Heat transfer from a constant heat source to the plate surface: an analytical solution

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Abstract. This paper proposes an analytical solution to the thermoelasticity problem for a plate heated by a source with a constant heat input on one surface. Numerous comparisons were made between the results obtained on the basis of the analytical solution and the numerical one. When comparing the analytical and numerical solutions, it was shown that the advantage of the analytical solution is that it makes it possible to determine the greatest stress values over the entire heating time range, while using the numerical method it is necessary to perform a large number of calculations at fixed time values. However, the finite element method allows for determining stresses in areas of the plate that are far from its centre, where the analytical solution proves to be unfair.

Key words: Analytical solutions, numerical solutions, thermoelasticity, surface.

1 Introduction

Composite materials are now widely used in rocket and space technology as thermal protection materials due to their unique properties arising from their manufacturing technology. A matrix of fine fibre fillers is impregnated with binders that readily degrade at moderate temperatures [1-9]. The result is a variety of filled plastics: glass-filled plastics, carbon-filled plastics, asbofilled plastics, etc. [10-19]. When using such materials as heat shields at hypersonic flight speeds (Mach number greater than 5) of aircraft, it is important to study the effect of temperature on the mechanical characteristics of materials and stress state on temperature distribution in structural elements such as thin heat shield plates.

Currently, there are a sufficient number of works on solving thermoelasticity problems [20-27]. The works [28-33] propose analytical and numerical methods for solving thermal conductivity problems in the flow of supersonic and hypersonic flows of aircraft. And in papers [34-38] various mechanical investigations of heat protective composite materials on the basis of both analytical and numerical methods are given. Analytical solutions of thermoelasticity problems are important, including the validation of numerical solutions [39-48]. In this paper, we propose an analytical solution to the thermoelasticity problem for a plate heated by a source with a constant heat input at one surface. The results obtained on the basis of the analytical solution are compared with those of the numerical model.
2 Problem statement

The problem is considered within the framework of linear thermoelasticity. Accordingly, the deformations are assumed to be small, so that the following equations are fulfilled.

**Equilibrium equations:**

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho X = 0,
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + \rho Y = 0,
\]

\[
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial y} + \frac{\partial \tau_{zy}}{\partial x} + \rho Z = 0,
\]

where \(\sigma_x, \sigma_y, \sigma_z\) — normal stresses, \(\tau_{xy}, \tau_{yz}\) — tangential stresses, \(\rho\) — material density, \(X, Y, Z\) — the volumetric force intensity components.

**Cauchy ratios:**

\[
\varepsilon_{xy} = \frac{\partial u}{\partial x} + \frac{\partial y}{\partial y},
\]

\[
\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},
\]

\[
\varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},
\]

\[
\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},
\]

\[
\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},
\]

**Generalised Hooke's law with regard to temperature change:**

\[
\varepsilon_x = \frac{E}{1 - \nu^2} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] + aT,
\]

\[
\varepsilon_y = \frac{E}{1 - \nu^2} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right] + aT,
\]

\[
\varepsilon_z = \frac{E}{1 - \nu^2} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] + aT,
\]

\[
\gamma_{xy} = \frac{\tau_{xy}}{G} + \gamma_{xz},
\]

\[
\gamma_{yz} = \frac{\tau_{yz}}{G} + \gamma_{zy},
\]

\[
\gamma_{zx} = \frac{\tau_{zx}}{G} + \gamma_{xz},
\]

\[
\frac{\partial \tau_{xy}}{\partial t} = \frac{\lambda}{\rho c_p} \frac{\partial^2 T}{\partial x^2},
\]

In the following calculations we will determine the thermal stresses occurring in the heat shield slab under non-steady-state heating conditions.
\[
\frac{\partial T}{\partial x} \bigg|_{x=0} = 0 \quad \frac{\partial T}{\partial x} \bigg|_{x=L} = \frac{h}{\lambda} (T_{as} - T \bigg|_{x=L}) \quad \frac{T}{t} = 0
\]

3 Solution method

![Diagram of a rectangular slab]

\[
\sigma_x = -\alpha ET(y)
\]

\[
P_x = \int_{-c}^{c} \alpha ET(y) y dy
\]

\[
\left( \sigma_x \right)_{\text{flexion}} = \frac{M_{z} y}{I_{z}} = \frac{M_{z} y}{c^2 b} = \frac{y}{c^2} \int_{-c}^{c} \alpha ET(y) y dy
\]

\[
\sigma_x = -aET(y) + \frac{1}{c} \int_{-c}^{c} aET(y) y dy + \frac{c}{c} \int_{-c}^{c} aET(y) y dy
\]
Let us consider a thick plate with a variable temperature across its thickness. Consider the value of $b$ significant and treat it as a width, and $2s$ as the thickness of the plate. As the dimension along the axis $z$ is significant and its elements expand in this direction in different ways due to the uneven distribution $T(y)$, then both stresses will be generated in the plate $\sigma_x, \sigma_y$ as well as voltages $\sigma_x, \sigma_y$.

When fully anchored at the edges in the directions $X$ and $Z$ in the equation $0 = 0_{xz} \varepsilon \varepsilon = 0_{x} \varepsilon$, what gives:

$$1 \left( \int_{0}^{b} \frac{-aET(y)}{\nu} \, dy \right) = \frac{-aET(y)}{\nu}$$

Equation (18) in the case of a free thick plate for points away from the edges, taking into account (19), takes the form:

$$\int_{0}^{b} \frac{-aET(y)}{\nu} \, dy = \int_{0}^{b} \frac{-aET(y)}{\nu} \, dy$$

The outer surface of a thick upholstery or plate is heated by heat transfer by convection from the boundary layer, and heat is dissipated through the thickness of the plate by heat conduction. Accordingly, the heat conduction and heat transfer equations for the interface between the air and the plate can be used.

To solve the partial differential equation (11), (12) according to the method of separation of variables, let us assume:

$$T(x,t) = F(x)F(t)$$

Hence:

$$\frac{F''}{F} = \frac{F''}{F} = -\gamma \frac{a}{c_{\rho} \rho}$$

The solution to the ordinary differential equations (22) is:

$$F(x) = C_1 \sin \gamma x + C_2 \cos \gamma x$$

$$F(t) = C_3 e^{-\gamma \alpha t}$$

Thus, the equation for the eigenvalues is:

$$\cot \gamma n \frac{h}{L} = \frac{\lambda}{c_{\rho} \rho}$$

From the boundary conditions (12) we find:

$$T(x,L) = A e^{-\gamma \alpha L} + B$$

$$B = T_E \gamma L = \frac{h}{\lambda} \frac{\lambda}{\gamma L}$$

$$p_n = \frac{hL}{\lambda} \frac{n}{p_n}$$

$$T = 0, t = 0$$
\[ T = T_E - \sum_{n=1}^{\infty} a_n \left( -\frac{p_n a^t}{t_n} \right) \frac{p_n x}{L} \]

\[ T_E = \sum_{n=1}^{\infty} a_n \left[ \frac{p_n x}{L} \right] \int_0^t \frac{p_n x}{L} \, dx + \frac{T_E}{p_n + \cdot \cdot \cdot p_n} \]

\[ \frac{T}{T_E} = \sum_{n=1}^{\infty} p_n \left( \frac{-p_n W t}{t_n} \right) \frac{p_n x}{L} \]

\[ T_{E} = T_{as} = \text{const} \]

\[ \frac{\alpha E T}{E} = -x \left( \frac{x}{L} - 1 \right) \]

\[ H_c = \sum_{n=1}^{\infty} p_n \left( \frac{-p_n W t}{t_n} \right) \]

\[ H_B = \sum_{n=1}^{\infty} \frac{p_n \left[ \left( \frac{-p_n W t}{t_n} \right) - p_n \right] \left( \frac{-p_n W t}{t_n} \right) - \frac{p_n W t}{t_n}}{p_n + \cdot \cdot \cdot p_n \cdot \cdot \cdot p_n} \]

\[ H_B = \frac{\lambda t}{\rho c_p L} \]

4 Comparison of numerical and analytical solutions for an isotropic plate

\[ \lambda = 0.061905 \frac{W}{m \cdot K} \]

\[ v = 0.33 \]

\[ c_p = 1000 \frac{J}{kg \cdot K} \]

\[ T_B = 1250^\circ C \]

\[ \rho = 144 \frac{kg}{m^3} \]
Immediately find the time and area in the slab in which the maximum and minimum stresses are realized. This has been done in Mathematica. It was found that the maximum tensile stresses occur in the plate at the 14th second near the heated surface of the plate.

**Table 1** Comparison of numerical and analytical solutions.

<table>
<thead>
<tr>
<th>Maximum tensile stresses, MPa</th>
<th>Analytical solution</th>
<th>Numerical solution</th>
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</thead>
<tbody>
<tr>
<td>0.026</td>
<td>0.022</td>
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**Fig. 2**

a) Distribution of normal stresses $\sigma_x = \sigma_y$ in a slab; b) Distribution of normal stresses by the thickness of the slab away from its edges.
As can be seen from figure 3, the normal stresses $\sigma_x = \sigma_y$ on the heated surface of the plate are not equally distributed in its central area and on the edges. From the figure it is possible to determine the area of the edge effect in terms of normal stresses; it is 0.911 of the thickness of the board.

Fig. 4. Tangential stress distribution $\tau_{xy}$ by the thickness of the slab away from its edges.
Fig. 5. Distribution of normal stresses $\sigma_x = \sigma_y$ along the thickness of the slab at the maximum distance from its central part

Fig. 6. Tangential stress distribution $\tau_{xy}$ along the thickness of the slab at the maximum distance from its central part
Comparing Figures 2 b) and 5 for normal stresses $\sigma_x = \sigma_y$ and Figures 4 and 6 tangentially $\tau_{xy}$, their difference between the centre of the plate and the edges of the plate becomes apparent.

**Fig. 7** (a) Distribution of normal stresses $\sigma_x = \sigma_y$ in slab thickness away from the edges (based on analytical solution); b) Distribution of normal stresses $\sigma_x = \sigma_y$ on the thickness of the thermal insulation board away from its edges (based on numerical solution).

**Fig. 8** Plot of temperature distribution across the thickness of the slab at different moments $t = 0; t = 0.5t_1; t = t_1$, where $t_1 = 300$ seconds warm-up time. Volumetric pore content 93.2%.
Figure 8 shows the temperature distribution over the thickness of the slab at various points in time, constructed using the analytical solution. At the initial moment of time some instability is visible due to the limited number of terms in the analytical solution.

5 Conclusions

In this paper analytical solution of the thermoelasticity problem for a plate heated by a source with constant heat input on one surface is obtained. The analytical solution obtained is compared with the numerical one. When comparing the analytical and numerical solutions, it may be noted that the advantage of the analytical solution is that it makes it possible to determine the largest stress values over the entire heating time range. However, the finite element method makes it possible to determine stresses in areas of the plate that are far from its centre where the analytical solution is not valid. Also, the numerical method clearly demonstrates the stress-strain state of the plate, which appeared to bend during heating and on its heated and "cold" surfaces, stresses of the same sign arise.

References