Debris flow seismo-acoustic wave in a finite layer waveguide

Shih-Chao
Ko-Fei
Chao
Sudhan

Abstract.

1 Introduction

The acoustic-based detection is a popular way to detect debris flows and has been used in different early warning systems worldwide for decades. From field observation, the high correlation between the energy of ground vibration signal and flow condition has been proved by many studies. Some of the studies also try to calibrate the empirical formula [1-3]. However, it only can be used in the location calibrated before and should be modified once the environment changes after a new event.

Some recent studies introduced the bed load seismicity conceptual model [4] to analyze debris flows [5-6]. These approaches are based on the vibration source generated by particle impact and use the Rayleigh-wave propagation function (Green's function) [7] to simulate the receiver's vibration energy. Although these models have been used in the field, the assumptions of the simplified process, the determination of field parameters, the applicability of Green's function, etc., still lack independent verification mechanisms [8].

Because the flow condition still cannot be quantified using a recording signal from a theoretical point of view. To bridge the gap, we aimed to develop a connecting theory between debris flow motion and seismo-acoustic wave propagation.

2 Ideal waveguide

2.1 Fundamental theory

We consider a debris flow flowing down a channel with a constant wave velocity \( C \) [12] and radiating acoustic waves from its boundaries, such as the free surface or channel bed. Below the channel bed, the underground layer is regarded as a homogeneous and isotropic elastic media. The \( x \)-axis coincides with a streamwise direction along the channel bottom. The \( y \)-axis is in the transverse direction, and the \( z \)-axis is perpendicular to both the \( x \)- and \( y \)-axis, as shown in Fig. 1(a). An ideal channel shape with radius \( r \) is assumed, as shown in Fig. 1(b).

In this underground layer, the seismo-acoustic wave propagation is governed by the elastic wave equation or Navier's equation [7, 9-10]

\[
\frac{\partial^2 \mathbf{u}}{\partial t^2} = c_p^2 \nabla (\nabla \cdot \mathbf{u}) - c_s^2 \mathbf{u} \times \nabla \times \mathbf{u},
\]

(1)

where \( \mathbf{u} \) is displacement vector \((u, u, u)\), \( c_p \) and \( c_s \) are compressional-wave and shear-wave velocity below

\[
c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}},
\]

(2)

where \( \lambda \) and \( \mu \) are Lamé's first and second parameter, respectively.

The seismo-acoustic source comes from the bottom stress of debris flow \( \tau_{yo}^{s} \). The stress in the channel layer \( \sigma_{yo}^{s} \) can be expressed as [11]

\[
\left. \tau_{yo}^{s} \right|_{yo} = \left. \sigma_{yo}^{s} \right|_{yo} = \lambda \left( \frac{\partial u}{\partial y} \right)_{yo} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{yo}.
\]

(3)
When the river channel radius \( r \) is much smaller than \( r_0 \), the debris flow along the \( x \)-axis can be simplified as one-dimensional, and only the leading order stress \( \tau_{zz}^{\sigma} \) will be survived [12]. Thus Eq.(3) can be simplified as

\[
\tau_{zz}^{\sigma} = \sigma_{zz}^{\sigma} \left( \begin{array}{c} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} \\
\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} \end{array} \right). \tag{4}
\]

![Image](https://doi.org/10.1051/e3sconf/202341503032)

Fig. 1 Sketch of ideal debris flow seismo-acoustic source and its waveguide; (a) debris flow seismo-acoustic source in stationary waveform (one-dimensional) acting on an ideal axisymmetric channel (three-dimensional). (b) the waveguide of seismo-acoustic waves induced by debris flow in cross-section direction, where dark gray axes denote the Cartesian coordinate system in \( x \)-\( y \)-\( z \)-\( t \), black axes describe the moving cylindrical coordinate system in \( \xi \)-\( \phi \)-\( \eta \)-\( t \), red slash-hatched is debris flow, light gray shadow indicate wave propagation domain.

### 2.2 Governing equations

Since \( r << r_0 \), we consider the debris flow as a line source and allow \( r \to 0 \) in the following section.

To simplify the problem, we transform the coordinate from the Cartesian system \( x \)-\( y \)-\( z \)-\( t \) to a moving cylindrical coordinate system \( \xi \)-\( \phi \)-\( \eta \)-\( t \), as the relation below

\[
\begin{align*}
\xi &= x - Ct, & r &= \sqrt{y^2 + z^2}, \\
\theta &= \arctan \frac{z}{y}, & \eta &= x + Ct, \tag{5}
\end{align*}
\]

where \( C \) is debris flow wave velocity, \( C \) is a constant related to wave velocity, \( y = r \cos \theta \), \( z = r \sin \theta \). The differential operator of function \( f \) with respect to \( x \)-\( y \)-\( z \)-\( t \) are listed below

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x}, \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y}, \\
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z}, \\
\frac{\partial^2 f}{\partial t^2} &= \frac{\partial^2 f}{\partial \xi^2} \frac{\partial \xi}{\partial t} + \frac{\partial^2 f}{\partial \theta^2} \frac{\partial \theta}{\partial t^2} + \frac{\partial^2 f}{\partial \eta^2} \frac{\partial \eta}{\partial t^2} + \frac{\partial^2 f}{\partial t^2}.
\end{align*}
\]

With the Helmholtz decomposition theorem, the displacement vector is expressed as

\[
\mathbf{u} = \nabla \phi + \nabla \times \Psi,
\]

where \( \phi \) and \( \Psi = (\psi_x, \psi_y, \psi_z) \) are the potential of compressional-wave (scalar) and the potential of shear-wave (vector), respectively. The gradient and curl operator are listed below

\[
\begin{align*}
\nabla \phi &= \frac{\partial \phi}{\partial \xi} \hat{\xi} + \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{\partial \psi}{\partial \eta} \hat{\eta}, \\
\nabla \times \Psi &= \left| \begin{array}{ccc}
\hat{\xi} & \hat{\theta} & \hat{\eta} \\
\frac{\partial}{\partial \xi} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \eta} \\
\psi_x & \psi_y & \psi_z
\end{array} \right|,
\end{align*}
\]

where \( \Psi \) must satisfy zero divergence condition

\[
\nabla \cdot \Psi = \frac{\partial \psi_x}{\partial \xi} + \frac{\partial \psi_y}{\partial \theta} + \frac{\partial \psi_z}{\partial \eta} = 0.
\]

Due to the nature of line sources, this problem can be reduced to an axisymmetric problem, i.e., \( \partial \phi/\partial \theta = 0 \) and \( \psi_x = 0 \). The displacement vector is simplified as

\[
\mathbf{u} = u_x \hat{\xi} + u_\theta \hat{\eta} = \nabla \phi + \nabla \times \psi_x \hat{\eta}.
\]

Substituting Eq. (11) to Eq. (1), the governing equation can be decoupled as a potential form

\[
\begin{align*}
\left( \frac{C^2}{c_p^2} - \frac{\partial \phi}{\partial \xi} \right) \frac{\partial \phi}{\partial \xi} &= \left( \frac{CC_p}{c_s^2} - \frac{\partial \psi_x}{\partial \xi} \right) \frac{\partial \phi}{\partial \xi} + \left( \frac{C^2}{c_p^2} - \frac{\partial \phi}{\partial \eta} \right) \frac{\partial \phi}{\partial \eta} + \left( \frac{C^2}{c_p^2} - \frac{\partial \phi}{\partial \theta} \right) \frac{\partial \phi}{\partial \theta}, \\
\frac{\partial \phi}{\partial \xi} &= \frac{\partial \phi}{\partial \xi} + \frac{\partial \phi}{\partial \theta}, \\
\frac{\partial \psi_x}{\partial \xi} &= \frac{\partial \psi_x}{\partial \xi} + \frac{\partial \psi_x}{\partial \theta}, \\
\frac{\partial \psi_x}{\partial \eta} &= \frac{\partial \psi_x}{\partial \eta} + \frac{\partial \psi_x}{\partial \theta}.
\end{align*}
\]

To simplify the governing equations, we introduce the \( \eta_r \)-axis and the \( \eta_\theta \)-axis below

\[
\eta_r = x - \frac{\hat{z}^2}{C}, \quad \text{i.e., } C_\eta = -\frac{\hat{z}^2}{C},
\]

\[
\eta_\theta = y - \frac{\hat{z}^2}{C}, \quad \text{i.e., } C_\theta = -\frac{\hat{z}^2}{C}.
\]

\[
\eta_r = x - \frac{\hat{z}^2}{C}, \quad \text{i.e., } C_\eta = -\frac{\hat{z}^2}{C},
\]

\[
\eta_\theta = y - \frac{\hat{z}^2}{C}, \quad \text{i.e., } C_\theta = -\frac{\hat{z}^2}{C}.
\]
\[ \eta_s = x - \frac{c_s}{C}, \text{ i.e. } C_s = -\frac{c_s}{C}. \]  
(15)

For Eq. (12), we shift the \( \eta \)-axis to the \( \eta_s \)-axis. Similarly, for Eq. (13), we move the \( \eta \)-axis to the \( \eta_s \)-axis. So the governing equations become

\[
\left( C_s \frac{d\phi}{d\eta_s} - \phi \right) + \frac{d^2\phi}{d\eta_s^2} = 0, 
\]
(16)

\[
\left( C_s \frac{d\psi_s}{d\eta_s} - \psi_s \right) + \frac{d^2\psi_s}{d\eta_s^2} = 0. 
\]
(17)

### 2.3 Boundary conditions

The corresponding stress boundary condition are listed below.

at \( r = r_1 \rightarrow \eta \):

\[ \sigma_s^* = S(r = \sqrt{\xi} \eta), \quad \sigma_{\phi \phi} = \sigma_{\theta \theta} = \sigma_{\phi \theta} = \sigma_{\theta \phi} = \sigma_{s s} = \sigma_{s \phi} = \phi = \eta = 0. \]
(18)

at \( r = r_0 \):

\[ \sigma_s^* = \sigma_{s \phi} = \sigma_{s \theta} = \sigma_{s \phi} = \phi = \eta = 0. \]
(19)

With the axisymmetric condition, the stress in Eq. (18) and (19) can be expressed below

\[
\sigma_s^* = \sigma_{\phi \phi} = \sigma_{\theta \theta} = \mu \left( \frac{\partial u_s}{\partial \xi} + \frac{\partial u_s}{\partial \eta} \right), 
\]
(20)

\[
\sigma_{s \phi} = \sigma_{s \theta} = \phi = \eta = 0. \]
(21)

\[
\sigma_{s s} = \sigma_{s \phi} = \sigma_{s \theta} = \phi = \eta = 0. \]
(22)

\[
\sigma_{s \phi} = \lambda \left( \frac{\partial u_s}{\partial \xi} + \frac{\partial u_s}{\partial \eta} \right) + \frac{\partial u_s}{\partial \xi} \right) + \mu \left( \frac{\partial u_s}{\partial \eta} + \frac{\partial u_s}{\partial \xi} \right). 
\]
(23)

\[
\sigma_{s \theta} = \lambda \left( \frac{\partial u_s}{\partial \xi} + \frac{\partial u_s}{\partial \eta} \right) + \mu \left( \frac{\partial u_s}{\partial \eta} + \frac{\partial u_s}{\partial \xi} \right), 
\]
(24)

\[
\sigma_{s \phi} = \lambda \left( \frac{\partial u_s}{\partial \xi} + \frac{\partial u_s}{\partial \eta} \right) + \mu \left( \frac{\partial u_s}{\partial \eta} + \frac{\partial u_s}{\partial \xi} \right), 
\]
(25)

### 2.4 Separation of variables

We adopted conventional separation of variables to solve the potential \( \phi \) and \( \psi_s \). First, we let

\[
\phi = R(r) X(\xi), \quad \psi_s = \bar{R}(\xi), \quad \bar{X}(\xi), \quad \bar{T}(\eta_s). 
\]
(26)

Substituting Eq. (26) to Eq. (16) and (17), the governing equations become

\[
\left( \frac{C}{r} \right) \frac{d^2 \phi}{d\eta_s^2} + \frac{d\phi}{d\eta_s} = \frac{R(r)}{r}, \quad \frac{d\phi}{d\eta_s} = \frac{R'(r)}{r}. 
\]
(27)

\[
\left( \frac{C}{r} \right) \frac{d^2 \psi_s}{d\eta_s^2} + \frac{d\psi_s}{d\eta_s} = \frac{R(r)}{r}, \quad \frac{d\psi_s}{d\eta_s} = \frac{R'(r)}{r}. 
\]
(28)

Substituting Eq. (26) to Eq. (16) and (18), the boundary condition at \( r = r_0 \) gives

\[ R(\xi) = \bar{R}(\xi), \quad \bar{R}(\xi) = \bar{R}(\xi). \]
(29)

\[ \bar{R}(\xi) \]
(30)

2.4.1 Solution of \( \phi \)

If we let

\[
\left( \frac{C}{r} \right) \frac{d^2 \phi}{d\eta_s^2} + \frac{d\phi}{d\eta_s} = \frac{R(r)}{r}, \quad \frac{d\phi}{d\eta_s} = \frac{R'(r)}{r}. 
\]
(27)

\[
\left( \frac{C}{r} \right) \frac{d^2 \psi_s}{d\eta_s^2} + \frac{d\psi_s}{d\eta_s} = \frac{R(r)}{r}, \quad \frac{d\psi_s}{d\eta_s} = \frac{R'(r)}{r}. 
\]
(28)

Eq. (27) can be separated as

\[
\frac{d^2 \phi}{d\eta_s^2} + \frac{d\phi}{d\eta_s} = \frac{R(r)}{r}, \quad \frac{d\phi}{d\eta_s} = \frac{R'(r)}{r}. 
\]
(29)

\[ \left( \frac{C}{r} \right) \frac{d^2 \psi_s}{d\eta_s^2} + \frac{d\psi_s}{d\eta_s} = \frac{R(r)}{r}, \quad \frac{d\psi_s}{d\eta_s} = \frac{R'(r)}{r}. \]
(30)

where \( k^- \) and \( k^+ \) are separation variables (positive real number). In Eq. (34), the general solution is composed by the Bessel functions of the first kind \( J_1(kr) \) and second kind \( Y_1(kr) \) below

\[
R(r) = R_J(kr) + R_Y(kr), 
\]
(36)

where \( R_J \) and \( R_Y \) are constant. With Eq. (29) and (31), one can solve \( R = \bar{R} \) and deduce \( k = \alpha_m/r \) where \( \alpha_m \) is the \( m \)-th positive root of \( J_1 \). Then Eq. (36) can be reduced as

\[
R(r) = R_J \left( \frac{\alpha_m}{r} \right). 
\]
(37)
Because the debris flow propagation speed $C$ (around 10 m/s) is much less than compressional-wave velocity $c_p$ (about 300-700 m/s), the term $k^2 C / C^2 - c^2_p$ in Eq. (33) must be a negative value, and the solution of $T(\eta_r)$ is

$$T(\eta_r) = T^\omega \left\{ [k \lambda \eta_r] + T^\delta \left\{ -k \lambda \eta_r \right\} \right\}$$  (38)

where $\lambda = \sqrt{k^2 C / C^2 - c^2_p}$ is a positive real number. Because $\eta_r = x - c_p C t$ is a negative value ($O \times x \rightarrow O \times c_p C t \leftarrow \downarrow$), $T^\delta$ should be zero when $\eta_r \rightarrow -\infty$. Then, Eq. (38) can be reduced as

$$T(\eta_r) = T^\omega \left\{ [k \lambda \eta_r] \right\}$$  (39)

Up to now, we still cannot distinguish the value of $k_1^+$ and $k_2^+$. Thus, the solution $X(\xi)$ will exist in two modes simultaneously, as below

1) $k_1^+ < k_2^+ = (\alpha_\infty / r)^\omega$

$$X(\xi) = X^\omega \left\{ i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\} + X^\delta \left\{ -i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\}$$  (40)

2) $k_1^+ > k_2^+ = (\alpha_\infty / r)^\delta$

$$X(\xi) = X^\omega \left\{ \sqrt{k_1^+ - k_2^+} \lambda \xi \right\} + X^\delta \left\{ -\sqrt{k_1^+ - k_2^+} \lambda \xi \right\}$$  (41)

where $\lambda = \sqrt{c_p^2 / c_p^2 - C^2}$ is a positive real number, $X^\omega$ and $X^\delta$ are constant. Because $\xi = x - Ct$ can be positive or negative ($O \times x \rightarrow O \times C t \leftarrow \downarrow$) and $X^\omega \xi$ should be finite when $\xi \rightarrow \pm \infty$, the second mode (Eq. (41)) should be vanished.

Substituting Eq. (37), (39), (40) to Eq. (26), the potential $\phi$ is solved as

$$\phi = \sum \alpha_{nW} R_j \left\{ \frac{\alpha_\infty}{r^W} \right\} \left\{ X^\omega \left\{ i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\} \left\{ k \lambda \eta_r \right\} + X^\delta \left\{ -i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\} \left\{ k \lambda \eta_r \right\} \right\}$$  (42)

where $k_1^+ < k_2^+ = (\alpha_\infty / r)^\cdot$

2.4.2 Solution of $\psi_\theta$

Similarly, if we let

$$\left( C^2 - c_p^2 \right) \tilde{X}(\xi) - \tilde{k} \tilde{X}(\xi) = \tilde{k} \tilde{X}(\xi) + \tilde{k} \tilde{X}(\xi) - \tilde{k} \tilde{X}(\xi),$$  and

Eq. (28) is separated as

$$\tilde{X}^\omega(\xi) + \tilde{k} \tilde{X}^\delta(\xi) = 0$$  (43)

$$\tilde{X}^\omega(\eta_r) + \tilde{k} \tilde{X}^\delta(\eta_r) = 0$$  (44)

$$\tilde{X}^\omega(\xi) + \tilde{k} \tilde{X}^\delta(\xi) = 0$$  (45)

where $\tilde{k}$ and $\tilde{k}$ are separation variables (positive real number). In Eq. (45), the general solution is

$$\tilde{X}(\xi) = \tilde{R}(\tilde{k} \xi)$$  (46)

where $\tilde{R}$ and $\tilde{R}$ are constant. With Eq. (30), we found $	ilde{R}(\cdot) = \square$ and deduced

$$\psi_\theta = \square.$$  (47)

The shear waves vanish from this ideal waveguide due to $r = r \rightarrow \square$.

2.4.3 Eigenvalue solutions

If we substitute Eq. (42) and (47) into Eq. (18), the seismo-acoustic source $S(\nabla \xi \eta_r \eta_s)$ can be deduced as

$$\frac{S}{\mu} = \sum -R_j \frac{\alpha_\infty}{r^W} \left\{ \frac{\alpha_\infty}{r^W} \right\} \left\{ k \lambda \eta_r \right\} \left\{ i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\} \left\{ -i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\}$$  (48)

The intensity of seismo-acoustic source $S$ decreases with Bessel functions of the first kind $J_1(k r)$ in $r$ direction. Alternatively, the seismo-acoustic source decay exponentially in the $\eta_r$ axis. Along the $\xi$ axis, the seismo-acoustic source propagates with periodic sinusoidal mode.

Using Eq. (5), the coordinate of Eq. (48) can be inversed from the moving cylindrical system $(\xi \tau \theta \eta)$ to the Cartesian system $(x \tau y \theta)$ as

$$\frac{S}{\mu} = \sum -R_j \frac{\alpha_\infty}{r^W} \left\{ \frac{\alpha_\infty}{r^W} \right\} \left\{ k \lambda \left( x - c_p C t \right) \right\} \left\{ i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\} \left\{ -i \sqrt{k_1^+ - k_2^+} \lambda \xi \right\}$$  (49)
where $\lambda_1 = \sqrt{C^2/c_r^2 - C^{-2}}$ and $\lambda_2 = \sqrt{C^2/c_p^2 - C^{-2}}$ are positive real numbers, and $k^0 < k^\pm = (2\pi/\lambda_1, \lambda_2)$. 

3 Discussion

With the term $i\sqrt{k^0 - k^\pm \lambda_c(x - C_t)}$ in Eq. (49), the wave number $k^\pm$ and angular frequency $\omega_\pm$ can be deduced as

$$k^\pm = k^0 - k^0 \pm \frac{c_p}{c_r - C}, \quad (50)$$

$$\omega_\pm = C \sqrt{k^0 - k^0 \pm \lambda_c} = C \sqrt{k^0 - k^0} \pm \frac{c_p}{c_r - C}. \quad (51)$$

With Eq. (51), the frequency $f^\pm = \omega_\pm / \pi$ at any fixed location $x$ will be

$$f^\pm = \pm \frac{C}{\pi} \sqrt{k^0 - k^0 \pm \lambda_c} \frac{c_p}{c_r - C}. \quad (52)$$

The frequency $f^\pm$ at any fixed location $x$ is not only the function of debris flow propagation speed $C$, but the function of compressional-wave velocity $c_r$. Therefore, it implies the measuring frequency $f^\pm$ is different from the frequency $f$ radiated from debris flow.

In reality, the $c_p$ in alluvium medium is around 300-700 m/s, much larger than the debris flow propagation speed $C$ observed from the field (5-20 m/s). By applying the Taylor series expansion to Eq. (52) with $K = \sqrt{k^0 - k^0 / \pi}$, we get

$$f^\pm = KC \left( \frac{C}{c_p} \right)^{\frac{1}{2}} = KC \left( \frac{C}{c_p} + \frac{C^2}{c_p^2} + \ldots \right) = KC + \frac{KCC}{c_p}. \quad (53)$$

The result shows that the measuring frequency of debris flow at a fixed location is proportional to the debris flow propagation speed $C$ in the leading order. If the underground media change in measuring cross-sections, the frequency band might be shifted due to the higher order term of Eq. (53), even if the debris flow passes these cross-sections at the same speed.

4 Conclusions

We propose an analytical solution for debris flow-induced seismo-acoustic wave propagation on an ideal finite waveguide. Through the finite domain cylindrical seismo-acoustic propagation problem, this research found that the measuring frequencies recorded by a fixed recorder are proportional to the debris flow propagation speed in the leading order. Furthermore, the underground media, such as the soil layer, rock layer, and dam, would affect frequencies with a slight shift in the higher-order term. This finding indicated a Doppler-like property from the theoretical point of view and showed the potential to evaluate flow conditions based on seismo-acoustic frequencies.

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References