma^{yz}_{ijk} + \sigma^{yz}_{i,j,k-1}}{h_3^2} + \frac{1}{1+v}\{\frac{\sigma^{x}_{i+1,j+1,k} - \sigma^{x}_{i-1,j+1,k} - \sigma^{x}_{i+1,j-1,k} + \sigma^{x}_{i-1,j-1,k}}{4h_2h_3} + \frac{\sigma^{y}_{i+1,j+1,k} - \sigma^{y}_{i-1,j+1,k} - \sigma^{y}_{i+1,j-1,k} + \sigma^{y}_{i-1,j-1,k}}{4h_2h_3} + \frac{\sigma^{z}_{i+1,j+1,k} - \sigma^{z}_{i-1,j+1,k} - \sigma^{z}_{i+1,j-1,k} + \sigma^{z}_{i-1,j-1,k}}{4h_2h_3}\} = 0 \tag{28}$$

Replacing the first term in equation (23) with a left finite relation, one can find that

$$\frac{\sigma^{x}_{i,j,k} - \sigma^{x}_{i-1,j,k}}{h_1} + \frac{\sigma^{xy}_{i,j+1,k} - \sigma^{xy}_{i,j-1,k}}{2h_2} + \frac{\sigma^{xz}_{i,j,k+1} - \sigma^{xy}_{i,j,k-1}}{2h_3} = 0, \tag{29}$$

From relations (23) and (29), we can find that

$$2\sigma^{x}_{ijk} = \sigma^{x}_{i+1,j,k} + \sigma^{x}_{i-1,j,k} \tag{30}$$

Similarly, from equation (4.2), we find

$$2\sigma^{y}_{ijk} = \sigma^{y}_{i,j+1,k} + \sigma^{y}_{i,j-1,k} \tag{31}$$

$$2\sigma^{z}_{ijk} = \sigma^{z}_{i,j,k+1} + \sigma^{z}_{i,j,k-1} \tag{32}$$

By resolving the finite difference equations (26-28) and (30-32), concerning $\sigma^{x}_{ijk}, \sigma^{y}_{ijk}, \sigma^{z}_{ijk}$ and $\sigma^{xy}_{ijk}, \sigma^{xz}_{ijk}, \sigma^{yz}_{ijk}$ and assuming the following renaming $\sigma^{x}_{ijk} = \sigma^{(t)}_{x\,ijk}$, $\sigma^{y}_{ij} = \sigma^{(t)}_{y\,ijk}$, $\sigma^{z}_{ij} = \sigma^{(t)}_{z\,ijk}$, and $\sigma^{xy}_{ijk} = \sigma^{(t)}_{xy\,ijk}$, $\sigma^{xz}_{ijk} = \sigma^{(t)}_{xz\,ijk}$, $\sigma^{yz}_{ijk} = \sigma^{(t)}_{yz\,ijk}$ can be found, respectively

$$\sigma^{(t+1)}_{x\,ijk} = \frac{\sigma^{(t)}_{x\,i+1,j,k} + \sigma^{(t)}_{x\,i-1,j,k}}{2}, \tag{33}$$

$$\sigma^{(t+1)}_{y\,ijk} = \frac{\sigma^{(t)}_{y\,i,j+1,k} + \sigma^{(t)}_{y\,i,j-1,k}}{2} \tag{34}$$

$$\sigma^{(t+1)}_{z\,ijk} = \frac{\sigma^{(t)}_{z\,i,j+1,k} + \sigma^{(t)}_{z\,i,j-1,k}}{2} \tag{35}$$

$$\sigma^{(t+1)}_{xy\,ijk} = \{\frac{\sigma^{(t)}_{xy\,i+1,j,k} + \sigma^{(t)}_{xy\,i-1,j,k}}{h_1^2} + \frac{\sigma^{(t)}_{xy\,i,j+1,k} + \sigma^{(t)}_{xy\,i,j-1,k}}{h_2^2} + \frac{\sigma^{(t)}_{xy\,i,j,k+1} + \sigma^{(t)}_{xy\,i,j,k-1}}{h_3^2} + \frac{1}{1+v}\{\frac{\sigma^{(t)}_{x\,i+1,j+1,k} - \sigma^{(t)}_{x\,i-1,j+1,k} - \sigma^{(t)}_{x\,i+1,j-1,k} + \sigma^{(t)}_{x\,i-1,j-1,k}}{4h_1h_2} + \frac{\sigma^{(t)}_{y\,i+1,j+1,k} - \sigma^{(t)}_{y\,i-1,j+1,k} - \sigma^{(t)}_{y\,i+1,j-1,k} + \sigma^{(t)}_{y\,i-1,j-1,k}}{4h_1h_2} + \frac{\sigma^{(t)}_{z\,i+1,j+1,k} - \sigma^{(t)}_{z\,i-1,j+1,k} - \sigma^{(t)}_{z\,i+1,j-1,k} + \sigma^{(t)}_{z\,i-1,j-1,k}}{4h_1h_2}\}\} / \{\frac{2}{h_1^2} + \frac{2}{h_2^2} + \frac{2}{h_3^2}\} \tag{36}$$

$$\sigma^{(t+1)}_{xz\,ijk} = \{\frac{\sigma^{(t)}_{xz\,i+1,j,k} + \sigma^{(t)}_{xz\,i-1,j,k}}{h_1^2} + \frac{\sigma^{(t)}_{xz\,i,j+1,k} + \sigma^{(t)}_{xz\,i,j-1,k}}{h_2^2} + \frac{\sigma^{(t)}_{xz\,i,j,k+1} + \sigma^{(t)}_{xz\,i,j,k-1}}{h_3^2} + \frac{1}{1+v}\{\frac{\sigma^{(t)}_{x\,i+1,j+1,k} - \sigma^{(t)}_{x\,i-1,j+1,k} - \sigma^{(t)}_{x\,i+1,j-1,k} + \sigma^{(t)}_{x\,i-1,j-1,k}}{4h_1h_3} + \frac{\sigma^{(t)}_{y\,i+1,j+1,k} - \sigma^{(t)}_{y\,i-1,j+1,k} - \sigma^{(t)}_{y\,i+1,j-1,k} + \sigma^{(t)}_{y\,i-1,j-1,k}}{4h_1h_3} + \frac{\sigma^{(t)}_{z\,i+1,j+1,k} - \sigma^{(t)}_{z\,i-1,j+1,k} - \sigma^{(t)}_{z\,i+1,j-1,k} + \sigma^{(t)}_{z\,i-1,j-1,k}}{4h_1h_3}\}\} / \{\frac{2}{h_1^2} + \frac{2}{h_2^2} + \frac{2}{h_3^2}\} \tag{37}$$

$$\sigma^{(t+1)}_{yz\,ijk} = \{\frac{\sigma^{(t)}_{yz\,i+1,j,k} + \sigma^{(t)}_{yz\,i-1,j,k}}{h_1^2} + \frac{\sigma^{(t)}_{yz\,i,j+1,k} + \sigma^{(t)}_{yz\,i,j-1,k}}{h_2^2} + \frac{\sigma^{(t)}_{yz\,i,j,k+1} + \sigma^{(t)}_{yz\,i,j,k-1}}{h_3^2} + \frac{1}{1+v}\{\frac{\sigma^{(t)}_{x\,i+1,j+1,k} - \sigma^{(t)}_{x\,i-1,j+1,k} - \sigma^{(t)}_{x\,i+1,j-1,k} + \sigma^{(t)}_{x\,i-1,j-1,k}}{4h_2h_3} + \frac{\sigma^{(t)}_{y\,i+1,j+1,k} - \sigma^{(t)}_{y\,i-1,j+1,k} - \sigma^{(t)}_{y\,i+1,j-1,k} + \sigma^{(t)}_{y\,i-1,j-1,k}}{4h_2h_3} + \frac{\sigma^{(t)}_{z\,i+1,j+1,k} - \sigma^{(t)}_{z\,i-1,j+1,k} - \sigma^{(t)}_{z\,i+1,j-1,k} + \sigma^{(t)}_{z\,i-1,j-1,k}}{4h_2h_3}\}\} / \{\frac{2}{h_1^2} + \frac{2}{h_2^2} + \frac{2}{h_3^2}\} \tag{38}$$

where indices change at interior points i.e., $1 \le i \le n_1 - 1$, $1 \le j \le n_2 - 1$, $1 \le k \le n_3 - 1$, t is iteration number.
In this case, according to (18), the boundary conditions have the form:
for $x = \mp a$:

$$\begin{matrix} \sigma^{(0)}_{x\,0j,k} = 0, & \sigma^{(0)}_{xy\,0j,k} = 0, & \sigma^{(0)}_{xz\,0j,k} = 0, \\ \sigma^{(0)}_{x\,N_1j,k} = 0, & \sigma^{(0)}_{xy\,N_1j,k} = 0 & \sigma^{(0)}_{xz\,N_1j,k} = 0 \end{matrix} \tag{39}$$

for $y = \mp b$:

$$\begin{matrix} \sigma^{(0)}_{y\,i,0,k} = 0, & \sigma^{(0)}_{xy\,i,0,k} = 0, & \sigma^{(0)}_{yz\,i,0,k} = 0, \\ \sigma^{(0)}_{y\,iN_2,k} = 0, & \sigma^{(0)}_{xy\,iN_2,k} = 0 & \sigma^{(0)}_{yz\,iN_2,k} = 0 \end{matrix} \tag{40}$$

for $z = \mp c$:

$$\begin{matrix} \sigma^{(0)}_{z\,i,j,0} = S, & \sigma^{(0)}_{xz\,i,j,0} = 0, & \sigma^{(0)}_{yz\,i,j,0} = 0, \\ \sigma^{(0)}_{z\,i,j,N_2} = S, & \sigma^{(0)}_{xz\,i,j,N_2} = 0 & \sigma^{(0)}_{yz\,i,j,N_2} = 0 \end{matrix} \tag{41}$$

According to relations (20-22), additional boundary conditions have the form:
For $x = \mp a$:

$$
\begin{aligned}
&\sigma_{y\ 0,j,k}^{(0)} = \sigma_{y\ 0,j+1,k}^{(0)} + h_2 \frac{\sigma_{yz\ 0,j,k+1}^{(0)} - \sigma_{yz\ 0,j,k-1}^{(0)}}{2h_3}, \\
&\sigma_{y\ n_1 j,k}^{(0)} = \sigma_{y\ n_1,j+1,k}^{(0)} + h_2 \frac{\sigma_{yz\ n_1,j,k+1}^{(0)} - \sigma_{yz\ n_1,j,k-1}^{(0)}}{2h_3} \\
&\sigma_{z\ 0,j,k}^{(0)} = \sigma_{z\ 0,j,k+1}^{(0)} + h_3 \frac{\sigma_{yz\ 0,j+1,k}^{(0)} - \sigma_{yz\ 0,j-1,k}^{(0)}}{2h_2}, \\
&\sigma_{z\ n_1 j,k}^{(0)} = \sigma_{z\ n_1,j,k+1}^{(0)} + h_3 \frac{\sigma_{yz\ n_1,j+1,k}^{(0)} - \sigma_{yz\ n_1,j-1,k}^{(0)}}{2h_2} \\
&\sigma_{yz\ 0,j,k}^{(0)} = \sigma_{yz\ 1,jk}^{(0)} \\
&\sigma_{yz\ n_1,j,k}^{(0)} = \sigma_{yz\ n_1-1,j,k}^{(0)}
\end{aligned}
\tag{42}
$$

for $y = \mp b$:

$$
\begin{aligned}
&\sigma_{x\ i,0,k}^{(0)} = \sigma_{x\ i+1,0,k}^{(0)} + h_1 \frac{\sigma_{xz\ i,0,k+1}^{(0)} - \sigma_{xz\ i,0,k-1}^{(0)}}{2h_3}, \\
&\sigma_{x\ i,n_2,k}^{(0)} = \sigma_{x\ i+1,n_2,k}^{(0)} + h_1 \frac{\sigma_{xz\ in_2,k+1}^{(0)} - \sigma_{xy\ i,n_2,k-1}^{(0)}}{2h_3} \\
&\sigma_{z\ i,0,k}^{(0)} = \sigma_{z\ i,0,k+1}^{(0)} + h_3 \frac{\sigma_{xz\ i+1,0,k}^{(0)} - \sigma_{xz\ i+1,0,k}^{(0)}}{2h_2}, \\
&\sigma_{z\ i,n_2,k}^{(0)} = \sigma_{z\ i,n_2,k+1}^{(0)} + h_3 \frac{\sigma_{xz\ i+1,0,k}^{(0)} - \sigma_{xz\ i-1,0,k}^{(0)}}{2h_2} \\
&\sigma_{xz\ i,0,k}^{(0)} = \sigma_{xz\ i,1,k}^{(0)} \\
&\sigma_{xz\ i,n_2,k}^{(0)} = \sigma_{xz\ i,n_2-1,k}^{(0)}
\end{aligned}
\tag{43}
$$

For $z = \mp c$:

$$
\begin{aligned}
&\sigma_{x\ i,\mathrm{j},0}^{(0)} = \sigma_{x\ i+1,\mathrm{j},0}^{(0)} + h_1 \frac{\sigma_{xy\ i,j+1,0}^{(0)} - \sigma_{xz\ i,j-1,0}^{(0)}}{2h_3}, \\
&\sigma_{x\ i,j,n_3}^{(0)} = \sigma_{x\ i+1,j,n_3}^{(0)} + h_1 \frac{\sigma_{xz\ i,j+1,n_3}^{(0)} - \sigma_{xy\ i,j-1,n_3}^{(0)}}{2h_3} \\
&\sigma_{y\ i,j,0}^{(0)} = \sigma_{y\ i,j+1,0}^{(0)} + h_2 \frac{\sigma_{xy\ i+1,j,0}^{(0)} - \sigma_{xy\ i-1,j,0}^{(0)}}{2h_3}, \\
&\sigma_{y\ i,j,n_3}^{(0)} = \sigma_{y\ i,j+1,n_3}^{(0)} + h_2 \frac{\sigma_{xy\ i+1,j,n_3}^{(0)} - \sigma_{xy\ i-1_1,j,n_3}^{(0)}}{2h_3} \\
&\sigma_{xy\ i,\mathrm{j},0}^{(0)} = \sigma_{xy\ i,\mathrm{j},1}^{(0)} \\
&\sigma_{xy\ i,\mathrm{j},n_3}^{(0)} = \sigma_{xy\ i,\mathrm{j},n_3-1}^{(0)}
\end{aligned}
\tag{44}
$$

Finite difference equations (33-44) can be solved by iteration method. These schemes converge because they satisfy the conditions of diagonal dominance. Initial values can be obtained as trivial.

5 Results and discussion

This section is devoted to the numerical solution and justification of the validity of the formulated boundary value problem on the equilibrium of a parallelepiped under stresses (10-17). Let the origin of the coordinate system be in the center of a parallelepiped of size $l_1 = 2a,\ l_2 = 2b\ l_3 = 2c$, and it is in equilibrium under the action of a domed load applied on opposite faces perpendicular to the OZ axis (Fig. 1). The remaining faces are free from loads. This problem using the variational method based on the Castigliano functional was considered in the works of Filonenko-Borodich [3].

In this case, the boundary conditions have the form

$$\begin{gathered} \text{for } x = \pm a: \ \sigma_x = 0, \quad \sigma_{xy} = 0, \quad \sigma_{xz} = 0 \\ \text{for } y = \pm b: \ \sigma_y = 0, \quad \sigma_{xy} = 0, \quad \sigma_{yz} = 0 \\ \text{for } z = \pm c: \ \sigma_z = S, \quad \sigma_{xz} = 0, \quad \sigma_{yz} = 0 \end{gathered} \tag{45}$$

Let the applied load have a domed shape, i.e., $S = \cos(1 - \frac{2\pi x}{a})\cos(1 - \frac{2\pi y}{b})$.

In this case, the initial data had the following values:

$$\begin{gathered} \lambda = 0.8,\ \mu = 0.5, \\ a = 0.5,\quad b = 0.5,\quad \mathrm{c} = 1, \\ h_1 = h_2 = 0.1,\ h_3 = 0.2, \end{gathered}$$

In our case, this problem is described with differential equations (10-17) with corresponding boundary conditions (18) and additional boundary conditions (20-22). The iterative method solves the finite-difference equations and boundary conditions corresponding to this boundary value problem (33-44).

The first line of Table 1 shows the stress values along the OZ (-c,+c) line passing through the center of the parallelepiped and compared with the results of other authors. It can be seen from the table that the maximum value of the applied load S in the center of the face $z = c$ is 4, and the calculated value also turned out to be $\sigma_z = 4.00$, and in [18], the considered value is equal to $\sigma_z = 3.9279$.

Table 1. Problems in y and x and corresponding stresses and didplacements

Problems y=0; x=0	*z=-1*	*z=-0.8*	*z=-0.6*	*z=-0.4*	*z=-0.2*	*z=0*
In stresses	4.0000	3.2065	2.5657	1.9248	1.6807	1.4365
In displacements	3.9279	2.7025	2.3333	1.8701	1.5437	1.4294

In [18], the problem was formulated concerning displacements, and when discretizing, the boundary conditions have the first order of approximation, so, apparently, underestimated values are obtained. At the center of the parallelepiped, the stress value is equal to $\sigma_z = 1.4365$, and in work [3] of Filonenko-Borodich $\sigma_z = 1.47$, and in work [18], it is equal to $\sigma_z = 1.4294$. It took t=14 iterations to obtain numerical results. The closeness of the obtained results shows the validity of the formulated boundary value problem in stresses and the method of their solution. On fig. 2 shows a graph of stress distribution σ_z in the central plane z=0 at y=0. In this figure, the solid line is built according to the results of the problem in stresses, and the dotted curve is according to the problem's solution in displacements [18]. As shown in Fig. 2, the results of the solved boundary value problems in stresses and displacements coincide with some errors. Some deviations of the dotted line from the solid line can be connected, apparently, with the error of differentiation of the boundary conditions since it has the first order of approximation.

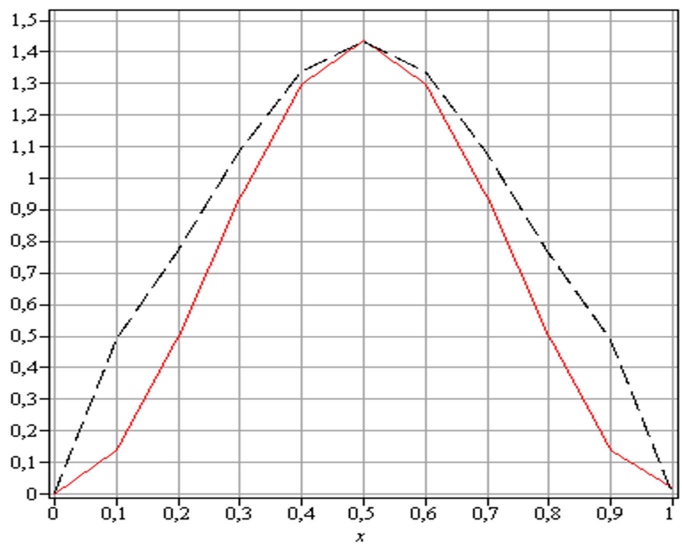

Fig. 2. Boundary value problems in stresses and displacements

The problem of parallelepiped compression under the action of a uniformly distributed load (S=1) applied along the faces perpendicular to the OZ axis is also solved. Then the stress values along the entire parallelepiped are equal to $\sigma_{33} = 1$, which is equal to the applied load (Table 2). This position also shows the validity of the formulated boundary value problems in stresses and numerical results.

Table 2. Problems y and z values for different x intervals

Problem y=0;	*x=-0.5*	*x=-04*	*x=-0.3*	*x=-0.2*	*x=-0.1*	*x=0*
z=1	1.000	1.000	1.000	1.000	1.000	1.000
z=0.5	1.000	1.000	1.000	1.000	1.000	1.000
z=0	1.000	1.000	1.000	1.000	1.000	1.000

6 Conclusions

In this work, based on the Beltrami-Michell equations, a spatial boundary value problem of the theory of elasticity in stresses for a parallelepiped is formulated. The boundary value problem consists of three equilibrium equations and three Beltrami Michell equations (of the second group). It is shown that on each face of the parallelepiped, it is necessary to set three ordinary surface boundary conditions and three additional conditions based on the equilibrium equation, which ensures the correctness of the formulation of the problem in stresses. Symmetric finite-difference equations are constructed for the spatial boundary value problem in stresses, which are solved by the iteration method. The problems of equilibrium of a parallelepiped under the action of dome-shaped and uniformly distributed loads applied on opposite sides of the parallelepiped are solved numerically. The reliability of the results is ensured by comparison with the numerical results of the Lame equations obtained in other works, as well as with the well-known Filonenko-Borodich solution.

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