The stress state of the rock mass with spherical cavity

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Abstract. One of the main hypotheses accepted in the mechanics of deformable solids is the assumption of the homogeneity of materials. This means that all mechanical characteristics of the material (modulus of elasticity, Poisson's ratio, yield strength, relaxation parameters, etc.) are constant over the volume of the body, in other words, these characteristics are constants. This hypothesis makes it possible not to take into account the natural inhomogeneity of materials at the microlevel - the presence of various fractions in composite materials (concrete, fiberglass, etc.), crystal lattice defects, etc. Examples can be given when various physical phenomena (temperature field, radiation exposure, explosive impact, etc.) lead to a change in the mechanical characteristics along the body. These changes can be quite significant. So, for example, in the presence of high-gradient temperature fields, the deformation characteristics of materials at different points of the body can change dozens of times. Thus, when calculating and designing structures, it is necessary to take into account such macroheterogeneity, since it leads to a significant change in the stress-strain state of bodies. This article considers the problem associated with the continuous inhomogeneity of materials. It means such a heterogeneity that arose in the process of creating an underground cavity with the help of an explosion. In contrast to the classical mechanics of a deformable solid body, the problems of which are reduced to differential equations with constant coefficients, in the mechanics of continuously inhomogeneous bodies we deal with equations with variable coefficients, which greatly complicates their solution. In this case, depending on the type of inhomogeneity functions - functions that describe the change in mechanical characteristics along the coordinates - differential equations turn out to be significantly different.

Keywords: heterogeneity, explosion, sphere, mechanical characteristics, underground cavity

1 Introduction

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\[ \sigma_r = \lambda \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \varphi} \right) + \mu \frac{\partial u}{\partial r} - \mu K_{\varepsilon, \alpha} \]
\[ \sigma_\theta = \lambda \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \varphi} \right) + \mu \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) - \mu K_{\varepsilon, \alpha} \]
\[ \sigma_\varphi = \lambda \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \varphi} \right) + \mu \left( \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} \right) - \mu K_{\varepsilon, \alpha} \]
\[ \tau_{r\theta} = \mu \left( \frac{\partial u}{\partial r} - \frac{\partial v}{\partial \theta} \right) \]
\[ \tau_{\theta\varphi} = \mu \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \varphi} \right) \]
\[ \tau_{\varphi r} = \mu \left( \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} \right) \]

\[ \mu V^2 u - \mu \left( u + \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \varphi} \right) + \frac{\partial \mu}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{\partial w}{\partial \varphi} \right) + \frac{\partial \mu}{\partial \theta} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) + \frac{\partial \mu}{\partial \varphi} \left( \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} \right) = 0 \]
\[\mu \mathbf{V} \cdot \nabla = \frac{\mu}{r^2} \left( \frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\mu}{r} \frac{\partial v}{\partial \theta} \right) + \frac{\mu}{r^2} \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} \right) + \frac{\mu}{r} \frac{\partial u}{\partial r} + \frac{\mu}{r} \frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} - w \frac{\partial \theta}{\partial \phi} = 0\]

\[\nabla \cdot \mathbf{w} + \frac{\mu}{r^2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} - w \right) + \frac{\mu}{r^2} \left( \frac{\partial u}{\partial r} + \frac{\partial w}{\partial \theta} - \frac{w}{r} \right) + \frac{\mu}{r} \frac{\partial u}{\partial r} + \frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial \phi} = 0\]

\[
\begin{align*}
\nabla &= \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( r \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \\
E_{cp} &= \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} + \frac{\partial w}{\partial \theta} + \frac{\partial \theta}{\partial \phi}
\end{align*}
\]

\[
\begin{align*}
\mu \mathbf{V} \cdot \nabla u &= \mu \left( \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \phi} \right) + \frac{\mu}{r^2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \phi} \right) + \mu \left( \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta} \right) \\
\mu \mathbf{V} \cdot \nabla v &= \mu \left( \frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial \phi} \right) + \frac{\mu}{r^2} \left( \frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial \phi} \right) + \mu \left( \frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta} \right)
\end{align*}
\]

\[
\begin{align*}
r &= a & \sigma_r &= -p_a \quad r &= b & \sigma_r &= -p_b \\
\tau_{r\theta} &= q_a & \tau_{r\theta} &= q_b
\end{align*}
\]

3.2. Numerical-analytical method of solution axisymmetric problem

\[
\begin{align*}
\sigma_r &= \lambda \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta} \right) + \mu \frac{\partial u}{\partial r} \\
\sigma_\theta &= \lambda \left( \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial \phi} \right) + \mu \left( \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta} \right) \\
\sigma_\phi &= \lambda \left( \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \theta} \right) + \mu \left( \frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta} \right)
\end{align*}
\]

\[
\begin{align*}
u(r; \theta) &= \sum_{n=1}^{\infty} u_n(r) P_n(\cos \theta) & \nu(r; \theta) &= \sum_{n=1}^{\infty} v_n(r) \frac{dP_n(\cos \theta)}{d\theta}
\end{align*}
\]
\[ P_n(\theta) \]

\[ \frac{d^2 P_n(\theta)}{d\theta^2} + \frac{dP_n(\theta)}{d\theta} \theta + n(n+1)P_n(\theta) = 0 \]

Equations (12), (13) must be supplemented with boundary conditions for the functions \( u \) and \( v \).

Here and below, the prime denotes differentiation with respect to \( r \).

Substitution of relations (9) into Equations (12), (13) allows, using Equation (10), to obtain

\[ \begin{cases}
P_a = \sum_{n=1}^{\infty} \left( \begin{array}{c}
\frac{\partial}{\partial n} P_n(\theta) \\
\frac{\partial}{\partial n} Q_n(\theta)
\end{array} \right) P_n(\theta), \quad n \neq m \\
\frac{\partial}{\partial n} P_n(\theta) + \frac{\partial}{\partial n} Q_n(\theta) = \sum_{n=1}^{\infty} \left( \begin{array}{c}
\frac{\partial}{\partial n} P_n(\theta) \\
\frac{\partial}{\partial n} Q_n(\theta)
\end{array} \right) P_n(\theta),
\end{cases} \]

The expansion coefficients are determined by the formulas:

\[ \sigma_r = \sum_{n=1}^{\infty} \left[ (\lambda + \mu) v'_n + \frac{\lambda}{r} u_n - \frac{\lambda n(n+1)}{r} v_n + \kappa g_n \right] P_n(\theta) \]

\[ \tau_{r\theta} = \sum_{n=1}^{\infty} \mu \left[ v'_n - \frac{v_n}{r} + \frac{u_n}{r} \right] P_n(\theta) \]

\[ \left( \lambda + \mu \right) v'_n + \left( \frac{\lambda}{r} + \mu \right) u_n' - \left[ \frac{n(n+1)}{r} + \left( \lambda + \mu \right) + \frac{\lambda'}{r} \right] u_n + \]

\[ + n(n+1) \frac{\lambda}{r} v'_n + \left( \frac{\lambda}{r} + \frac{\mu}{r} \right) v_n = 0 \]

\[ \mu v''_n + \left( \frac{\mu}{r} + \mu' \right) v'_n - \left[ n(n+1) \frac{\lambda}{r} + \frac{\mu}{r} \right] v'_n + \frac{\lambda}{r} u'_n + \left( \frac{\lambda}{r} + \frac{\mu}{r} \right) u_n = 0 \]
\begin{equation}
\frac{\partial}{\partial r} \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = \begin{pmatrix}
\lambda + \mu \\
-\lambda
\end{pmatrix} u_n + \frac{\lambda}{r} u_n - \frac{\lambda(n+1)}{r} v_n = \begin{pmatrix}
-p_{a,1} \\
-p_{b,1}
\end{pmatrix}
\end{equation}

\begin{equation}
\mu \left( u_n + \frac{\lambda}{r} u_n + \frac{\lambda}{r} v_n \right) = \begin{pmatrix}
q_{a,1} \\
q_{b,1}
\end{pmatrix}
\end{equation}

3.3. Numerical solution algorithm

\[ \frac{dY_n}{dr} = A_n Y_n + F_n \]

\[ y_{r,n} = u_n \quad y'_{r,n} = u'_n \quad y_{v,n} = v_n \quad y'_{v,n} = v'_n \]

\[ A \begin{pmatrix}
\alpha_1 \\
\beta_1 \\
\alpha_2 \\
\beta_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix} \]

\[ a_{1,1} = \frac{n(n+1)}{r} + \frac{\lambda}{r} + \frac{\lambda'}{r} \]

\[ \dot{a}_{1,1} = \frac{n(n+1)}{r} + \frac{\lambda}{r} + \frac{\lambda'}{r} \]

\[ a_{2,2} = -n(n+1) \frac{\lambda + \mu}{r} \]

\[ a_{2,3} = a_{3,2} = a_{3,3} = a_{3,4} = a_{4,4} \]

\[ f_1 = f_2 = \frac{(K_{g,n})' - R_n}{\lambda + \mu} f_3 = -\frac{T_n}{\mu} \]
\[ r = a \cdot b \] \[ B_n \gamma_n = \Phi_n \]

\[
B_n = \begin{bmatrix} b & b & b \\ b & b & b \end{bmatrix} \quad \Phi_n = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}
\]

\[
b_1 = \frac{\lambda}{r} \quad b_2 = (\lambda + \mu) \quad b_3 = -\frac{\lambda n(n+1)}{r} \quad b_4 = \frac{\mu}{r} \quad b_5 = -\frac{\mu}{r} \quad b_6 = \mu
\]

\[
\varphi = \begin{bmatrix} K_g \gamma_n + \begin{pmatrix} -p_{a,n} \\ -p_{b,n} \end{pmatrix} \\ q_{a,n} \\ q_{b,n} \end{bmatrix}
\]

4 Results

\[ p_b(\theta) = \gamma (H - b) \begin{pmatrix} \frac{v}{-v} + \frac{\theta}{-\theta} \end{pmatrix} \]

\[ q_b(\theta) = \gamma (H - b) \begin{pmatrix} -\frac{v}{-v} \end{pmatrix} \]

Fig. 1. Calculation scheme of an array with a spherical cavity.
\[ R = -\gamma \Theta \quad \Theta = \gamma R \]

\[
T(r) = \frac{b-a}{b-a} \left[ (T_a - T_b) \frac{ab}{r} + T_b b - T_a a \right]
\]

\[
T = \frac{T_a - T_b}{r} a + T_b
\]

\[
P_n = A_n \sum_{k=\infty}^{n-1} B_{nk} t^{n-k} \frac{dP_n}{dt} = A_n \sum_{k=\infty}^{n-1} C_{nk} t^{n-k}
\]

\[
A_n = \frac{1}{n!} \sum_{k=\infty}^{n-1} B_{nk} = \frac{n(n-k)!}{k!(n-k)!} C_{nk} = \frac{(-1)^n (n-k)!}{k!(n-k)!}
\]

\[
t = \Theta
\]

\[
P_1 = 0 \quad P_2 = \Theta \quad P_3 = -\left( t^2 - 1 \right) \quad P_4 = -\left( t^3 - t \right)
\]

\[
p_b = p_b + P_1 + p_b + P_2 + p_b + P_3
\]

\[
q_b = q_b \frac{dP_1}{d\theta} + q_b \frac{dP_2}{d\theta} + q_b \frac{dP_3}{d\theta}
\]

\[
p_{bn} = \frac{\gamma H}{-v} \quad q_{bn} = -\frac{\gamma H}{-v}
\]
The remaining coefficients for $4 \geq n$ are equal to zero. Similarly, comparing representations for $R$ and $\varphi$ with formulas (18), taking into account (20), we find $\gamma - \gamma = 1$. At the same time, $R = \varphi$ and $n = n$.

Note that, in contrast to the problems in which the numerical-analytical method was used, where the accuracy of the numerical solution was mainly determined by two parameters: $N$ – the number of terms of the Fourier series and $M$ – the number of steps into which the integration interval was divided, in this case, the accuracy depends only on $M$, since the finite sums of the series exactly satisfy the boundary conditions. Thus, the solution of the problem under consideration can be obtained by numerically solving 4 boundary value problems (for $n=0,1,2,3$) described by the matrix differential equation (15) with boundary conditions (16).

In this case, the solution of the problem for $n=0$ is simplified, since $r \equiv r$.

The calculation was carried out using the MOPVU program, in Fortran IV.

Let us consider an example of calculation, when the array is only under the action of external surface loads and the inhomogeneity of the material is due only to the explosive effect. The obtained results in comparison with some analytical data allow us to determine the required values of the number of steps $b$, $a$ and the radius of the outer surface of the cut array $b$.

Considering that at $H < a < b$ volume forces can be neglected, we will carry out the calculation without taking into account the self-weight of the cut-out array ($R = \varphi = 0$). In the absence of temperature influence, temperature inhomogeneity can be ignored. The modulus of elasticity, depending on $r$, is changed by the formula:

$$E(r) = E \left[ 1 + \left( \frac{a}{r} \right)^m \right]$$

where $k$ and $m$ are parameters that allow an increase or decrease in the modulus of elasticity. The accuracy of the results obtained can be estimated by comparing the calculated stresses at some points of the surface with analytical values, which partially follow from the boundary conditions, as well as from the solution of the problem of loading a solid mass, since at a sufficient distance from the cavity, the stress concentration near the cavity and the influence of local inhomogeneity can be neglected.

In table 1. formulas for stresses at characteristic points are given (see Fig. 2).
Table 1. Formulas for stresses on the outer surface:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sigma_r$</th>
<th>$\sigma_\theta$</th>
<th>$\sigma_\phi$</th>
<th>$\tau_{r\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$-\gamma(H - b)$</td>
<td>$-k\gamma(H - b)$</td>
<td>$-k\gamma(H - b)$</td>
<td>$q_b(\theta)$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$-p_b(\theta)$</td>
<td>$q_b(\theta)$</td>
<td>$q_b(\theta)$</td>
<td>$q_b(\theta)$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$-k\gamma H$</td>
<td>$-\gamma H$</td>
<td>$-k\gamma H$</td>
<td>$q_b(\theta)$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$-p_b(\theta)$</td>
<td>$q_b(\theta)$</td>
<td>$q_b(\theta)$</td>
<td>$q_b(\theta)$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$-\gamma(H + b)$</td>
<td>$-k\gamma(H + b)$</td>
<td>$-k\gamma(H + b)$</td>
<td>$q_b(\theta)$</td>
</tr>
</tbody>
</table>

Table 2. Comparison of stresses calculated by numerical-analytical method, with analytical values.

<table>
<thead>
<tr>
<th>Points</th>
<th>Stress</th>
<th>Method</th>
<th>Material</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\sigma_\theta$</td>
<td>numerical</td>
<td>homogeneous</td>
<td>7.09 MPa</td>
</tr>
<tr>
<td>A</td>
<td>$\sigma_\phi$</td>
<td>analytical</td>
<td>homogeneous</td>
<td>7.06 MPa</td>
</tr>
<tr>
<td>C</td>
<td>$\sigma_\theta$</td>
<td>numerical</td>
<td>homogeneous</td>
<td>30 MPa</td>
</tr>
<tr>
<td>C</td>
<td>$\sigma_\phi$</td>
<td>analytical</td>
<td>homogeneous</td>
<td>29.99 MPa</td>
</tr>
<tr>
<td>E</td>
<td>$\sigma_\theta$</td>
<td>any option</td>
<td>homogeneous</td>
<td>10.82 MPa</td>
</tr>
<tr>
<td>E</td>
<td>$\sigma_\phi$</td>
<td>any option</td>
<td>homogeneous</td>
<td>10.52 MPa</td>
</tr>
</tbody>
</table>

Some differences in the given stress values can be explained by the fact that when calculating $\sigma_\theta$ and $\sigma_\phi$, the functions $u$ and $v$ are numerically differentiated, which always leads to a decrease in accuracy.

In general, it can be noted that with the numerical-analytical method of calculation, the choice of the ratio $\frac{a}{b}$ and the division of the segment $[a, b]$ into 100 steps gives quite satisfactory results.

On fig. 3 shows diagrams of normal stresses built along the horizontal radius ($\theta = 90^\circ$) and along the contour of the cavity.
Fig. 3. Stress diagrams in an array with a spherical cavity:

(a) along the radius $r = \theta$

(b) along the angular coordinate $r = a$

As in the one-dimensional problem of calculating an array with a spherical cavity, as well as in the plane problem of calculating an array with a cylindrical hole, in the zone closest to the cavity, significant differences are observed in the stress values in homogeneous and inhomogeneous arrays. In the presence of a spherical cavity, stress decay occurs faster than in the case of a cylindrical hole. It can be noted that at $r \geq a$, the stresses in homogeneous and inhomogeneous arrays coincide. Approximately at the same distance, the stress diagrams approach the asymptotic values corresponding to the stresses on the outer surface of the cut out spherical region. Hence, we can conclude that the assumption about the local influence on the stress state of the spherical concentrator and inhomogeneity is quite reasonable. The change in stresses along the contour of the cavity also qualitatively agrees with the results obtained for an array with a cylindrical hole.

5 Conclusions

Summing up the results of the article, it should be noted that the questions of linear and nonlinear mechanics of inhomogeneous bodies have been of interest to scientists for almost 90 years. At the beginning of the article, the authors mentioned the initiators of the development of the mechanics of inhomogeneous bodies. The volume of the article does not allow listing scientists who work in the field of mechanics of inhomogeneous bodies. At the end of the conclusions, a small list of works [11-20] of Russian scientists who are currently devoting their efforts to the development of the mechanics of inhomogeneous bodies is given.

Reference


