Accounting for tension in transverse bending of statically indeterminate elastic rods

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Abstract. For statically indeterminate rod structures with excess constraints in the horizontal direction, tension forces arise under transverse loading, significantly affecting the stress-strain state of the rods compared to similar rods caused by bending alone (in the absence of excess longitudinal constraints). This fact requires computational analysis, and in this work, analytical dependencies of the deflection of the rod and the magnitude of tension due to transverse loading were obtained for cases with clamped-clamped and two hinged supports for the ends of the rod based on the derived Lagrangian principles of mechanics. Based on analytical dependencies, a graphical method was developed to determine the axial force under given stiffness and transverse loading for the considered computational cases. These results can be applied to the study of the stress-strain state of rod structures with excess constraints in the horizontal direction under transverse loading.

1 Introduction

Accounting for tension in the transverse bending of statically indeterminate elastic rods is an important task, as tension can significantly affect the stress-strain state. This allows for more accurate results and increased reliability in the calculation of structures. When a rod is bent under transverse load and longitudinal force, there is a complex interaction between bending stress and longitudinal stress (tension or compression) in the rod. This leads to changes in deformation and deflection of the rod compared to the deflection caused by the influence of transverse load alone.

To account for the influence of tensile or compressive forces on the deflection of the rod, methods from the theory of elasticity and structural mechanics [1-6], or numerical modeling [7-15], can be used. In the theory of elasticity, the deflection of the rod can be calculated using an equation that describes the interaction between transverse load and longitudinal force:

$$\frac{d^2y}{dx^2} + \frac{P}{AE}y = -\frac{M}{EI}, \tag{1}$$

where y – the deflection of the rod, x – is the coordinate along the beam, P – the force in the direction of the longitudinal axis of the beam, A – the cross-sectional area, E – the modulus of elasticity, M – the bending moment, and I – the moment of inertia of the cross section.

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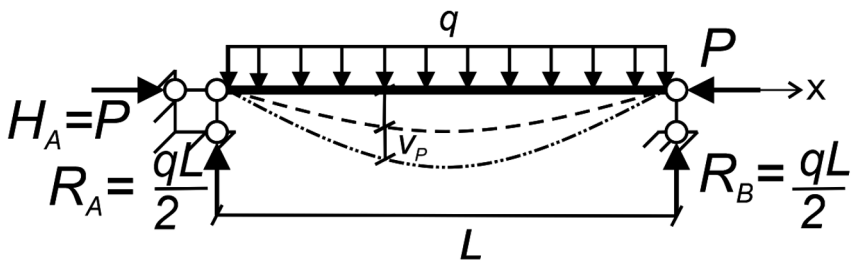

Fig. 1. The calculation scheme for longitudinal-transverse bending, which arises from the combined action of transverse and longitudinal loads.

From the numerical modeling perspective, the solution to this problem is only possible in a geometrically nonlinear formulation. The finite element method (FEM) is often used as an effective method for studying such mechanical problems based on rod, plate, and volume elements. In [15], the authors investigated the solution to the problem of longitudinal-transverse bending of a beam based on numerical solutions in various software packages that are most commonly used in Russia for structural mechanics problems. The main conclusion was that not all commonly used software packages are capable of determining the behavior of the rod in the critical displacement range during longitudinal-transverse bending, although this work only considered a pre-specified longitudinal force.

However, the aforementioned works do not take into account the effects associated with determining the tensile force caused by transverse load on rod structures with excessive connections in the longitudinal direction. As a result of this fact, the present work and its scientific novelty are related to determining the dependence of the longitudinal force magnitude, which arises in such rod structures, on the transverse load magnitude.

Let's describe the mechanics of such structures. With zero transverse load, there is no deflection or tension (resistance), and the length of the element remains unchanged. However, as the transverse load gradually increases, bending occurs, which is also counteracted by the increasing longitudinal force (tension), which will increase the length of the rod (stretch), depending on the magnitude of the transverse load, as well as the mechanical properties of the structure (stiffness parameters). In structural elements such as beams, the presence of a sliding support allows for small longitudinal displacements (relative to the beam deflection), which cause deformation without increasing the length of the beam. However, in the case of clamping on both sides, preventing horizontal displacement, bending occurs due to elongation of the rod caused by tension. Thus, the tension force affects the deflection, which affects the length, while the elongation of the element affects the tension force.

This work explores the issue of tension arising from the bending of statically indeterminate elastic rods without sliding supports, caused by transverse loads, as well as determining the dependencies that affect the magnitude of tension. The influence of tension force from the transverse load on various types of rods is studied, starting from flexible rod elements (strings or threads) that do not work on bending, to medium rod elements, which satisfy the condition, where h is the height of the rod, l is length of span. The analytical solutions obtained will allow us to draw conclusions about the influence of transverse load and stiffness parameters on the magnitude of rod tension, as well as to determine the resulting contribution to the stress-strain state of rods with double-sided fixation (hinged-hinged and rigid-rigid) without sliding supports.

2 Methods

Let us derive the differential equation for the deflection of a beam from variational principles. We will consider a beam that is deflected in the direction of the axis x. The Lagrangian density for the beam [16] has the form:

$$\mathscr{L} = \mathcal{T} - \mathcal{U} = \frac{\rho}{2}\left(x^2\left(X'\right)\dot{}^2\right) - \frac{E^2}{2}\left(X''\right)^2 \frac{\sigma_T}{2}\left(X'\right)^2\Big), \tag{2}$$

where $\mathcal{T}$ — density of kinetic energy, $\mathcal{U}$ — density of potential energy, f — density of mass force per unit volume, σ_T — longitudinal normal stress created by tension force T, ρ — density of material of the rod, E — modulus of elasticity of the material of the rod, $X(y,t)$ — function deflection of the rod z, point — variable$t$.

By substituting this Lagrangian into the variational principle of least action by Hamilton, we obtain the Lagrange equations:

$$\frac{\delta\mathscr{L}}{\delta X} = \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial^2 X}{\partial x^\mu \partial x^\nu}\right)}\right) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial X}{\partial x^\mu}\right)}\right) + \frac{\partial\mathscr{L}}{\partial X} = 0, \tag{3}$$

where $\frac{\delta\mathscr{L}}{\delta X}$ — variational derivative of the Lagrangian $\mathscr{L}$ to a function $X(y,t)$.

These equations are simplified for the considered case:

$$\frac{\delta\mathscr{L}}{\delta X} = \left(\frac{\partial\mathscr{L}}{\partial\left(X''\right)}\right)'' + \left(\frac{\partial\mathscr{L}}{\partial\left(X'\right)}\right)'' - \left(\frac{\partial\mathscr{L}}{\partial(\dot{X})}\right)^{\bullet} \frac{\partial\mathscr{L}}{\partial X}\Big) \tag{4}$$

Substitution of partial derivatives:

$$\frac{\partial\mathscr{L}}{\partial X} = f; \quad \frac{\partial\mathscr{L}}{\partial(\dot{X})} = \rho\dot{X}; \quad \frac{\partial\mathscr{L}}{\partial(X')} = -\sigma_T X'; \quad \frac{\partial\mathscr{L}}{\partial\left(X'\right)\rho^2 \frac{\partial\mathscr{L}}{\partial(X'')}^{2''}} \tag{5}$$

get:

$$\rho\ddot{X} - \rho x^2\ddot{X}'' + Ex^2 X^{IV} - \sigma_T X'' = f. \tag{6}$$

Integrating over the cross-sectional area, we obtain:

$$\rho A\ddot{X} - \rho I\ddot{X}'' + EIX^{IV} - TX'' = q, \tag{7}$$

where A — cross-sectional area, I — moment of inertia of a section.

Setting the time derivatives to zero, we obtain an ordinary differential equation for the static deflection of the rod under the action of transverse load and tensile longitudinal force, described by an ordinary differential equation for the function $X(y)$:

$$EIX^{IV} - TX'' = q. \tag{8}$$

The solution of this differential equation for a homogeneous beam is found as the sum of the general solution of the homogeneous equation and the particular solution of the non-homogeneous equation:

$$X = C_1 + C_2 y + C_3 ch(ky) + C_4 sh(ky) - \frac{q}{2T}y^2, \tag{9}$$

where: $k = \sqrt{\frac{T}{EI}}$.

Further, in all cases of interest to us, we will obtain a transcendental algebraic equation to find the solution T.

3 Results

3.1 Double rigid fixation

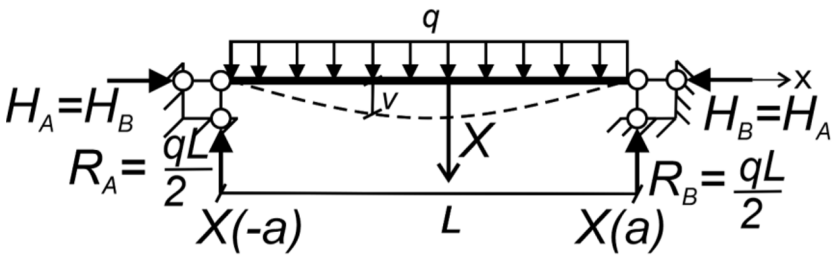

Fig. 2. The calculation scheme of a rod with double rigid fixing under the action of a uniformly distributed load

Assuming that the rod is located from $y=-a$ to $y=a$ and is clamped rigidly on both sides, we obtain the following boundary conditions:

$$X(-a)=X(a)=X'(-a)=X'(a)=0. \tag{10}$$

Taking into account the boundary conditions, the solution takes the form:

$$X=\frac{qa}{Tksh(ka)}[ch(ky)-ch(ka)]+\frac{q}{2T}(a^2-y^2). \tag{11}$$

Then:

$$X'=\frac{q}{T}\left[a\frac{sh(ky)}{sh(ka)}-y\right]. \tag{12}$$

For small deflections in the linear approximation, elongation is described by the expression [4]:

$$\Delta L=\int_{-a}^{a}\sqrt{1+X'^2}dy-2a\approx\frac{1}{2}\int_{-a}^{a}X'^2dy, \tag{13}$$

where, according to Hooke's law:

$$T=ES\frac{\Delta L}{2a}=\frac{ES}{4a}\int_{-a}^{a}X'^2dy. \tag{14}$$

Substituting to (14) X' in view $k=\sqrt{\frac{T}{EI}}$, the result is a transcendental algebraic equation for finding T:

$$T=\frac{ES}{4a}\frac{q^2}{T^2}\int_{-a}^{a}\left[a\frac{sh(ky)}{sh(ka)}-y\right]^2dy=\frac{ESq^2a^2}{4T^2}\int_{-1}^{1}\left[\frac{sh(kat)}{sh(ka)}-t\right]^2dt=$$
$$=\frac{ESq^2a^2}{4T^2}\left[\frac{5}{3}+\frac{4}{(ka)^2}-\frac{3}{ka}cth(ka)-cth^2(ka)\right]. \tag{15}$$

from here:

$$\frac{E^2I^3}{Sq^2a^8}=\frac{1}{4(ka)^6}\left[\frac{5}{3}+\frac{4}{(ka)^2}-\frac{3}{ka}cth(ka)-cth^2(ka)\right]\equiv F(ka). \tag{16}$$

Then:

$$k=\frac{1}{a}F^{-1}\left(\frac{E^2I^3}{Sq^2a^8}\right),\quad T=EIk^2=\frac{EI}{a^2}\left[F^{-1}\left(\frac{E^2I^3}{Sq^2a^8}\right)\right]^2, \tag{17}$$

where: F^{-1} — function inverse to$F$.

The maximum deflection will be at the center of the rod and is equal to:

$$X(0)=\frac{qa^4}{EI(ka)^3}\left[\frac{ka}{2}-th\left(\frac{ka}{2}\right)\right]. \tag{18}$$

Special attention is paid to extreme particular cases. Let's consider the case where the deflection is much smaller than the thickness of the rod. Then $ka\ll 1$, and the leading terms of the $X(0)$ Taylor series expansion are as follows:

$$X(0)\simeq\frac{qa^4}{24EI}\left[1-\frac{1}{10}(ka)^2\right] \tag{19}$$

The leading term of the expansion of the function F have view$F(ka) \simeq \frac{1}{945}(ka)^{-2}$, thus:

$$(ka)^2 \simeq \frac{1}{945}\frac{Sq^2a^8}{E^2I^3}, \quad T = k^2EI \simeq \frac{1}{945}\frac{Sq^2a^6}{EI^2}, \tag{20}$$

In this approximation, the maximum deflection at the center of the rod is equal to:

$$X(0) \simeq \frac{qa^4}{24EI}\left[1 - \frac{1}{9450}\frac{Sq^2a^8}{E^2I^3}\right]. \tag{21}$$

Another special case is when the deflection is much greater than the thickness of the rod. In this case$ka \gg 1$, which means that the resistance of the rod to bending is much smaller compared to its resistance to tension, and the rod behaves like a string. Then:

$$F(ka) \simeq \frac{1}{6}(ka)^{-6}, \quad k^2 \simeq \left(\frac{Sq^2a^2}{6E^2I^3}\right)^{\frac{1}{3}}, \quad T = k^2EI \simeq \left(\frac{ESq^2a^2}{6}\right)^{\frac{1}{3}}, \tag{22}$$

In this approximation, the maximum deflection at the center of the rod is equal to:

$$X(0) \simeq \frac{1}{2}\left(\frac{6qa^4}{ES}\right)^{\frac{1}{3}}. \tag{23}$$

3.2 Double hinged support

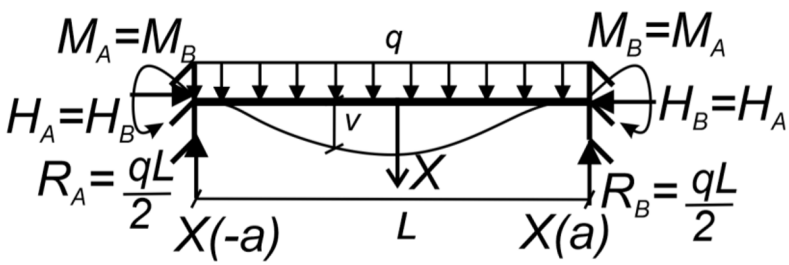

Fig. 3. The calculation scheme of a bar with double hinged support under the action of uniformly distributed load

Assuming that the rod is located from $y = -a$ to $y = a$ assuming that the rod is located from:

$$X(-a) = X(a) = X''(-a) = X''(a) = 0. \tag{24}$$

Taking into account the boundary conditions, the solution takes the form:

$$X = \frac{q}{Tk^2}\left[\frac{ch(ky)}{ch(ka)} - 1\right] + \frac{q}{2T}(a^2 - y^2). \tag{25}$$

Then:

$$X' = \frac{q}{T}\left[\frac{1}{k}\frac{sh(ky)}{ch(ka)} - y\right]. \tag{26}$$

According to [4]:

$$T = \frac{ES}{4a}\int_{-a}^{a} X'^2 dy. \tag{27}$$

Substituting toX' in view $k = \sqrt{\frac{T}{EI}}$, We obtain a transcendental algebraic equation to find T:

$$T = \frac{ES}{4a}\frac{q^2}{T^2}\int_{-a}^{a}\left[\frac{1}{k}\frac{sh(ky)}{ch(ka)} - y\right]^2 dy = \frac{ESq^2a^2}{4T^2}\int_{-1}^{1}\left[\frac{1}{ka}\frac{sh(kat)}{ch(ka)} - t\right]^2 dt =$$
$$= \frac{ESq^2a^2}{4T^2}\left[\frac{2}{3} - \frac{5}{(ka)^2} + \frac{1}{(ka)^2}th^2(ka) + \frac{5}{(ka)^3}th(ka)\right]. \tag{28}$$

According to:

$$\frac{E^2I^3}{Sq^2a^8} = \frac{1}{4(ka)^6}\left[\frac{2}{3} - \frac{5}{(ka)^2} + \frac{1}{(ka)^2}th^2(ka) + \frac{5}{(ka)^3}th(ka)\right] \equiv G(ka). \quad (29)$$

Then: $k = \frac{1}{a}G^{-1}\left(\frac{E^2I^3}{Sq^2a^8}\right)$, $T = EIk^2 = \frac{EI}{a^2}\left[G^{-1}\left(\frac{E^2I^3}{Sq^2a^8}\right)\right]^2$, where G^{-1} — function inverse to G.

The maximum deflection will be at the center of the rod and is equal to:

$$X(0) = \frac{qa^4}{EI(ka)^4}\left[\frac{1}{ch(ka)} - 1 + \frac{(ka)^2}{2}\right]. \quad (30)$$

Special attention is paid to extreme particular cases. Let's consider the case where the deflection is much smaller than the thickness of the rod. Then $ka \ll 1$, and the leading terms $X(0)$ of the Taylor series expansion are as follows:

$$X(0) \simeq \frac{5qa^4}{24EI}\left[1 - \frac{61}{150}(ka)^2\right]. \quad (31)$$

The leading term of the function expansion G has the form $G(ka) \simeq \frac{17}{630}(ka)^{-2}$, thus:

$$(ka)^2 \simeq \frac{17}{630}\frac{Sq^2a^8}{E^2I^3}, \quad T = k^2EI \simeq \frac{17}{630}\frac{Sq^2a^6}{EI^2}, \quad (32)$$

In this approximation, the maximum deflection at the center of the rod is equal to:

$$X(0) \simeq \frac{5qa^4}{24EI}\left[1 - \frac{1037}{94500}\frac{Sq^2a^8}{E^2I^3}\right] \quad (33)$$

Another special case is when the deflection is much greater than the thickness of the rod. Then$ka \gg 1$, which means that the resistance of the rod to bending is much smaller compared to its resistance to tension, and the rod behaves like a string. Then:

$$G(ka) \simeq \frac{1}{6}(ka)^{-6}, \quad k^2 \simeq \left(\frac{Sq^2a^2}{6E^2I^3}\right)^{\frac{1}{3}}, \quad T = k^2EI \simeq \left(\frac{ESq^2a^2}{6}\right)^{\frac{1}{3}}, \quad (34)$$

In this approximation, the maximum deflection at the center of the rod is equal to:

$$X(0) \simeq \frac{1}{2}\left(\frac{6qa^4}{ES}\right)^{\frac{1}{3}} \quad (35)$$

This corresponds to a double hard fixing, as the string is capable of abruptly changing the angle of its inclination.

3.3 Graphical method for calculating the axial force and deflection at the center of the rod

The graph in figure 4 shows a graphical method for determining the axial force and deflection at the center of the beam for given stiffness and load q, obtained based on the analytical solution for double rigid and double hinged supports. By setting the values in the expression $\frac{E^2I^3}{Sq^2a^8}$ can determine the magnitude of the axial force from the equation $a \cdot \sqrt{\frac{T}{EI}}$.

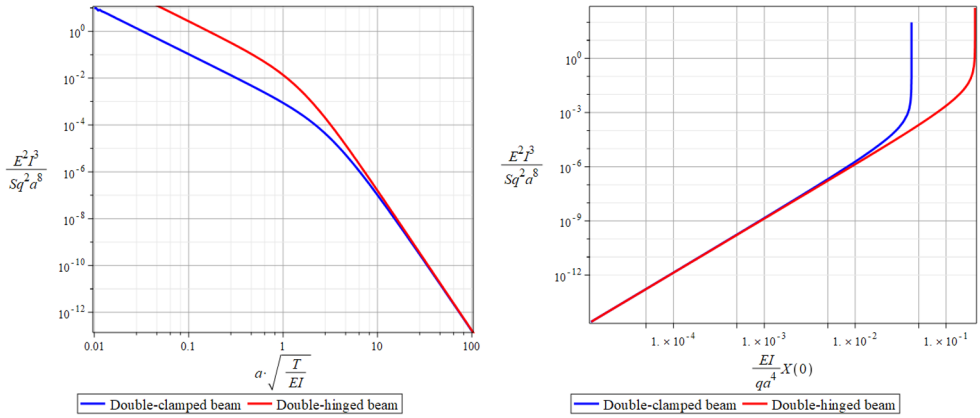

Fig. 4. Graphical method for determining the longitudinal force (left) and deflection at the center of the beam (right) for given stiffness and load q, obtained based on the analytical solution for the double rigid fixing.

4 Conclusions

Based on the conducted research, the following conclusions can be formulated:

1. The mechanics of accounting for tension in statically indeterminate elastic beams in transverse bending were described and the importance of its consideration was justified, affecting the stress-strain state of the beams compared to the stress-strain state caused only by transverse loading.
2. A differential equation for the beam deflection was derived based on the Lagrangian principles of mechanics, taking into account the influence of longitudinal force on bending.
3. Analytical dependencies were obtained for the beam deflection and tension force magnitude due to transverse load q for cases with double rigid support and double hinged support for the beam ends.
4. A graphical method was developed for determining the longitudinal force and deflection at the center of the beam under given stiffness and load q for cases with double rigid support and double hinged support for the beam ends.

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