Models of dynamics of complex heterostructures under pulsed force effects

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Abstract. Relevance is due to insufficient knowledge and development of models of dynamics of complex heterostructures. Modern mechatronic systems include hybrid components consisting of complex heterogeneous structures: mechanical, electrical, electronic, etc. Heterostructures function in extreme conditions due to kinematic, dynamic, temperature, vibration and other external factors and are exposed to external influences. To create a reliable system for protecting complex heterostructures, it is necessary to build appropriate mathematical models that allow you to adequately describe the processes occurring in complex heterostructures. The purpose of the work is to study and develop heterostructural models based on the equations of point dynamics, plate and multilayer structures under the action of force and impulse loads in the presence or absence of internal friction in heterostructures. Particular attention is paid to the study of models of heterostructures with dissipation, with many degrees of freedom under the action of impulse loads.

1 Introduction

Complex hybrid heterostructures are constantly exposed to external influences, among which vibrational and shock influences are most powerful. They subject the system to significant overloads. Overloads of several tens of g can cause irreversible degradation processes in heterostructure materials, leading to the formation of cracks [1], lamination of printed circuit boards, violation of solder contacts, destruction of welds, etc. For example, many mechatronic systems of Western manufacturers are vulnerable in climatic conditions of the Russian Federation, or do not withstand extreme loads. The lack of knowledge of complex hybrid heterostructures and, as a result, the problem of developing models for studying the behavior of heterostructures under the influence of extreme influences makes the work relevant.

A lot of articles by various authors are devoted to improving the reliability of complex heterostructures. To protect against external influences, various methods of protecting heterostructures from shock and vibration overloads [2-3] are used, including in the form of multilayer fills with high dissipative properties and allowing to transfer the energy of shock loads or periodic force into thermal radiation [4]. Heterogeneous layers should have not...
2 Materials and methods

To develop the protection of complex heterostructures in modern conditions, it is necessary to use mathematical models to conduct computational experiments. When developing technologies for vibration shock protection of heterostructures, fundamental models of heterostructures [5] can be used without taking into account and taking into account internal friction, which allow analyzing the results of effects of impulse loads on technical systems.

In most cases, heterostructures can be represented as rod, plate and multilayer structures. In the simplest case, these are heterostructures with one, and in general with many degrees of freedom, for which it is necessary to compose systems of linear or nonlinear differential equations, the latter of which have no analytical solutions.

To describe the processes occurring under the action of shock loads, a numerical and analytical modeling apparatus [6,7] is used, which involves the use of numerical methods and computer technology.

The purpose of this study is to study and develop heterostructural models of mechatronics under the influence of impulse loads.

3 Results

3.1 Fundamental models of heterostructure dynamics

When a load of mass \( m \) moving horizontally at a speed \( v_0 \) is hit by a spring with mass \( m_s = 0 \), the process of joint movement of the load and the spring is described by the equation

\[
z = C_1 \sin p_0 t + C_2 \cos p_0 t,
\]

where \( p_0 = \sqrt{\frac{\varsigma}{m}} \) is the frequency of natural vibrations of the load attached to the spring, and \( \varsigma \) is the spring stiffness.

When \( t = 0 \), \( z = 0 \) and \( \dot{z} = v_0 \), the constants \( C_1 = \frac{v_0}{p_0} \) and \( C_2 = 0 \) are constant.

The largest value of the force \( P_{max} \), compressing the spring (dynamic load) is determined by the expression

\[
P_{max} = z_{max} \varsigma = \frac{v_0 \varsigma}{p_0}.
\]

According to the energy balance condition [8], we equate the kinetic energy of the moving load \( T \) to the potential energy of the compressed spring \( U \). We get

\[
T = U = \frac{m v_0^2}{2} = \frac{1}{2} m_{max} \varsigma, \quad P_{max} = v_0 \sqrt{m \varsigma} = \frac{v_0 \varsigma}{p_0}.
\]

When a load of mass \( m \), moving vertically at a speed \( m \), hits a spring with mass \( m_s = 0 \), the spring receives a dynamic deflection \( z_d \).
\[ z_A = z_c \left[ 1 + \sqrt{1 + \frac{2T}{cz_c^2}} \right] \]

\[ \chi = 1 + \sqrt{1 + \frac{2T}{cz_c^2}} \]

\[ \chi = \frac{z_A}{z_c} \]

\[ \sigma_A = \chi \sigma_c \]

\[ \chi = 1 + \sqrt{1 + \frac{2T}{cz_c^2}(1 + \frac{m_1}{m})} \]

\[ z_{\text{max}} = 2z_c, \sigma_A = 2\sigma_c \]

\[ v_1 = \frac{mv_0}{(m + m_1)} \]

\[ \chi = \frac{m_1}{m} = 0.25; 0.5; 1 \]

\[ c = 100 \, N/m \]

Fig. 1. Dependence of the dynamic coefficient of \( \chi \) on the initial speed of the \( v_0 \).
3.2 Model of pulsed effects on heterostructures

The differential equation of the motion of the system under the force \( P(t) \) without taking into account internal friction is

\[
m \ddot{z}(t) + c \dot{z}(t) = P(t),
\]

where \( z \) – moving the system; \( m \) – its mass; \( c \) – stiffness.

Solving the equation under zero initial conditions \( z(0) = 0, \dot{z}(0) = 0 \) for the time interval \( 0 \leq t \leq \tau \) has the form

\[
z = \frac{P_0}{mp_0} \int_0^\tau f(t') \sin p_0 (t - t') dt',
\]

for \( t > \tau \)

\[
z = \frac{P_0}{mp_0} \int_\tau^\tau f(t') \sin p_0 (t - t') dt',
\]

where \( P_0 \) – maximum power; \( \tau \) – the moment of time for which the movement is determined; \( T_1 \) – frequency period.

\[
p_0 = \sqrt{\frac{c}{m}} = \frac{2\pi}{T_1},
\]

\[
p_0 \tau = \alpha, \quad \frac{P_0}{c} = z_c, \quad \frac{P_0}{c} = z_c, \quad \frac{P_0}{c} = z_c,
\]

\[
z_\alpha = \chi \left( \frac{T_1}{\tau} \right) z_c,
\]

\[
x = \chi, \quad \frac{P_0}{c} = z_c, \quad \frac{P_0}{c} = z_c.
\]

\[
0 \leq t \leq \tau
\]

1. \( f(t) = 1 \)
2. \( f(t) = \frac{t}{\tau} \)
3. \( f(t) = 1 - \frac{t}{\tau} \)
4. \( f(t) = \sin(\pi \frac{t}{\tau}) \)

\[
T = \frac{\tau}{T_1}
\]
3.3 Exposure to periodic pulses

When determining the motion of the system under the action of periodic pulses, it is necessary to take into account the internal friction in the system [11]. Consideration of frequency-independent internal friction in problems of free oscillations of dissipative systems is realized using the hypothesis of complex rigidity [12].

\[ m\ddot{Z}(t) + (a + ib)cZ(t) = 0, \]

where \( Z(t) \) – complex system movement; \( i \) – imaginary unit, \( a = (1 - \alpha^2)(1 + \alpha^2) \), \( b = 2\alpha(1 + \alpha^2) \), \( \alpha = \frac{\gamma}{2} \).

Substitute \( Z = Ae^{p^*t} \) in (5), we obtain the characteristic equation

\[ p^2(a + ib)p_0^2 = \frac{c}{\sqrt{m}} \]

\( p^* = \pm i(1 + i\alpha)p \), \( p = (1 + \alpha^2)^{-1/2} \cdot p_0 \).

\[ Z = (A - iB)e^{-\alpha pt}e^{ipt} \]

\( Z = Re(Z) \)

\[ z = e^{-\frac{\gamma}{2}pt}(A \cos pt + B \sin pt) \]
\[
p = \frac{p_0}{\sqrt{1 + \gamma^2/4}}, \quad \delta = \pi \gamma.
\]

\[
z(0) = z_0, \quad \dot{z}(0) = v_0
\]

\[
z = e^{-\frac{\gamma}{2}pt}\left(z_0 \cos pt + \left(\frac{v_0}{p} + \frac{\gamma z_0}{2}\right) \sin pt\right)
\]

\[
S_0 = 0 \quad \Rightarrow \quad \delta = \pi \gamma.
\]

Under initial conditions
\[
\begin{align*}
z(0) &= z_0, \\
\dot{z}(0) &= \pi
\end{align*}
\]

\[
z = \frac{S_0}{m \pi} e^{-\frac{\gamma}{2}pt} \sin pt
\]

Substituting in (6) the initial \(z_0 = 0, \quad \upsilon_0 = S_0 m^{-1}\) corresponding to the application of the \(S_0\) pulse to the stationary system at the moment \(t = 0\), we obtain
\[
z = S_0 m \pi e^{-\frac{\gamma}{2}pt} \sin pt.
\]

For a finite number of \(n + 1\) periodic pulses with a \(T_0\) period, the solution is built by superimposing functions (7) with different time starts
\[
\begin{align*}
z_n &= S_0 m \pi e^{-\frac{\gamma}{2}pt} \sin pt \\
&= S_0 m \pi e^{-\frac{\gamma}{2}p(t - nT_0)} \sin p(t - nT_0),
\end{align*}
\]

where
\[
\begin{align*}
t^* &= \frac{t - nT_0}{T_1} \quad (0 \leq t^* \leq \theta), \\
A_n &= \sum_{k=0}^{n} e^{-b'_k} \cos a'_k = \\
&= e^{b'} - \cos a' - e^{-nb'} \cos(n + 1) a' + e^{-(n+1)b'} \cos n a' \\
&= \frac{2 \left(\cosh b' - \cos a'\right)}{n} \\
B_n &= \sum_{k=0}^{n} e^{-b'_k} \sin a'_k = \\
&= \sin a' - e^{-nb'} \sin(n + 1) a' + e^{-(n+1)b'} \sin n a' \\
&= \frac{2 \left(\sinh b' - \cos a'\right)}{n} \\
a' &= 2\pi \theta, \quad \theta = \frac{T_0}{T_1}, \quad b' = \gamma \pi \theta, \quad n - r' = k.
\end{align*}
\]

The global maximum is set from the analysis of \(n\)-values \(z_{\text{max}}\).

\[
z_{n_{\text{max}}} = \frac{S_0}{m \pi} e^{-\gamma \pi t^*_0} \sqrt{(A_n^2 + B_n^2)}
\]

\[
t^* = \frac{1}{2\pi} \arctg \frac{2A_n - \gamma B_n}{2B_n + \gamma A_n}
\]

\[
z_{n_{\text{max}}} = \frac{S_0}{m \pi} \sqrt{(A_n^2 + B_n^2)}
\]
When $\frac{T_0}{T} = N$, pulsed resonance occurs, the amplitude of which can be approximately determined by the formula

$$z_{n \text{ max}} \approx \frac{S_0}{mp_0} \frac{1 - e^{-\gamma n N(n+1)}}{1 - e^{-\gamma n N}}.$$

The largest of the $z_{n \text{ max}}$ at $N = 1$.

With a large $n$, the fluctuations will be practically steady. Then in case of $n \to \infty$

$$z_{n \text{ max}} = \frac{S_0}{mp_0} e^{b_1/2} e^{-\gamma n t_0} \sqrt{2(chb' - cosa')},$$

at $\gamma \leq 0.1$,

$$z_{n \text{ max}} \approx \frac{S_0}{mp_0} e^{b_1/2} e^{-\gamma n t_0} \sqrt{2(chb' - cosa')}.$$

At $\frac{T_0}{T} = N$, impulsive resonance occurs.

The resonance amplitude can be determined by the formula

$$z_{\text{res max}} = \frac{S_0}{mp_0} e^{b_1/2} e^{-\gamma n t_0} \sqrt{2(chb' - cosa')},$$

Expression (11) is a decomposition of solutions of the dissipative system of equations (9) by the forms of natural oscillations. Imagine $z_{ik}$ as a decomposition of $z_{ik} = c_k \phi_{ik}$, where $\phi_{ik}$ -- coefficients of the forms of natural oscillations determine $d$ from $N(N - 1)$

$$\phi_{ik} = \frac{1}{2 \sum_{i=1}^{N} \delta_{ir} \phi_{kr}}.$$

From $2N$ equations $\cos v_i = 0$,

$$m_i \sum_{k=1}^{N} c_k \varphi_{ik} p_i = -S_0, \quad i = 1, 2, ..., N,$$

$$z_k = \sum_{i=1}^{N} z_{ik} e^{-\frac{\gamma}{2} p_i t} \cos(p_i t + v_i),$$

$z_{ik} = c_k \varphi_{ik}$, $\varphi_{ik} = \frac{p_i}{\sum_{i=1}^{N} \varphi_{ir} c_i},$ $p_i = \left[1 + \frac{\gamma^2}{4}(1/2)p_i^0\right]^{-1/2}$

4 Multiple Degrees of Freedom Model

The movement of a system with $N$ degrees of freedom is described by $N$ differential equations of the form

$$m_k \ddot{z}_k + \sum_{r=1}^{N} (a + ib) c_{kr} z_r = 0, (k = 1, 2, ..., N).$$

$z_k(0) = 0$, $m_k \dot{z}_k(0) = S_0^0$.

$$z_k = \sum_{i=1}^{N} z_{ik} e^{-\frac{\gamma}{2} p_i t} \cos(p_i t + v_i).$$

$z_{ik} = c_k \varphi_{ik}$, $\varphi_{ik} = \frac{p_i}{\sum_{i=1}^{N} \varphi_{ir} c_i},$ $p_i = \left[1 + \frac{\gamma^2}{4}(1/2)p_i^0\right]^{-1/2}$

Solution (11) will take the form $z_k = \sum_{i=1}^{N} z_{ik} e^{-\gamma p_i t} \sin(p_i t + \varphi_{ik}).$
4.1 Shock processes in rod systems

\[
\frac{\partial^2 Z}{\partial t^2} + (a + ib)E_j \frac{\partial^4 Z}{\partial x^4} = 0.
\]

\[
Z = \sum_{i=1}^{\infty} c_i X_i(x) e^{-\frac{c_i}{2} t^2} e^{i(\omega t + \nu)}.
\]

\[
z(x, 0) = 0, \quad \mu \ddot{z}(x, 0) = -S_0(x).
\]

\[
p_i = \left(1 + \frac{y^2}{4}\right)^{-1/2} \cdot p_i^0 - i - \sum_{j=1}^{n} \frac{\partial \rho_j}{\partial \lambda_j} - \sum_{k=1}^{\infty} \frac{\partial \rho_k}{\partial \zeta_k}.
\]

\[
p_i^0 = \lambda_i^2 \sqrt{\frac{E_j}{\mu_i}} \cdot \frac{1}{\mu_i}.
\]

\[
X_i(x) = A_i \left[\sin \lambda_i \left(\frac{x}{L}\right) + B_i \sinh \lambda_i \left(\frac{x}{L}\right) + \frac{C_i \cos \lambda_i \left(\frac{x}{L}\right) + D_i \cosh \lambda_i \left(\frac{x}{L}\right)}{\lambda_i \left(\frac{x}{L}\right)}\right].
\]

\[
X_i^4 - \left(\frac{\lambda_i^4}{L^4}\right) X_i = 0,
\]

\[
\frac{1}{t} \int_0^t X_i^2 dx = 1.
\]

\[
z(x, t) = \sum_{i=1}^{\infty} c_i X_i(x) e^{-\frac{c_i}{2} t^2} \sin p_i t.
\]

\[
c_i = \frac{(\tau_{i/\mu_i})}{\int_0^t s(x) X_i(x) dx}.
\]

\[
\varepsilon_i = \varepsilon_i \left(\frac{1}{\lambda_i}\right) = \varepsilon_i(T); \quad \varepsilon_i(T) = \frac{X_i(T)}{2\pi \int_0^T f(t) dt}.
\]

\[
c_i = (\varepsilon_i s/\mu_i^2 p_i) \int_0^t X_i(x) dx.
\]
\[ c_i = \left( \frac{\varepsilon_i S}{\mu l p_i} \right) X_i(x_0). \]

Some values of \( \chi \) and \( \varepsilon \) are given in Table 1.

<table>
<thead>
<tr>
<th>T</th>
<th>Pulse shapes (*)</th>
<th>( f(t) )</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( \varepsilon )</td>
<td>( \chi )</td>
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<td>1</td>
<td>( \varepsilon_1 )</td>
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<td>2</td>
<td>( \varepsilon_5 )</td>
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<td>3</td>
<td>( \varepsilon_9 )</td>
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<tr>
<td>4</td>
<td>( \varepsilon_{13} )</td>
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</table>

4.2 Shock processes in rectangular plate systems

The equation of natural vibrations of plates taking into account internal friction for an instantaneous pulse has the form

\[ \mu \frac{\partial^2 Z}{\partial t^2} + D(a + ib) \Delta \Delta Z = 0, \Delta = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}. \]

The solution of the problem of a rectangular plate on a two-parameter basis is sought in the form of a double row of beam functions and is proposed in [15]

\[ z(x, y, t) = \sum_{i=1}^\infty c_i F_i(x, y) e^{-\frac{\gamma}{2} p_i t} \sin p_i t, \]

where \( F_i(x, y) \approx X_r(x)Y_j(y) \),

\[ X_r(x) = A_r \left[ \sin \lambda_r \left( \frac{x}{l} \right) + B_r \sinh \lambda_r \left( \frac{x}{l} \right) + C_r \cos \lambda_r \left( \frac{x}{l} \right) + D_r \cosh \lambda_r \left( \frac{x}{l} \right) \right] \]

\[ Y_j(y) = A_j \left[ \sin \lambda_j \left( \frac{y}{b} \right) + B_j \sinh \lambda_j \left( \frac{y}{b} \right) + C_j \cos \lambda_j \left( \frac{y}{b} \right) + D_j \cosh \lambda_j \left( \frac{y}{b} \right) \right] \]

\[ \frac{1}{l} \int_0^l X_r^2 dx = 1, \frac{1}{b} \int_0^b Y_j^2 dy = 1. \]
\( r, j \) – indexes of beam functions \( X_r(x) \) and \( Y_j(y) \) corresponding to the rods - strips cut from the plate along the \( X, Y \) axes and having the appropriate fastening conditions.

The correspondence between the number of the \( i \)–th oscillation frequency of the plate and the indices \( r, j \) are established according to Table 2.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( r )</th>
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</tr>
<tr>
<td>16</td>
<td>8</td>
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</tbody>
</table>

Circular frequencies of the plates \( p_i = p_i^0 \sqrt{1 + \gamma^2/4} \), \( p_i^0 = \lambda_i^2 \sqrt{D/\mu l^4} \), (16)

\( \lambda_i^2 = \sqrt{\lambda_r^2 + \lambda_r^2 \eta^2 + \lambda_4 \eta^4}, \eta = \frac{l}{b} \)

\( c_i = \frac{\epsilon_i}{\mu b p_i^0} \int_0^l \int_0^b s(x, y)X_r(x)Y_j(y)\,dx\,dy \) for an evenly distributed over the plate \( s(x, y) = \text{const} \)

\( c_i = \frac{\epsilon_i s}{\mu b p_i^0} \int_0^l X_r(x)\,dx \int_0^b Y_j(y)\,dy \) for a concentrated load \( s(x, y) \) at the point \( (x_0, y_0) \)

\( c_i = \frac{\epsilon_i s}{\mu b p_i^0} \int X_r(x)Y_j(y_0)\,dx \) for a concentrated load \( s(x, y) \) at the point \( (x_0, y_0) \)

\( \epsilon_i = \epsilon_i(\tau/T_i) \)

\( |A_0| = \sum_{i=1}^{\infty} |A_i| e^{-\frac{\gamma p_i^0 l}{4}} A_i(z_0) = c_i X_r Y_j \)

5 Conclusion

The developed models have a place to find wide application in technology and technology. One of the applications is building beam and rod structures, mast structures. Important is the ability to simulate the behavior of dissipative systems taking into account friction (9).

Models allow you to investigate dynamic processes that occur under the influence of impulse loads.

The theory of building models of heterostructures of complex systems has been developed. Fundamental mathematical models of heterostructures are proposed. Models of heterostructures were built in the presence of exposure in the form of periodic and impulse loads. The results of modeling heterostructures under the influence of dynamic loads of various types are presented. General model of multilayer heterostructures and with multiple degrees of freedom are described.

References
Green’s functions stiffness method for Euler–Bernoulli beams. A macroelement model.