On the unique solvability of a nonlocal boundary value problem with the Poincaré condition

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Abstract. As is known, it is customary in the literature to divide degenerate equations of mixed type into equations of the first and second kind. In the case of an equation of the second kind, in contrast to the first, the degeneracy line is simultaneously the envelope of a family of characteristics, i.e. is itself a characteristic, which causes additional difficulties in the study of boundary value problems for equations of the second kind. In this paper, in order to establish the unique solvability of one nonlocal problem with the Poincaré condition for an elliptic-hyperbolic equation of the second kind, we develop a new principle of extremum, which helps to prove the uniqueness of resolutions as signed problem. The existence of a solution is realized by reducing the problem posed to a singular integral equation of normal type, which is known by the Carleman-Vekua regularization method developed by S.G. Mikhlin and M.M. Smirnov equivalently reduces to the Fredholm integral equation of the second kind, and the solvability of the latter follows from the uniqueness of the solution delivered problem.

1 Introduction

Boundary value problems for degenerate equations of elliptic and equations of mixed types are in the center of attention of mathematicians and mechanics due to the presence of numerous applications in the study of problems in mechanics, physics, engineering and biology. Starting from [1], [2], a new direction has appeared in the theory of equations of elliptic and mixed types, in which nonlocal boundary value problems (problems with a shift) and Bitsadze-Samarskii problems are considered. Further, it turned out that nonlocal boundary conditions arise in problems of predicting soil moisture [3], in modeling fluid filtration in porous media [4], in mathematical modeling of laser radiation processes and problems of plasma physics [5], as well as in mathematical biology [6].

Solving various boundary value problems with the Poincaré conditions or with a conormal derivative for the Tricomi, Lavrentiev-Bitsadze and more general equations devoted to a large number of articles [7-13]. We note that the results of all the
2 Statement of the problem

\[ \text{sgn} y \left| y \right|^m u_{xx} + u_{yy} \quad m \in \mathbb{R} \]

\[ \Omega = \Omega_{1} \cup \Omega_{2}, \quad \sigma \cup J \]

\[ J = \{ x, y \mid x < y < 0 \} \]

\[ \beta = \frac{m}{m+1} \]

\[ \Omega = \Omega_{1} \cup \Omega_{2} \cup \sigma \cup J \]

\[ u(x, y) \in C(\Omega) \cup C(\Omega_{1} \cup \Omega_{2} \cup \sigma \cup J) \]

\[ u_{x} \quad u_{y} \]

\[ A(\Omega_{1}) \quad B(\Omega_{2}) \]

\[ R \]

\[ y \rightarrow x \quad y \rightarrow y \]

\[ \left\{ \delta \varphi f s \right\} \]

\[ \frac{d}{dx} \left[ \Theta_{x} \right] + b \frac{d}{dx} \left[ \Theta_{c} x \right] \]

\[ x \in J \]
\[ l - \text{the length of the whole curve} \]
\[ \rho \delta s \geq 0 \leq s \leq l \]
\[ \rho \delta s \in C \quad \phi \delta s \in C \cap C \]
\[ \Theta \left(x + \left(m + \right)^{m+1} \right) \Theta \left(x + \left(m + \right)^{m+1} \right) \]
\[ \Theta \left(x + \left(m + \right)^{m+1} \right) \Theta \left(x + \left(m + \right)^{m+1} \right) \]
\[ x \in J \quad AC \quad BC \quad A_s \quad u \]
\[ A_s [u] \equiv y^m \frac{dy}{as} \frac{\partial u}{\partial x} - \frac{dx}{as} \frac{\partial u}{\partial y} \]
\[ T(x) = \frac{\sin \frac{\pi \beta}{\pi \beta}}{\frac{\pi \beta}{\pi \beta}} \frac{d}{dx} \int_{x}^{t} (x-t)^{\beta} \tau'(t) dt \]
\[ \tau(x) \in C \quad C^{(k)} \quad k > -\beta \]
\[ \tau(x) \in C \quad C^{(k)} \quad k > -\beta \]
\[ E(x) = \frac{1}{\Gamma(1 - \beta)} \frac{d^{\beta} \tau(x)}{dx^{\beta}} \bigg|_{x = x_0} \]

\[ E(x) = -x_0 \frac{d^{\beta} \cos \beta \pi + (\beta - 1)x_0^{\beta - 2}}{x^{\beta - 2}} \tau(x) - \frac{1}{\beta} \frac{d^{\beta - 1} \tau(x)}{dx^{\beta - 1}} + \]

\[ + \left( \frac{1}{\sin \beta \pi} \right) \right] \int_{x_0}^{\beta} \tau(x_0) - \tau(t) \frac{d \tau(t)}{dt} dt - \int_{x_0}^{\beta} \frac{\tau(t) - \tau(x)}{(x - t)^{\beta - 2}} d \tau(t) dt \]

\[ x = x_0 \left( x_0 \in [0,1] \right) \]

**Lemma 3.** Let conditions (2), (10) be satisfied and the function \( f(x) \) at the point \( x = x_0 \) can be represented as

\[ f(x) = \frac{T(x)}{\beta} - \frac{1}{\beta} \frac{d^{\beta - 1} \tau(x)}{dx^{\beta - 1}} \]

**Theorem 1.** (An analogue of the extremum principle of A.V. Bitsadze).

If conditions (2) are satisfied and \( b > 0 \), then the solution \( u(x,y) \) inside the region \( \Omega \) cannot reach its refinery \( \Omega \) and \( \Omega \). Let us show that the solution \( u(x,y) \) inside the region \( \Omega \) does not reach its OR and \( \Omega \) on the segment \( J \). Assume the opposite, let \( u(x,y) \) at some point \( x_0 \) segment \( J \) reaches its refinery \( \Omega \). Based on Lemma 2, if the function \( f(x) \) at the point \( x = x_0 \) accepts the refinery \( \Omega \), then \( \tau(x) \) cannot reach its refinery \( \Omega \) on \( x \).
But on the other hand, by virtue of the Zaremba-Giraud principle [24], [26], for the solution of equation (1), taking into account (15), we have

\[ (14) \]

Taking into account (4) from (14) we find\[ (15) \]

This inequality contradicts inequality (15).

In this way, \( (\mathbf{1}, 0) \) does not reach its refinery (SNV) in the open section \( J \).

Theorem 1 is proved.

Theorem 2. If the conditions of Theorem 1 are satisfied, and the functions \( s \) \( \delta \) and \( s \) \( \rho \) near points \( (0, 0), (1, 0) \) satisfy conditions (7) and (17),

\[ (16) \]

then in the area \( D \) there cannot be more than one solution to the problem \( C \).

Proof of Theorem 2. Let \( s\phi \equiv 0 \), then, by virtue of Theorem 1, it suffices to show that the solution of the problem \( C \) cannot reach its positive maximum and negative minimum on \( \sigma \).

Assume that a positive maximum (negative minimum) is reached at some point \( s \in (0, 0), (1, 0) \). Then at this point, due to the Zaremba-Giraud principle \([24, 27]\),

\[ (17) \]

But this is impossible due to condition (7).

Therefore, at interior points \( (\mathbf{1}, 0) \) function \( u \) \( \mathbf{x} \mathbf{y} \) does not reach its positive maximum (negative minimum).

At points \( (0, 0), (1, 0) \), taking into account (2), (3), (17) we have respectively.

\[ (18) \]
Hence, taking into account (16), we obtain \( u(A) = u(B) = \tau \). Based on the extremum principle (see Theorem 1), we conclude that \(const \in \Omega\). Therefore, taking into account (19), we have \( \tau = \frac{\partial u(x, y)}{\partial y} = v^-(x) \) in \( \Omega \). Due to the uniqueness of the solution of the Cauchy problem in the domains \( j = 1, 3 \) for equation (1), we obtain that \( \tau = \frac{\partial u(x, y)}{\partial y} = v^+(x) \) in \( \Omega \). This proves the uniqueness of the solution of the problem C. Theorem 2 is proved.

When studying the problem C an important role is played by the functional relationships between \( \xi \) and \( \tau \) from the elliptic and hyperbolic parts of the domain \( \Omega \), where \( \nabla \) and \( \nabla \) are the derivatives in the directions of the elliptic and hyperbolic parts, respectively. Therefore, taking into account (20), (21) for equation (1), we have \( \tau = \frac{\partial u(x, y)}{\partial y} = v^+(x) \) in \( \Omega \). Hence it follows that \( \tau = \frac{\partial u(x, y)}{\partial y} = v^-(x) \) in \( \Omega \).
\[ u[\Theta_\beta(x)] = \int (x-t)^{-\beta} t^{-\beta} N(t) dt \]
\[ u[\Theta_\beta(x)] = \int (x-t)^{-\beta} (x-t)^{-\beta} T(t) dt + \int (t-x)^{-\beta} (x-t)^{-\beta} N(t) dt \]

\[ \gamma \left( x^{-\beta} - b \pi \beta \cdot x^{-\beta} (\beta-1) + b^{-\beta} (\beta-1)^{-\beta} \right) v - x^{-\beta} + b^{-\beta} \pi \beta \cdot x^{-\beta} T(x) \]

\[ T(x) = V^-(x) \]

We put (25) and (26) in the boundary condition (6), by virtue of the fractional integration operators and (23) we obtain a functional relation between \( T(x) \) and \( x \), transferred from the area \( J \):

\[ \frac{1}{2} \int_0^1 x^{-\beta} \frac{d}{ds} G(x, y, s) ds + \int \frac{\partial}{\partial \eta} G(x, y, s) ds \]

\[ u[\Theta_\beta(x)] = \int \tau \frac{\partial}{\partial \eta} G(x, y, s) ds + \int \frac{\phi}{\delta s} G(x, y, s) ds \]

\[ \chi \eta \xi x y + \int \frac{\partial}{\partial y} q (\xi, \eta, x, y, s) ds \]

\[ \chi \xi + \int \frac{\partial}{\partial \eta} \chi \eta x y + \int \frac{\partial}{\partial \eta} q (\xi, \eta, x, y, s) ds \]

\[ q \left( \xi, \eta, x, y \right) \]
\[ q_{\pm}(\xi, \eta, x, y) = k \left( \frac{1}{m + i} \right)^{\beta} \left( r - w \right)^{\beta} F(-\beta) \left( 1 - \beta \right) \left( -w \right)^{\beta} \]

\[ r = (\xi - x) + \left( \frac{m + i}{\eta + y} \right) \]

\[ w = \frac{r}{r'} \beta = \frac{m}{m + i} \quad \text{for} \quad \frac{1}{\beta} < \beta < k < \frac{1}{\pi \left( m + i \right)} \frac{\Gamma(-\beta)}{\Gamma(-\beta)} \]

\[ F(a, b; \alpha; z) \]

\[ v^+(x) = -\frac{\pi k \tan \beta \pi}{x - x'} \frac{T}{T(x)} T(x) \frac{k}{k - \beta} \left( \frac{1}{x - x'} \right)^{-\beta} \int_{t-x}^{t} \left( t - x \right)^{\beta} \left( t - x + t - x \right) dt + \]

\[ + \left( \frac{1}{\beta} \right) \partial \phi \partial \eta \partial y \frac{H(x)}{\eta \partial y} + \frac{k}{k - \beta} \int_{t-x}^{t} \left( t - x \right)^{-\beta} T(x) dt + \]

\[ + \left( \frac{1}{\beta} \right) \partial q \frac{\xi - s}{\partial y} \frac{s}{s} \frac{\eta - s}{\eta} \frac{x}{x} \in J \]

\[ P(T(x))^+ = \frac{\pi k \tan \beta \pi}{x - x'} \frac{T}{T(x)} \frac{k}{k - \beta} \left( \frac{1}{x - x'} \right)^{-\beta} \int_{t-x}^{t} \left( t - x \right)^{\beta} \left( t - x + t - x \right) dt - \]

\[ K(x) T(x) dt = F(x) \quad \text{for} \quad x < x' \]

\[ P(x) = \frac{\pi k \tan \beta \pi}{x - x'} \frac{d}{dx} \]

\[ K(x) = \frac{b}{x - x'} \frac{d}{dx} \left( \frac{1}{x - x'} \right)^{\beta} \]

\[ F(x) = \frac{b}{x - x'} \frac{d}{dx} \left( \frac{1}{x - x'} \right)^{\beta} \]

\[ \frac{1}{\Gamma(-\beta)} D_{-\beta} \left( \right) c(x) + \frac{b}{\Gamma(-\beta)} D_{-\beta} \left( \right) c(x) \]
equation (31) is an equation of normal type [23, 24]. Applying the well-known Carleman-Vekua regularization method [23], we obtain the Fredholm integral equation of the second kind, the solvability of which follows from the uniqueness of the solution of the problem. Theorem 3 is proved. 

3 Conclusions 

Thus, with the help of the new extremum principle developed by the authors of the article for an equation of the second kind, the uniqueness of the problem posed is proved. When studying the existence of a solution to the problem under study with the help of functional relations, a singular integral equation of normal type is obtained, the solvability of which follows from the uniqueness of the solution to the problem. The article presents new mathematical results. Which are of interest to a person skilled in the art. What can be used to build some models of gas and hydrodynamic processes, when predicting soil moisture, when modeling fluid filtration in porous media.

References

for a Mixed Type Equation with Two Lines of Degeneracy.


