Spectral and cross-spectral densities expressions’ refinement for rail vehicle oscillations

Anatoly Savoskin and Natalia Lavinskaya

1 Russian University of Transport, 127994 Moscow, Russia

Abstract. In this paper, we derive analytical expressions describing spectral and mutual spectral densities of random processes by performing analytical Fourier transform of the corresponding correlation functions. On the basis of accepted analytic expressions for correlation functions of differentiable random processes, analytical expressions for spectral density and mutual spectral density components, which is a complex frequency function, are derived. The influence of the accepted analytic expressions on the correlation functions, spectral densities, and mutual spectral density components (real, imaginary, amplitude, and phase) on the parameters is presented. Plots obtained using obtained analytical expressions have showed full convergence with those obtained by direct integration of the analytical expression of the correlation function. The analytical expressions given in this paper may be used for investigating various random processes. In particular, spectral and reciprocal spectral densities can be used for approximation of experimentally received spectral densities, including equivalent geometrical irregularities of a rail track and random fluctuations of a rail vehicle. The parameters of analytical expression obtained by such an approximation can be used for generation of analogous experimental multidimensional random processes in the tasks of mathematical modelling.

1 Introduction

Mathematical modelling techniques are widely used to solve engineering problems, especially those related to dynamic processes. The real processes to be modelled are often of a random nature. An example of such processes is oscillations of rail vehicles that arise when they move over irregularities of rail track, which have a random nature. Previously, it was shown in [1-3] that geometric track unevenness should be considered as stationary ergodic random processes. When modelling random oscillations of rail vehicles, generated random processes of equivalent geometric track irregularities, the probability characteristics of which are similar to the real processes, are used as a perturbation [4-7]. An effective way to generate such processes is to use the impulse characteristic of a shaping filter [1,8]. To implement it, one

* Corresponding author: lav.nata@mail.ru

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first has to determine the parameters of analytical expressions that describe auto-
and mutual correlation functions and spectral densities of a multidimensional random
perturbation process through which pulse characteristics are expressed [9].
In addition, such analytical expressions may be useful in solving other problems related
to the study of random processes in the dynamics of rail vehicles [10-12]. In this
connection, the issues related to the derivation of analytical expressions of spectral and
mutual spectral densities are relevant.

2 Materials and methods

\[
R_{xy}(\tau) = S_x S_y \left[ -\pi \alpha (\tau - \tau_0) \right] \left[ \pi \theta (\tau - \tau_0) \right]
\]

(1, a)

\[
R_{xy}(\tau) = S_x S_y \left[ -\pi \alpha (\tau - \tau_0) \right] \left[ \pi \theta (\tau - \tau_0) \right]
\]

(1, b)

where \( x \) and \( y \) – standard deviations of random processes \( X_t \) and \( Y_t \),
\( S_x \) and \( S_y \) – random process variance \( X_t \).

Note also that the expressions (1, b) and (2, b) can be seen as special cases of
(1, a) and (2, a) by \( \theta = 0 \).

These correlation functions correspond to spectral densities \( \Phi_x(j\omega) \) and
\( \Phi_{xy}(j\omega) \), which are found as the Fourier transform of the corresponding correlation
functions [10-13]:

\[
\Phi_x(j\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega \tau} d\tau
\]

\[
\Phi_{xy}(j\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j\omega \tau} d\tau
\]

For autocorrelation functions (2, a) and (2, b), which are even, the spectral density
\( \Phi_x(j\omega) \) becomes a real function of frequency and the Fourier transform formula takes the form:
\[
\Phi_x(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{j\omega \tau} d\tau
\]

\[
\Phi_x(f) = \int_{-\infty}^{\infty} R(\tau) e^{j2\pi f \tau} d\tau
\]

\[
\Phi_x(f) = \frac{S_x}{\alpha \sqrt{\pi}} \left\{ \left[ \left( \frac{f - \theta}{\alpha} \right)^2 \right] + \left[ \left( \frac{f + \theta}{\alpha} \right)^2 \right] \right\}
\]

\[
\Phi_x(f) = \frac{S_x}{\alpha \sqrt{\pi}} \left[ \left( \frac{f}{\alpha} \right)^2 \right]
\]

Fig. 1. Graphs (a) and (b) of random process \( X(t) \), constructed from analytical expressions (2, a) and (4, a).

Graphs (Fig. 1 and 2, curves 1÷3) normalised (divided by the variance \( S_x^2 \)) autocorrelation functions \( r_x(\tau) \) and their respective spectral densities \( \phi_f(\omega) \) are constructed with the following parameter values: \( \theta = 2\text{Hz} \), \( \alpha = 0.3\text{Hz} \) – curves 1, \( \alpha = 0.6\text{Hz} \) – curves 2 and \( \alpha = 0.9\text{Hz} \) – curves 3.
Fig. 2. Graphs $r_x(\tau)$, $\phi_x(f)$, $X(t)$, $\delta(\tau)$. As can be seen from these figures, the autocorrelation function (1, a) when reducing $\alpha_t$ tends towards the cosine. At the same time, with increasing $\alpha$ autocorrelation functions (2, a) and (1, a) taper and tend to a unit momentum function $\delta(\tau)$. The maxima of these functions are narrowed in width and increased in height, but the area bounded by the graph remains constant:

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau =$$

Contrary to the graphs of the normalised spectral densities (Fig. 1, b and 2, b), the normalised mutual correlation functions $r_{xy}(\tau)$ (1, a) and (1, b) of two random processes $X(t)$ and $Y(t)$, plotted at $0.7 \, \text{Hz}$ and $0.3 \, \text{s}$, are also damped; their maximum is shifted by $0.3 \, \text{s}$ in relation to the origin.

Fig. 3. Graphs of the normalised mutual correlation functions $r_{xy}(\tau)$, $\phi_{xy}(f)$, $X(t)$, $Y(t)$, built according to:

- the formula (1, a);
- the formula (1, b)

For these mutual correlation functions defined by the expressions (1, a) and (1, b), reciprocal spectral density $\Phi_{xy}(f)$ will be a complex function of frequency [14-16]. (From now on we will use the expression (1, a) as describing a general case).

The function $\Phi_{xy}(f)$ can be represented by the real (in-phase) $\Re \Phi_{xy}(f)$ and imaginary (quadrature) $\Im \Phi_{xy}(f)$ or via amplitude $A_{xy}(f)$ and phase $\phi_{xy}(f)$ components:

$$\Phi_{xy}(f) = S_x \delta(f - \theta)$$

$$\Phi_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j \omega \tau} d\tau = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j \omega \tau} d\tau$$
These components, based on (6), are defined by the following expressions:

- real:

\[
\Phi_{xy}(if) = \int_{-\infty}^{\infty} R_{xy}(\tau) \pi f \tau d\tau
\]

- imaginary:

\[
\Phi_{xy}(if) = -\int_{-\infty}^{\infty} R_{xy}(\tau) \pi f \tau d\tau
\]

- amplitude:

\[
\Phi_{xy}^2(f) = \left[ \Phi_{xy}(if) \right]^2 + \left[ \Phi_{xy}(if) \right]^2
\]

- phase:

\[
\Phi_{xy}^\phi(f) = \text{Arg} \Phi_{xy}(if)\]

\[
\Phi_{xy}^\phi(f) = \begin{cases} 
\text{arctg} \left[ \Phi_{xy}(if) / \Phi_{xy}(if) \right] & \Phi_{xy}(if) > 0 \\
\text{arctg} \left[ \Phi_{xy}(if) / \Phi_{xy}(if) \right] + \pi & \Phi_{xy}(if) < 0 \\
\text{arctg} \left[ \Phi_{xy}(if) / \Phi_{xy}(if) \right] - \pi & \Phi_{xy}(if) < 0 \\
\end{cases}
\]

\[
\Phi_{xy}(if) = \sqrt{\left[ \Phi_{xy}(if) \right]^2 + \left[ \Phi_{xy}(if) \right]^2}
\]

The need to correct for the phase component in the left half-plane arises because the tangent function is periodic with a rad period, so that the arctangent gets correct values only in the interval \([\pi, \pi]\).

Note that the calculation of integrals included in these expressions is difficult and in [1] it is performed with some inaccuracies, let us perform these calculations more precisely. For this purpose, we present expression (3) for \(\Phi_{xy}(if)\) based on (1, a) as follows:

\[
\Phi_{xy}(if) = \int_{-\infty}^{\infty} R_{xy}(\tau) \pi f \tau d\tau - j \int_{-\infty}^{\infty} R_{xy}(\tau) \pi f \tau d\tau =
\]

\[
= S_x S_y \cdot \left\{ \int_{-\infty}^{\infty} \left[ -\pi \alpha (\tau - \tau) \right] \left[ \pi \theta (\tau - \tau) \right] \pi f \tau d\tau - j \int_{-\infty}^{\infty} \left[ -\pi \alpha (\tau - \tau) \right] \left[ \pi \theta (\tau - \tau) \right] \pi f \tau d\tau \right\}
\]
\[\Phi_{x y} (j') = S_x S_y \left[ -\pi \alpha (\tau - \tau') \right] \times \left\{ \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau + \pi \alpha \tau' \right] \left[ \pi f(\tau - \tau') \right] d\tau - \right\} \]

\[= \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau + \pi \alpha \tau' \right] \left[ \pi f(\tau - \tau') \right] d\tau \]

\[I_0 = \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau + \pi \alpha \tau' \right] \left[ \pi f(\tau - \tau') \right] d\tau \]

\[I = I_{-\infty} + I_{+\infty} \]

\[I_{-\infty} = \left\{ \left[ \pi \theta \right] \right\} \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau + \pi \alpha \tau' \right] \left[ \pi f(\tau - \tau') \right] d\tau \]

\[I_{+\infty} = \left\{ \left[ \pi \theta \right] \right\} \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau + \pi \alpha \tau' \right] \left[ \pi f(\tau - \tau') \right] d\tau \]

\[I_0 = \left\{ \left[ \pi \theta \right] \right\} \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau + \pi \alpha \tau' \right] \left[ \pi f(\tau - \tau') \right] d\tau \]

\[\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} f(-x) dx = \int_{-\infty}^{\infty} f(x) dx \]

\[I_{-\infty} = \left\{ \left[ \pi \theta \right] \right\} \times \left\{ \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau - \pi \alpha \tau' \right] \left[ -\pi (f + \theta) \tau \right] d\tau + \right\}

\[+ \int_{-\infty}^{\infty} \left[ -\pi \alpha \tau + \pi \alpha \tau' \right] \left[ \pi (f + \theta) \tau \right] d\tau \]
\[ I_{+\omega} = \sqrt{\frac{-\pi \tau}{\pi \alpha}} \left( \pi \alpha \tau \right) \frac{i \pi}{\pi \alpha} \left[ \pi (f + \theta) \tau \right] d\tau + \right. \\
+ \int \frac{i \pi}{\pi \alpha} \left[ -\pi \alpha \tau + \pi \alpha \tau \tau \right] \left[ \pi (f + \theta) \tau \right] d\tau \]
\[ \Phi_{xy}^\phi(f) = \begin{cases} -\pi \tau_c f + \text{sign} \tau_c k + \pi \tau_c f & \text{if } \tau_c \leq f \leq |\tau_c| \\
-\pi \tau_c f - \text{sign} \tau_c k + \pi \tau_c f & \text{if } -|\tau_c| < f < -\tau_c \end{cases} \]

where \( k \) is odd (1, 3, 5...). Graphs of these normalised components (divided by \( SS_{xy} \)) of the reciprocal spectral density function (Fig. 4), plotted for frequency \( f = \frac{\omega}{\pi} \), configured as \( \alpha = \text{Hz} \). Function \( \phi_{xy}^\text{Re} \) is symmetrical (Fig. 4, a) relative to the ordinate axis. Frequency \( \omega \) is between its maximum and minimum, shifting proportionally towards the larger absolute value of the extremum. Function \( \phi_{xy}^\text{Im} \) is asymmetrical (Fig. 4, b); the frequencies of its maximum and minimum are \( \omega \). Function \( \phi_{xy}^A \) is symmetrical (Fig. 4, c). Its maximums also occur at frequencies \( \omega \). From its graph you can determine the attenuation coefficient \( \alpha = \text{Hz} \) as half the width of the characteristic maximum at half its ordinate.
The function $\phi_{xy}(f)$ is asymmetrical (Fig. 4, d). It varies linearly from $[-\pi, \pi]$; its breaking points occur at frequencies $c_2 f_k = \pm \tau$, where $k$ is an odd number. These changes are periodically repeated in both the positive and negative frequency ranges, so the frequencies corresponding to the fractures in the graph (Fig. 4, d) by the $c_0.3s \tau = \pm 0.3s$, are equal to $1,667 f_k$ Hz.

For a more detailed study of the properties of the mutual spectral density functions described by the analytical expressions obtained in this paper, let us consider one more example. Let us change some parameters of the analytical expression (1, b) $\alpha = 0.1$ Hz, $c_0.3s \tau = -0.3s$, and $\theta = 0.83$ Hz. The appearance of the graph of the normalised correlation function (Figure 5), as well as the graphs of the components of the normalised mutual spectral density (Figure 6), will then change significantly.
Let us note some features of the components of the reciprocal spectral density function with changed parameters. The function \( \phi_{xy}(jf) \) (Fig. 6, a) due to changes in frequency \( \theta \) has the same absolute maximum and minimum, so that this frequency \( \theta = \pm 0.83 \text{ Hz} \) lies strictly between them at the intersection of the graph of the function with the abscissa axis. Furthermore, the functions \( \text{Im}\ xy \phi_{ff}(f) \) and \( \phi_{xy}(f) \) (Fig. 6, b and d) have changed sign compared to the example in Figure 4. This is due to the oddness of the sine in their analytical expressions:

\[
\sin(2\pi(f-\tau)/\theta) = -\sin(2\pi\tau/\theta)
\]

Thus, from the graphs of the functions \( \text{Im}\ xy \phi_{ff}(f) \) and \( \phi_{xy}(f) \), the nature of the shift in the mutual correlation function can be judged.

Increase the shift of the first variant of the parameters to \( c = 2 \) and plot the reciprocal spectral densities (Fig. 7).
4 Discussion

The reciprocal spectral densities are complex frequency functions whose analytical expressions found for the real and imaginary components are adequate, since they provide convergence with the results of calculations of these components based on direct integration of the reciprocal correlation function expression.
\[ \phi_{sx}(f) \] and \[ \phi_{sy}(f) \] also change the sign.

The graph of the phase component of the mutual spectral density varies linearly in the range from \([-\pi, \pi]\). Its breaking points occur at frequencies \( f = \pm k/\tau \), where \( k \) - odd number.

5 Conclusion

The analytical expressions of the mutual spectral density components and their identified properties obtained in this work can be useful in the study and modelling of random processes, including those occurring in the dynamics of rail vehicles. These expressions have shown absolute convergence with integration results and can be used for the approximation of the experimental spectral and mutual spectral densities of the unevenness of the rail track by analytical expressions that are required for the generation of random perturbation processes in the modelling of rail carriage oscillations.

References


