Penalty function method for geometrically nonlinear buckling analysis of imperfect truss with multi-freedom constraints based on mixed FEM

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Abstract. This paper is concerned with the approach to implementing the Penalty function method for imposing multi-freedom constraints in geometrically nonlinear analysis of imperfect trusses based on mixed finite element formulation. Using a finite element model based on displacement formulation, it is required to incorporate both the dependent boundary relations and initial length imperfection to the nonlinear master stiffness system of equations for solving the geometrically nonlinear problem of imperfect truss with multi-freedom constraints. For decreasing the mathematical complexion of the incorporating process, the author proposes a novel mixed finite truss element considering initial imperfection, used in building the model for solving the geometrically nonlinear problem of truss with multi-freedom constraints. The modified nonlinear stiffness equation is constructed by employing the penalty function method to convert a constrained problem into an unconstrained problem by extremizing the augmented energy function established based on the proposed mixed finite element formulation.

1 Introduction

Investigating the buckling and post-buckling behavior of truss structures arises in many engineering designs. Many research works focus on the computational formulation and solution of geometrically nonlinear buckling problems of the truss system, mostly using the finite element model based on displacement formulation. Building FEM mathematical model for geometrically nonlinear buckling analysis imperfect truss [1-3] with multi-freedom constraints requires incorporating both the dependent boundary relations and
initial length imperfection to the nonlinear master stiffness system of equations. The simultaneous implementation of both dependent boundary conditions at nodes (due to initial imperfection) and support connections (due to multi-freedom constraints) increases the mathematical difficulties of building nonlinear incremental equations and calculation algorithms. In this research, the author proposes a novel approach to modeling and building a system of nonlinear equations based on the mixed finite element formulation for overcoming the mathematical difficulties of establishing solving algorithms. The proposed approach in this work getting from developing the idea of research work published in preceding papers [4-5]. The equilibrium equation of the imperfect truss system gets by assembling mixed finite truss elements constructed considering the initial imperfection. For imposing a dependent boundary relationship due to multi-freedom constraints to the nonlinear master stiffness equation of an imperfect truss system, the optimization technique based on the penalty function method [6-7] is used to convert a constrained problem into an unconstrained problem by extremizing the augmented energy function. The incremental equilibrium equation is constructed for solving the nonlinear equilibrium equation of imperfect trusses with multi-freedom constraints, and the incremental-iterative algorithm for calculation is established utilizing the arc-length method. Based on the established incremental-iterative algorithm, the calculation program for investigating geometrically nonlinear buckling behavior of imperfect truss with multi-freedom constraints is written using Matlab software. The results of the numerical test show the significant influence of initial imperfection and choosing weight values on the equilibrium path of the truss.

**2 Building equilibrium equation and incremental equilibrium equation for imperfect truss element based on mixed finite element formulation**

Consider an imperfect truss system with multi-freedom boundary constraints expressed by an equation \( \mathbf{g} \mathbf{u} = 0 \) (shown in fig.1.a). Based on mixed finite element formulation, this work proposes establishing an equilibrium equation for the truss element with initial length imperfection for building the truss system's equilibrium equation. For establishing the mixed finite imperfect truss element consider a two-node element (shown in fig.1.b), having: distance between the \( i \)th and \( j \)th node before and after deformation \( \Delta r \) and \( L \); initial length (manufactured length) and length imperfection of the truss element - \( e \) and \( \Delta \); nodal displacements and forces in global coordinates - \( u_1, u_2, u_3, u_4 \) and \( P_1, P_2, P_3, P_4 \); resultant external force at the \( i \)th cross-section after deformation - \( P \); an axial load of truss element - \( N \); resultant external force at the \( i' \) cross-section of the truss element - \( P_5 \); area of cross-section of the truss element - \( A \); modulus of elasticity of the material - \( E \). According to the geometrical relation, the truss element length after deformation is calculated by the formula (1)

\[
L_\Delta = L - e = L + \varepsilon L - e
\]

(1)

Based on (1), the longitudinal deformation of the truss element can be defined by the expression

\[
\Delta L = L_\Delta - L = \varepsilon L + \Delta L - e
\]

(2)

The longitudinal strain is computed as follows

\[
\varepsilon = \frac{\Delta L}{L} = \frac{\varepsilon L + \Delta L}{L}
\]

(3)
Δ E.ε dV - P.u + P.Δ

\[ \Pi (\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6) = \int \varepsilon \cdot E \cdot dV - \sum \left( P \cdot u_i + P \cdot \Delta_i \right) \]  

(4)

Based on the principle of minimum potential energy, the system of equations is developed by minimizing the total potential energy function as follows

\[ \frac{\partial \Pi}{\partial u_i} = \int \varepsilon_i \cdot \frac{\partial \varepsilon}{\partial u_i} \cdot dV - \sum \frac{\partial P}{\partial u_i} = 0 \]
\[ \frac{\partial \Pi}{\partial \Delta_i} = \int \varepsilon_i \cdot \frac{\partial \varepsilon}{\partial \Delta_i} \cdot dV - \sum \frac{\partial P}{\partial \Delta_i} = 0 \]  

(5)

Replacing the longitudinal strain from equation (3) to equation (5), getting

\[ \begin{cases} (\Delta_i - \Delta_0) + \Delta_i \cdot \frac{\partial}{\partial \Delta_i} = 0 \iff \Delta_i = \Delta_0 \iff \Delta_i = \Delta_0 \ \text{for} \ \Delta_i = \Delta_0 \\ (\Delta_i - \Delta_0) + \Delta_i \cdot \frac{\partial}{\partial \Delta_i} = 0 \iff \Delta_i = \Delta_0 \iff \Delta_i = \Delta_0 \end{cases} \]  

(6)

Setting

\[ \begin{align*}
\Delta_1 &= \Delta_0 \\
\Delta_2 &= \Delta_0 \\
\Delta_3 &= \Delta_0 \\
\Delta_4 &= \Delta_0 \\
\Delta_5 &= \Delta_0
\end{align*} \]

\[ \begin{align*}
\Delta_6 &= \Delta_0 \\
\Delta_7 &= \Delta_0 \\
\Delta_8 &= \Delta_0 \\
\Delta_9 &= \Delta_0 \\
\Delta_{10} &= \Delta_0 \\
\Delta_{11} &= \Delta_0
\end{align*} \]

Designate truss element vector of unknowns \( u = \{ u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11} \} \), including \( \{ u_1, u_2, u_3, u_4 \} \) - nodal displacement unknowns and \( \{ u_{11} \} \) - nodal axial force unknown.
Writing equilibrium equations of imperfect truss element (6) in vector form

\[ \mathbf{q} \mathbf{u} \Delta = \mathbf{P} \]  

(7)

where

\[ \mathbf{q} \mathbf{u} \Delta = \left[ \begin{array}{c} \mathbf{q}^1 \mathbf{u} \Delta \\ \vdots \\ \mathbf{q}^5 \mathbf{u} \Delta \end{array} \right] \]

\[ \mathbf{P} = \left[ \begin{array}{c} \mathbf{P}^1 \\ \vdots \\ \mathbf{P}^5 \end{array} \right] \]

Equation (7) is the equilibrium equation of imperfect truss element (e) established based on mixed finite element formulation considering large displacement.

Expanding the Taylor series for a sort of \( \delta \mathbf{u} \) around variable point keeping only linear term in \( \delta \mathbf{u} \), getting the incremental equation

\[ \mathbf{q} \mathbf{u} \Delta + \frac{\partial \mathbf{q}^{(e)} \mathbf{u} \Delta}{\partial \mathbf{u}} \delta \mathbf{u} = \mathbf{P}^{(e)} + \Delta \mathbf{P}^{(e)} - \mathbf{q}^{(e)} \mathbf{u} \Delta \]

(8)

Setting \( \mathbf{k}^{(e)} \mathbf{u} \Delta = \frac{\partial \mathbf{q}^{(e)} \mathbf{u} \Delta}{\partial \mathbf{u}} \), incremental equilibrium equation (8) can be written as

\[ \mathbf{k}^{(e)} \mathbf{u} \Delta = \mathbf{q}^{(e)} \mathbf{u} \Delta \]

(9)

The matrix \( \mathbf{k}^{(e)} \mathbf{u} \Delta \) is a mixed matrix of imperfect truss element (e), considering the initial length imperfection \( \Delta \), [...].

3 Penalties function method for incorporating multi-freedom constraints into the mixed equilibrium equation

For imposing the multi-freedom constraints to the mixed equilibrium equation, this research utilized the Penalty function method, one of the most efficient methods for converting the constrained problem into an unconstrained problem. Consider \( n^{th} \) degree of freedom imperfect truss system with multi-freedom boundary constraints (fig.1.a). Designating:

- Vector of nodal unknowns in global coordinate system \( \mathbf{u} = \{u_1, u_2, ..., u_n\} \in \mathbb{R}^n \);
- Vector of length imperfections \( \mathbf{\Delta} = \{\Delta_1, \Delta_2, ..., \Delta_5\} \in \mathbb{R}^5 \);
- Linear boundary constraints \( \mathbf{u}^b = \{u_{1b}, u_{2b}, ..., u_{nb}\} \), expressed as linear equation \( \mathbf{g} \mathbf{u} = \mathbf{A} \mathbf{u} - \mathbf{b} = 0 \), where \( \mathbf{g} \mathbf{u} = \{g(u_1, u_2, ..., u_n)\} \quad \mathbf{A} = \mathbf{A}_{ab}, \mathbf{b} = \mathbf{b}_{ab} \);
- The function of total potential energy of truss system \( \Pi(\mathbf{u} \mathbf{\Delta}) \)

The equilibrium equation incorporated dependent boundary relations is developed by minimizing the system’s total potential energy function as follows

\[ \min \{ \Pi(\mathbf{u} \mathbf{\Delta}) | \mathbf{g} \mathbf{u} = \mathbf{A} \mathbf{u} - \mathbf{b} = 0, \mathbf{u} \in \mathbb{R}^n \} \]

(11)

The total potential energy of the truss system is defined by summing the strain energy of the truss system \( \Pi(\mathbf{u} \mathbf{\Delta}) \) and the potential energy of external forces \( \mathbf{u} \mathbf{P} - \Delta \mathbf{P} \), written in equation (13)

\[ \Pi(\mathbf{u} \mathbf{\Delta}) = \mathbf{\Pi}(\mathbf{u} \mathbf{\Delta}) - \mathbf{u} \mathbf{P} - \Delta \mathbf{P} \]

(12)

Where

\( \mathbf{P} = \{P_1, P_2, ..., P_5\} \) - Vector of nodal forces
- \( \mathbf{P} = \{P_1, P_2, \ldots, P_n\} \) - Vector of nodal force unknowns corresponding to the length imperfection \( \Delta \).

Based on the Penalty function method to convert the constrained optimization problem (11) into a non-constrained optimization problem by utilizing the penalty objective function, expressed in (13)

\[
\begin{aligned}
\mathbf{u} + \mathbf{P} - \Delta \mathbf{P} - \mathbf{A} (\mathbf{u} - \mathbf{b}) &= 0 \\
\mathbf{u} + \mathbf{P} - \Delta \mathbf{P} - \mathbf{b} &= 0
\end{aligned}
\]

(13)

Where \( w \) is numerical weight or penalty parameter.

Taking minimization of the penalty objective function (14)

\[
\begin{aligned}
\mathbf{w} \in \mathbb{R}^n, \\
\min \mathbf{Q}, \mathbf{R} (\mathbf{u}, \mathbf{P})
\end{aligned}
\]

(14)

by zeroing the partial derivative of the penalty function with variables \( \mathbf{w} \), getting equation (15)

\[
\begin{aligned}
\frac{\partial \mathbf{u} + \mathbf{P} - \Delta \mathbf{P} - \mathbf{A} (\mathbf{u} - \mathbf{b})}{\partial \mathbf{u}} = 0 \\
\frac{\partial \mathbf{u} + \mathbf{P} - \Delta \mathbf{P} - \mathbf{b}}{\partial \mathbf{P}} = 0
\end{aligned}
\]

(15)

Setting \( \mathbf{u} = \{u_1, u_2, \ldots, u_n\} \), \( \mathbf{P} = \{P_1, P_2, \ldots, P_n\} \), \( \mathbf{w} = \{w_1, w_2, \ldots, w_n\} \), \( \mathbf{A} = \{A_{jk}\} \), \( \mathbf{b} = \{b_1, b_2, \ldots, b_n\} \), the established equilibrium equation (16) is nonlinear with variable \( \mathbf{u} \).

\[
\begin{aligned}
\frac{\partial \mathbf{u} + \mathbf{P} - \Delta \mathbf{P} - \mathbf{A} (\mathbf{u} - \mathbf{b})}{\partial \mathbf{u}} = \mathbf{K} \\
\frac{\partial \mathbf{u} + \mathbf{P} - \Delta \mathbf{P} - \mathbf{b}}{\partial \mathbf{P}} = \mathbf{P}
\end{aligned}
\]

(16)

The vector function \( \frac{\partial \mathbf{u} + \mathbf{P} - \Delta \mathbf{P} - \mathbf{A} (\mathbf{u} - \mathbf{b})}{\partial \mathbf{u}} \) is a nonlinear function with variable \( \mathbf{u} \), consequently, the established equilibrium equation (16) is nonlinear with variable \( \mathbf{u} \). For solving the nonlinear equilibrium equation of the truss system, the incremental equilibrium equation (17) is established by expanding the Taylor series keeping only linear term in \( \delta \mathbf{u} \).

\[
\begin{aligned}
\mathbf{q} \mathbf{u} + \mathbf{K} \delta \mathbf{u} = \mathbf{P} + \Delta \mathbf{P}
\end{aligned}
\]

(17)

Setting \( \mathbf{K} = \frac{\partial \mathbf{q}}{\partial \mathbf{u}} \), called the mixed matrix of the truss system, equation (17) can be written as

\[
\begin{aligned}
\mathbf{K} \delta \mathbf{u} = \mathbf{P} + \Delta \mathbf{P} - \mathbf{q} \mathbf{u}
\end{aligned}
\]

(18)

Where: \( \delta \mathbf{u} = \{\delta u_1, \delta u_2, \ldots, \delta u_n\} \) - vector of incremental unknowns.

For solving the nonlinear equation (18), the arc length method is adopted to establish an incremental-iterative algorithm [8-15]. Based on the established incremental-iterative algorithm, the calculation programs had been written for geometrically nonlinear stability analysis of imperfect truss systems with multi-freedom constraints.
4 Numerical investigation

4.1 Example formulation

Analysing geometrically nonlinear buckling behaviour (finding the equilibrium path) of imperfect truss system with multi-freedom constraints (shown in fig.2.a). All truss bars are made of the same material and have equal cross-section areas $A$. The parameters are given as follows

$$\Delta_e^1 = \Delta_e^2 = \Delta_e^3 = \Delta_e^4 = \Delta_e^5 = 0.5 \text{ cm}; \Delta_e^6 = 1 \text{ cm}; \Delta_e^7 = 0.5 \text{ cm}; \Delta_e^8 = 1 \text{ cm}; \Delta_e^9 = 0 \text{ cm}.$$

![Investigated imperfect truss system with multi-freedom boundary constraints](image)

Fig. 2. Investigated imperfect truss system with multi-freedom boundary constraints

The equation expressed multi-freedom boundary constraints are constructed according to the geometrical relationships between nodes of the investigated truss system shown in fig 2.b. Expressing in matrix form

$$\mathbf{g} \mathbf{u} = \mathbf{A} \mathbf{u} - \mathbf{b}.$$

The numerical investigation is realized by employing the calculation program, written based on the proposed algorithm. For observing the converged rate, the problem was solved with different weight values as followings

$$w_1 = 10, w_2 = 10, w_3 = 10, w_4 = 10.$$

4.2 Numerical results

The calculation results are load-displacement and load-normal force relationships, described as equilibrium paths, corresponding to different cases of weight values mentioned in 4.1. The geometrically nonlinear analysis was done for the truss system with non-length imperfection (perfect truss system) to investigate the influence of length imperfection on
the behaviour of the truss system. Numerical results for the perfect truss system and imperfect truss system according to different cases of “w” are shown in fig.3 and fig.4.

![Fig. 3. Load-displacement equilibrium path (P - u)](image)

![Fig. 4. Load-normal force equilibrium path (P - N)](image)

Comment: The calculation results show a significant influence of length imperfection on the equilibrium path.

5 Conclusion

In this research, the author proposed a novel approach for building a model and establishing solving algorithm for geometrically nonlinear buckling analysis of the imperfect truss system with multi-freedom constraints. Using the proposed mixed finite imperfect truss element in building a finite element model shows the remarkable advantage in simplifying mathematical techniques in establishing equilibrium equations for solving the geometrically nonlinear problem of the truss systems with multi-freedom constraints. The penalty function method was effectively utilized for imposing multi-freedom constraints on the mixed finite element model of the truss system. The presented incremental-iterative
algorithm has been successfully realised in the numerical example. The presented incremental-iterative algorithm has been successfully used in the numerical investigation, showing the efficiency in defining the equilibrium path and limit points for investigating the buckling and post-buckling behavior of the system.

References


