The problem of propagation of a one-dimensional plastic wave in the environment with linear and polyline unloading

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Abstract. Problems of propagation of plane and spherical waves in a nonlinearly compressible medium with linear and broken line unloading under intense loads are considered. The solutions of the problems are constructed in the opposite way, assuming that the medium at the shock wave front is instantly loaded in a nonlinear manner, and behind the front in the perturbed region, the medium is irreversibly unloaded. For a specific structure of the medium, the results of calculations are presented in the form of graphs of pressure, velocity of the medium at the layer boundary, at the shock wave front and in the disturbed region as a function of time. The influence of the nonlinear properties of the medium on the distribution of the dynamic characteristics of shock-wave processes in it has been studied.

1 Introduction

The problems of the propagation of various waves in elastic-plastic and soil media and their interaction with underground structures are topical. The problems of wave propagation in elastic, elastoplastic, and soil media are the subject of extensive literature [1–24].

In the following article, we consider the problems of the propagation of plane and spherical waves in a nonlinearly compressible medium with linear and broken line unloading under intense loads. Solutions of the problems are constructed in the reverse way [1] under the assumption that the medium at the shock wave front is instantly loaded in a nonlinear manner, and behind the front in the perturbed region, the medium is irreversibly unloaded. The problem of propagation and reflection of an elastoplastic wave in a rod of finite length for the Prandtl scheme with broken line unloading was solved by the method of characteristics in [2].

In contrast to [2], in this paper, one-dimensional nonstationary problems of a flat and spherical layer are solved analytically inversely, and the propagation of a nonlinear shock wave of load-unload is considered. It should be noted that this work is a continuation of [1] for a medium with broken unloading. In the case of linear unloading of the medium, the finiteness of the time interval of the impact of the load applied to the boundary of the layer is taken into account and solutions of problems are given in areas outside its action. The reverse method consists in determining the wave field in the soil layer and the profile of the

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load applied to its boundary from the explosion products for a given law of shock wave motion.

2 Methods

The soil under intense impacts, as in [3], is taken as a non-linearly compressible ideal medium. A similar approach was previously used in [4] in the study of the mechanical impact of an underground explosion. For a specific structure of the medium, the results of calculations are presented in the form of graphs of pressure, velocity of the medium at the layer boundary, at the shock wave front and in the disturbed region as a function of time. A detailed analysis of the kinematic parameters of the medium for the case of linear unloading and a comparison with acoustics is given. The influence of the nonlinear properties of the medium on the distribution of the dynamic characteristics of shock-wave processes in it is studied. The calculations are made for the case when the shock wave front velocity is given as a linearly decreasing function of time, and the corresponding load profile is determined in the course of solving the problem. The surface of the pressure isobar is constructed.

Propagation of plane and spherical waves in a nonlinearly compressible medium with linear unloading. Let a monotonically decreasing load $p_0(t)$ be applied at the layer boundary $r = R_0$. The equations in the unloading region, the relations at the front $r=R(t)$ and the boundary condition (zero initial conditions) have the form [1]

$$\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial r}=0,\quad \frac{\partial \rho}{\partial t}+\rho_0\left(\frac{\partial u}{\partial r}+\nu\frac{u}{r}\right)=0, \tag{1}$$

$$p\,(r,t)=p^*+E(\varepsilon-\varepsilon^*),\ \ \varepsilon=1-\frac{\rho}{\rho},\quad E=c_p^2\rho_0;$$
$$u^*(t)=\varepsilon^*\dot{R},\qquad p^*=\rho^*\varepsilon^*\,\dot{R}^2, \tag{2}$$

$$p^*\,(t)=\alpha_1\,\varepsilon^*+\alpha_2\,\varepsilon^{*2}\,(\dot{R}=dR|dt)\ \ \text{at}\ \ r=R(t)$$
$$p(r,t)=p_0\,(t)\qquad \text{at}\quad r=R_0\,, \tag{3}$$

where u-mass velocity; ρ-density; p-pressure; ε-volumetric strain; ν = 0.2 refer respectively to a flat and spherical layer; the parameters of the medium related to the front are marked with an asterisk above. If we set the front velocity as a decreasing function of time, then all parameters of the medium at $r = R(t)$ will be known and relations (2) will be the boundary condition for (1). In this case, for a plane one-dimensional wave ($\nu = 0$), from (1) we obtain the equation

$$\frac{\partial^2 u}{\partial t^2}-c_p^2\frac{\partial^2 u}{\partial r^2}=0, \tag{4}$$

which, taking into account (2), has a solution in the form

$$u(r,t)=u*(0)-1/(2c\_p)\sum_{i=1}^{2}(-1)^{i-1} \tag{5}$$

$$\int_{R_0}^{r-(-1)^{i-1}c_p t}\ddot{R}[F(z_i)]\ x\left\{\Delta_1[F(z_i)]+\frac{\frac{\rho_0}{\alpha_2}\left[2\dot{R}(F(z_i))+(-1)^{i-1}c_p\right]\ \dot{R}[F(z_i)]}{\Delta_2[F(z_i)]}\right\}dz_i,$$

$$\text{Where } \Delta_1(t) = \frac{\left(1-\frac{\alpha_1}{\alpha_2}\right)}{2} - \Delta_2(t); \qquad \Delta_2(t) = \sqrt{\frac{\left(1-\frac{\alpha_1}{\alpha_2}\right)^2}{4} - \frac{\rho_0 \dot{R}^2(t) - \alpha_1}{\alpha_2}}$$

Substituting (5) into the first equation (1) and integrating over r from $r = R_0$ to $r = R(t)$, to determine the load $p_0(t)$, we have

$$p_0(t) = p^*(t) + \frac{\rho_0}{2}\int_{R_0}^{R(t)} \sum_{i=1}^{2} \ddot{R}[F(z_i)] \left\{ \Delta_1[F(z_i)] + \frac{\frac{\rho_0}{\alpha_2}[2\dot{R}(F(z_i)) + (-1)^{i-1} c_p]\, \dot{R}(F(z_i))}{\Delta_2[F(z_i)]} \right\} dr, \quad (6)$$

There $z_{1,2} = r \pm c_p t$; $F(z_{1,2})$ is the root of the equation $R(t) \pm c_p\, t = z_{1,2}$ with respect to t. Note that expression (6) is more accurate than in [1] and is valid as long as $p_0(t) \geq 0$. Next, the corresponding boundary value problems are solved. The area under consideration is divided into n = 1, 2, 3, . . . areas, each of which, for n≥2, is limited by the characteristics *AB, BC, CD*, etc. (Fig. 1) of positive, negative directions, the layer boundary or part of the front $r = R(t)$.

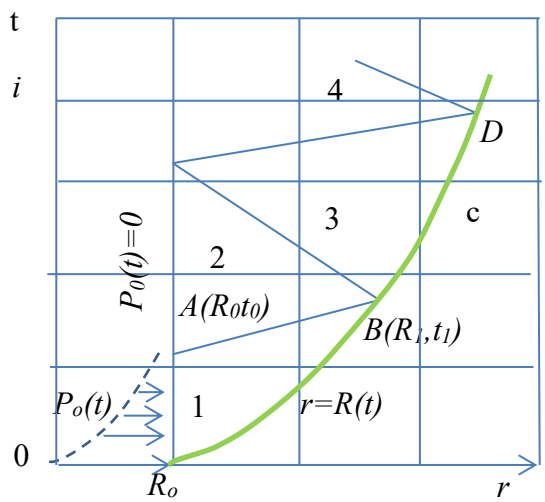

Fig. 1. Image of a shock wave on a half-plane

Let us represent the solution of the problem for region 2. From (1.1) for $v = 0$ we obtain the equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{\partial^2 p}{\partial r^2} = 0,$$

for the solution of which we have the conditions

$$p(r, t) = p_1(t) \qquad \text{at} \quad r - c_p t = R_0 - c_p t_0,$$
$$p(r, t) = 0 \qquad \text{at} \quad r = R_0, \ t \geq t_0.$$

Then, based on the d'Alembert formula, we obtain

$$p(r,t)=-p_1\left[\frac{(R_0+c_pt_0)-(r-c_pt)}{2c_p}\right]+p_1\left[\frac{(r+c_pt)-(R_0-c_pt_0)}{2c_p}\right].$$

Integrating the first equation (1) over t from $t = t^*(r)$ to t, we have

$$u(r,t)=u_1\left[r,\frac{r-(R_0-c_pt_0)}{c_p}\right]-\frac{1}{\rho_0c_p}\left\{p_1\left[\frac{(R_0+c_pt_0)-(r-c_pt)}{2c_p}\right]+p_1\left[\frac{(r+c_pt)-(R_0-c_pt_0)}{2c_p}\right]-p_1(t)-p_1(t_0)\right\},$$

where $p_1(t)$, $u_1(t)$ are the pressure and velocity of the medium on the characteristic AB, determined from the solution of the problem in region 1. The solution of Eq. (4) in region 3 can be represented as

$$u(r, t) = f_5(r - c_pt) + f_6(r + c_pt). \quad (7)$$

To find the functions f_5 and f_6, the problem has a boundary condition on the BC and relations at the front $r = R(t)$. However, as calculations show [5], the front of a two-dimensional stationary plastic wave varies slightly depending on the depth of the half-plane. The curvature of the front in comparison with the original form is approximately 15-20%, and even less at significant depths. In addition, the BD line has a finite length. Therefore, in the first approximation, the discontinuity relations are satisfied with respect to the initial front shape corresponding to the point $B(R_1, t,)$. Then we have

$$u(r, t) = u_2(t) \text{ at } r + c_p t = R_1 + c_p t_1 ; \quad (8)$$

$$\left.\begin{aligned} u(r,t) &= \dot{R}(t)\Delta_1(t) \\ \frac{\partial u}{\partial r} &= -\frac{\frac{\rho_0}{\alpha_2}\dot{R}(t)\ \ddot{R}(t)}{\Delta_2(t)} \end{aligned}\right\} \text{ at } r - R_1 \approx a_1(t - t_1), \quad (9)$$

where $\alpha_1 = dR/dt$, $t = t_1$; u_2 - velocity of the medium on the BC, determined from the solution in region 2. Substituting (7) into (8), we obtain

$$u(r,t)=u_2\left[\frac{r_1+c_pt_1-(r-c_pt)}{2c_p}\right]+f_6(r+c_pt)-f_6(r_1+c_pt_1). \quad (10)$$

System (9), taking into account (10), allows us to obtain with respect to f6(t) and $\dot{R}(t)$ (in region 3, in contrast to region 1, $\dot{R}$ is the desired parameter) a system of two equations of the form

$$u_2\left[\frac{\left(1+\frac{a_1}{c_p}\right)t_1+\left(1-\frac{a_1}{c_p}\right)t}{2}\right]-u_2(t_1)=\frac{\left(1-\frac{\alpha_1}{\alpha_2}\right)}{2}(\dot{R}(t)-a_1)-\left(\dot{R}(t)+c_p+a_1\right)\Delta_2[\dot{R}(t)]+$$
$$+(c_p+2a_1)\Delta_2(a_1); \quad (11)$$

$$u_2\left[\frac{\left(1+\frac{a_1}{c_p}\right)t_1+\left(1-\frac{a_1}{c_p}\right)t}{2}\right]+f_6(t_1)\left[(R_1-a_1t_1)+(a_1+c_p)t\right]--f_6(R_1+c_pt_1)=$$
$$\dot{R}(t)\Delta_1[\dot{R}(t)]. \quad (12)$$

Equation (11) with respect to $\dot{R}(t)$ is easily solved by graphical analysis way. After finding $\dot{R}(t)$, using (11) from (12), we determine $f_6(t)$, and then, using formula (10), the mass velocity. Further, integrating the equation of motion of system (1) with respect to r from $r = —c_pt + (R_1 + c_pt_1) = R_2(t)$ to r, we obtain

$$p(r,t)=p_2(t)-p_0\int_{R_2(t)}^{r}\frac{\partial u}{\partial t}dr,$$

2. Propagation of waves in a medium with broken line unloading**.** If the diagram of the state of the medium (Fig. 2, a) during unloading has a broken line, consisting of two straight lines, then the results of paragraph 1 are valid as long as $p(r, t) \geq p^{**}$ и $\varepsilon \geq \varepsilon^{**}$. Therefore, on the basis of the above results in the physical plane (r, t), the surface is first determined, in which $p = p^{**}$, $\varepsilon = \varepsilon^{**}$, and the velocity distribution on it is found. Calculations show that the pressure at the shock wave front decays more weakly than at the cavity. In this regard, the pressure isobar turns out to be elongated towards the spatial coordinate *r* (Fig. 2b). Further, at $p<p^{**}$, the medium becomes less rigid, having the Young's modulus E_1($E_1< E$), and then it becomes necessary to solve the problem of propagation of plane and spherical waves for region 2, bounded by the surface *AB*, the characteristic of the positive direction BC and the boundary of the *AC* layer (see Fig. 2, b).

Let us proceed to the presentation of the solution of a plane problem with the following boundary conditions:

$$\left.\begin{aligned} u(r,t) = u^{**}(t),\ \frac{\partial u}{\partial t}(r,t) = \dot{u}^{**}(t) \\ p = p^{**} = \text{const},\qquad \varepsilon = \varepsilon^{**} = \text{const}\end{aligned}\right\}\ \text{at } r = R_1^*(t). \tag{13}$$

The equation of state of the medium in this case has the form

$$p(r,t)=p^{**} + E_1(\varepsilon - \varepsilon^{**}), \tag{14}$$

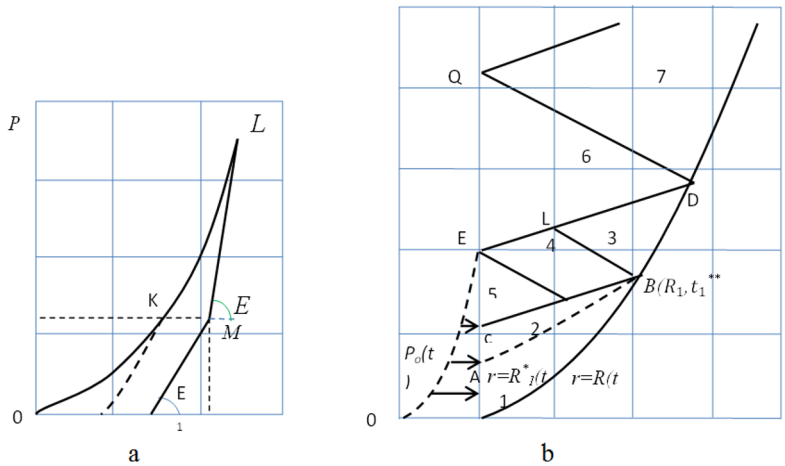

Fig. 2. a-diagram of the environment for broken line unloading; b-image of the shock wave of a broken unloading.

where $E_1 = \rho_0, c_{p_1}^2$; p^{**}, ε^{**} are given values determined from the diagram $p \sim \varepsilon$. Then (4), taking into account (13), (14), admits the solution

$$u(r,t) = u^{**}(R_0, t_0^{**}) - \frac{1}{2c_{p1}}\left\{\int_{z_{30}}^{r-c_{p1}t}\dot{u}^{**}[F_3(z_3)]dz_3 - \int_{z_{40}}^{r+c_{p1}t}\dot{u}^{**}[F_4(z_4)]dz_4\right\} \tag{15}$$

where $z_{i0} = R_0 \pm F\, c_{p1}t_0^{**}$; $F_i(z_i)$ $(i = 3, 4)$ is the root of the equation $R_0^*(t) \pm c_{p1}t = z_i$ with respect to time t. In this case, from (1), taking into account (13), (15)

$$p_0(t) = p^{**} - \frac{\rho_0}{2}\int_{R_1^*(t)}^{R_0}\{\dot{u}^{**}[F_3(r - c_{p1}t)]dz_3 + \dot{u}^{**}[F_4(r + c_{p1}t)]\}dr. \quad (16)$$

3. Results of calculations. Some calculation results for a medium with linear unloading with initial parameters [7]

$$\alpha_1 = 12{,}127\cdot 10^2 \text{ kg/sm}^2, \quad \alpha_2 = 58{,}73\cdot 10^3 \text{ kg/sm}^2, \quad (17)$$
$$E = 14\cdot 10^3 \text{ kg/sm}^2, \quad E_x = 391 \text{ m/s};$$

$$\alpha_1 = 18\cdot 10^2 \text{ kg/sm}^2, \quad \alpha_2 = 82\cdot 10^3 \text{ kg/sm}^2, \quad E = 18\cdot 10^3 \text{ kg/sm}^2, \quad (18)$$
$$R_1 = 440 \text{ m/s } (p_0 = 105 \text{ kg/sm}^2, \rho_0 = 200 \text{ kg}\cdot\text{s}^2/\text{m}^4, \quad R_2 = 2R_1\cdot 10^2 \text{ m/s}^2$$

For the case when the shape of the front surface is given as a polynomial of the second degree

$$R(t) = R_0 + R_2\frac{t^2}{2}$$

where $\dot{R}(t) \geq 0$are shown in Fig. 3–5 in dimensionless form, respectively, with respect to the maximum value of pressure, velocity, units of length, and time. And in Fig. 3a shows the graphs of changes in the load $p_0(t)$ and the mass velocity of the medium u(t) at the boundary of the flat and spherical (dashed lines) layer and at the front R(t) as a function of time. From this it can be seen that in order to maintain the same pressure at the corresponding points of the flat and spherical front, it is necessary to apply a larger value on the spherical cover, compared to the flat one; load. This is a consequence of the reverse formulation of the problem, since in the direct formulation (if a load is given), the pressure on a spherical front drops faster than on a flat one. In this case, the process of damping the pressure (velocity) at the wave front occurs more slowly than at the layer boundary. In Figure 3b shows the change in $p(r, t)$ and u(r, t) depending on the spatial coordinate r at a fixed time t. Note that the pressure varies with r in a linear fashion, while the velocity is mostly non-linear. In order to study the dependence of the load $p_0(t)$ and pressure p*(t) on the shape of the shock front

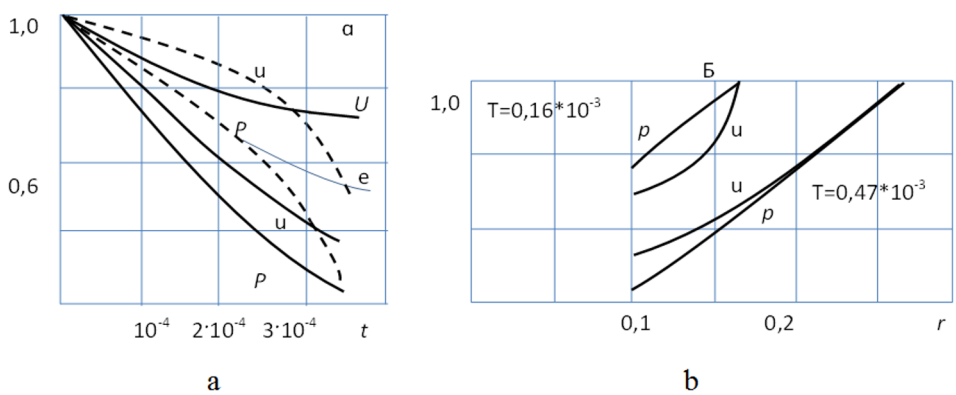

Fig. 3. A-change in speed and pressure depending on time; b-change in speed and pressure for fixed points in time

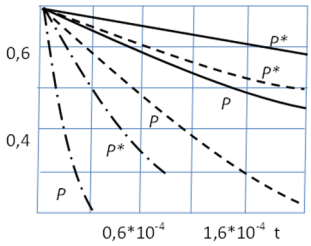

Fig. 4. Change in pressure over time

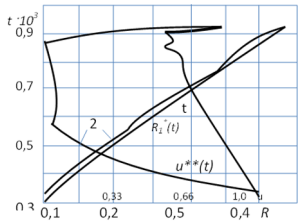

Fig. 5. Constant pressure surface and velocity versus time curve

Fig. 4 shows the curves of $p_0(t)$ and $p^*(t)$ as a function of t for the case (17) for the values $R_2 = 2R_1 \cdot 10^2$; $4R_1 \cdot 10^2$; $2R_1 \cdot 10^3$ by solid, dashed, and dash-dotted lines, respectively. The curves in Figure 4 show that in a plane problem with $R_2 = 4R_1 \cdot 10^2$ and $R_2 = 2R_1 \cdot 10^3$ the pressure $p^*(t)$ and the load $p_0(t)$ decrease non-linearly with increasing t. In Fig. 5 plots the surface of constant pressure p^{**} = const and the velocity distribution curve $u^{**}(t)$ on it depending on t, which serve as the boundary condition for studying the propagation of a plastic wave in a nonlinearly compressible medium in subsequent regions (curves 2 refer to a spherical wave).

3 Conclusion

The results of calculations for a specific structure of the medium, when the shape of the front surface is given by a polynomial of the second degree, are given in a dimensionless form, respectively, relative to the maximum pressure, velocity, length and time. This shows that in order to maintain the same pressure at the corresponding points and on the spherical front, it is necessary to apply a greater load to the spherical cavity compared to the flat one. This is a consequence of the reverse formulation of the problem, since in the direct formulation (if a load is given), the pressure on a spherical front drops faster than on a flat one. In this case, the process of pressure (velocity) at the wave front occurs more slowly than at the layer boundary.

References

1. N. Mamadaliev, Sh. Mamatkulov, On one method of solving wave problems for a nonlinearly compressible medium, PMTF, **6** (1977)
2. A. Phillips, E. R. Wood, et al., On the theory of plastic wave propagation in a bar-unloading waves, Intern. J. Non-Linear Mech., Rus. nep.— Sat. Mechanics, **8**, 1-16. (1973)
3. Kh. A. Rakhmatulin, A. Ya. Sagomonyan, N. A. Alekseev, Questions of soil dynamics. M., ed. Moscow State University, (1964)
4. M. A. Sadovsky. M., Nedra, Mechanical effect of underground explosion (1971)
5. N. Mamadaliev, V. P. Molev, On the propagation of a two-dimensional plastic wave in a nonlinearly compressible half-plane, PMTF, 4 (1977)
6. S. K. Godunov, V. S. Ryaben'kiy, Difference schemes, Moscow (1973)
7. N. A. Alekseev, Method for determining the dynamic characteristics of soils at high pressures, In the book: Dynamics of soils, Moscow, Gosstroyizdat (1961)

8. L. I. Slepyan, Non-stationary elastic waves, Leningrad, Shipbuilding (1972)
9. N. I. Shishkin, Seismic efficiency of contact explosion and high-velocity impact. Journal of Applied Mechanics and Technical Physics (PMTF) Novosibirsk, Russia, **2**, 3-12 (2007)
10. A. F. Revuzhenko, O. A. Mikenina, Elastic-plastic rock model with a linear structural parameter. Journal of Applied Mechanics and Technical Physics (PMTF), Novosibirsk, Russia, **2**, 167-176 (2018)
11. Yu. M. Volchkov, I. O. Bogulskiy, Numerical study of wave propagation in a nonlinear dissipative material. Journal of Applied Mechanics and Technical Physics (PMTF), Novosibirsk, Russia, **5**, 114-118 (2021)
12. G. M. Lyakhov, *Waves in soils and porous media*, Moscow, Nauka, 288 (1982)
13. K. S. Sultanov, Wave theory of seismic resistance of underground structures, Tashkent, Fan, 392 (2016)
14. V. G. Bazhenov, V. L. Kotov, et al., Investigation of wave processes in the soil medium during the explosion of a superimposed charge. Proceedings of the Russian Academy of Sciences, Journal of Solid State Mechanics (MTT), **2**, 70-77 (2001)
15. V. N. Kukudzhanov, Propagation of waves in elastic-viscoplastic materials with a general diagram. Proceedings of the Russian Academy of Sciences, Journal of Solid State Mechanics (MTT), **5**, 96-111 (2001)
16. A. A. Bakhodirov, K. S. Sultanov, Waves in a viscoelastic rod surrounded by a soil medium under smooth loading. Proceedings of the Russian Academy of Sciences, Journal of Solid State Mechanics (MTT), **3**, 132-144 (2014)
17. P. V. Loginov, Z. R. Salikhova, K. S. Sultanov, Experimental-theoretical method for determining the mechanical characteristics of soils under dynamic loading. Proceedings of the Russian Academy of Sciences, Journal of Solid State Mechanics (MTT), **4**, 76-93 (2019)
18. M. Sh. Israilov, H. Hamidu, Underground explosion action: rapid expansion of a spherical cavity in an elastic medium. Proceedings of the Russian Academy of Sciences, Journal of Solid State Mechanics (MTT), **3**, 100-117 (2021)
19. A. M. Bragov, V. V. Balandin, et al., Solution of the problem of expansion of a spherical cavity in relation to the assessment of the resistance to penetration of a solid body into the soil. Proceedings of the Russian Academy of Sciences, Journal of Solid State Mechanics (MTT), **3**, 110-121 (2022)
20. I. Mirzaev, E. A. Kasimov, The impact of harmonic and seismic waves on the pipeline during different earthquakes. Uzbek Journal of Problems of Mechanics, **3**, 80-83 (2020)
21. K. Weinberga, C. Wienersb, Dynamic phase-field fracture with a first-order discontinuous Galerkin method for elastic waves.Computer Methods in Applied Mechanics and Engineering, February, 389, 114330 (2022)
22. S. Gollob, G. K. Kocur, Analysis of the wave propagation paths in numerical reinforced concrete models, Journal of Sound and Vibration, **494**, 115861 (2021)
23. A. Aghaei, N. Bochud, G. Rosi, S. Naili, Wave propagation across a functionally graded interphase between soft and hard solids: Insight from a dynamic surface elasticity model. Journal of the Mechanics and Physics of Solids, **151**, 104380 (2021)