Mathematical modeling of the eigenvibrations for the loaded shallow shell

Anton Samsonov[1] and *Sergey* Solov'ev[1*]

[1]Institute of Computational Mathematics and Information Technologies, Kazan Federal University, 18 Kremlevskaya Street, Kazan, 420008, Russian Federation

Abstract. The eigenvalue problems modeling of the eigenvibrations of the loaded shallow shell are studied. The asymptotic properties of the eigenvalues and eigenvectors are investigated in dependence on the load parameters. The original problem is approximated by the finite element method with Bogner-Fox-Schmit finite elements in the square domain. The new efficient algorithms for solving finite element eigenvalue problems are developed and investigated. A classification of resonance curves for one-dimensional and two-dimensional loaded mechanical systems is given. The results of numerical experiments are presented. Key words: shallow shell, eigenvibration, load, eigenvalue, eigenvector, eigenvalue problem, finite element method, Bogner-Fox-Schmit finite element.

1 Introduction

Problems on the natural vibrations of thin-walled structures with attached loads arise in mathematical modeling in the aviation and space technology, in designing of surface and submarine vessels, in designing of various tanks and reservoirs in the chemical and petroleum industries. Numerous publications are devoted to the study of these problems [1–10].

The monograph [6] presents a broad review of publications, studies performed, and an analysis of the state of the art in the study of mechanical systems with added masses. In the monograph [6], untenable conclusions were also noted, to which analytical solutions of problems led, the main unsolved problems were identified, among them there are the influence of geometric and wave parameters of the supporting structure on the fundamental natural frequency, the influence of inhomogeneities of the supporting structure on the fundamental natural frequency, the inconsistency of analytical solutions to the change in the fundamental frequency with an increase in the value of added mass. To adequately identify the specific features of the solutions of this class of problems, in the monograph [6] it is indicated the need to create an exact theory of such problems and theoretically substantiated methods of solution.

In the present paper, on the basis of the mathematical apparatus of modern functional analysis, the construction of such a theory for parametric problems of natural oscillations of the mechanics of loaded systems and the theoretical justification of numerical methods of solution are carried out. In Section 2, the eigenvalue problem modeling of the eigenvibrations

* Corresponding author: sergey.solovev.kpfu@mail.ru

of the loaded shallow shell are formulated [11]. The asymptotic properties of the eigenvalues and eigenvectors are also investigated in dependence on the load parameters. The obtained results generalize and develop the results of papers [7–10] for two-dimensional problems. In Section 3, the original problem is approximated by the finite element method with Bogner-Fox-Schmit finite elements in the square domain [12–17]. New error estimates of the approximate solutions are established here. The new efficient algorithms for solving finite element eigenvalue problems are developed and investigated in Section 4. A classification of resonance curves for one-dimensional and two-dimensional loaded mechanical systems is also given in Section 5. To illustrate this classification, the results of numerical experiments are presented.

2 Variational statement of the problem

Let $\Omega = (0,1) \times (0,1)$ be a square region in the plane with boundary Γ, $\overline{\Omega} = \Omega \cup \Gamma$. Set $\Lambda = (0, \infty)$, $\overline{\Lambda} = [0, \infty)$. The symbols $L_2(\Omega)$ and $H_0^m(\Omega)$ denote the Lebesgue and Sobolev spaces equipped with the norms $|.|_0$ and $|.|_m$, $m = 1,2$. The symbol $H^\alpha(\Omega)$ denotes the Sobolev space of the fractional order $\alpha \in (0,4]$. By $L_\infty(\Omega)$ denote the space of measurable functions bounded almost everywhere on Ω. Define the Hilbert spaces $H = (L_2(\Omega))^3$ and $V = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$ with the norms $\|\bar{v}\|_H^2 = |v_1|_0^2 + |v_2|_0^2 + |v_3|_0^2$, $\|\bar{v}\|_V^2 = |v_1|_1^2 + |v_2|_1^2 + |v_3|_2^2$, $\bar{v} = (v_1, v_2, v_3)^{\mathrm{T}}$.

For gived functions ρ, d, E, ν, R_1, and R_2 from the space $L_\infty(\Omega)$, assume that the following inequalities hold $\rho_1 \le \rho(x) \le \rho_2$, $d_1 \le d(x) \le d_2$, $E_1 \le E(x) \le E_2$, $R_{11} \le R_1(x) \le R_{12}$, $R_{21} \le R_2(x) \le R_{22}$, $0 \le \nu(x) \le 1/2$, for almost all $x \in \Omega$ and for some positive constants $\rho_1, \rho_2, d_1, d_2, E_1, E_2, R_{11}, R_{12}, R_{21}, R_{22}$. Set $D = d^3S/12$, $G = E/(2(1+\nu))$, $S = E/(1-\nu^2)$, $\alpha_{11} = \alpha_{22} = dS$, $\alpha_{12} = \alpha_{21} = \nu dS$, $\alpha_{66} = dG$, $\beta_{11} = \beta_{22} = D$, $\beta_{12} = \beta_{21} = \nu D$, $\beta_{66} = d^3G/12$,

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & O \\ O & \mathcal{A}_{22} \end{pmatrix}, \quad \mathcal{A}_{11} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{66} \end{pmatrix}, \quad \mathcal{A}_{22} = \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ 0 & 0 & 4\beta_{66} \end{pmatrix},$$

$$(\bar{v}, \bar{w}) = \bar{v}^{\mathrm{T}}\bar{w}, \quad J\bar{w} = (J_1\bar{w}, J_2\bar{w}, J_3\bar{w}, J_4\bar{w}, J_5\bar{w}, J_6\bar{w})^{\mathrm{T}},$$

$$J_1\bar{w} = \partial_1 w_1 + \varkappa_1 w_3, \quad J_2\bar{w} = \partial_2 w_2 + \varkappa_2 w_3, \quad J_3\bar{w} = \partial_2 w_1 + \partial_1 w_2,$$

$$J_4\bar{w} = -\partial_{11} w_3, \quad J_5\bar{w} = \partial_{22} w_3, \quad J_6\bar{w} = -\partial_{12} w_3, \quad \varkappa_1 = R_1^{-1}, \quad \varkappa_2 = R_2^{-1},$$

$$\partial_i = \partial/\partial x_i, \quad \partial_{ij} = \partial_i \partial_j, \quad i, j = 1,2.$$

For given $z \in \overline{\Omega}$, $\xi \in \overline{\Lambda}$, and $\mu \in \overline{\Lambda}$, define the symmetric bilinear forms:

$$a(\bar{u}, \bar{v}) = \int_\Omega (\mathcal{A}J\bar{u}, J\bar{v})dx \quad \forall \bar{u}, \bar{v} \in V, \quad b(\bar{u}, \bar{v}) = \int_\Omega \rho d(\bar{u}, \bar{v})dx \quad \forall \bar{u}, \bar{v} \in H,$$

$$c(\bar{u}, \bar{v}) = u_3(z)v_3(z) \quad \forall \bar{u}, \bar{v} \in V,$$

$$a(\xi, \bar{u}, \bar{v}) = a(\bar{u}, \bar{v}) + \xi c(\bar{u}, \bar{v}), \quad b(\mu, \bar{u}, \bar{v}) = b(\bar{u}, \bar{v}) + \mu c(\bar{u}, \bar{v}) \quad \forall \bar{u}, \bar{v} \in V.$$

The point z is the attachment point of the load of mass μ, ξ is the support elasticity coefficient of the point z.

Introduce the following problems.

For $\xi \in \overline{\Lambda}$ and $\mu \in \overline{\Lambda}$, find $\lambda = \lambda(\xi, \mu) \in \mathbb{R}$, $\bar{u} = \bar{u}^{\xi,\mu} \in V\backslash\{0\}$, such that

$$a(\xi, \bar{u}, \bar{v}) = \lambda b(\mu, \bar{u}, \bar{v}) \quad \forall \bar{v} \in V. \tag{1}$$

For $\xi \in \overline{\Lambda}$ and $\mu \in \Lambda$, find $\eta = \eta(\xi, \mu) \in \mathbb{R}$, $\bar{u} = \bar{u}^{(\xi,\mu)} \in V\backslash\{0\}$, such that

$$a(\xi, \bar{u}, \bar{v}) = \eta\mu c(\bar{u}, \bar{v}) \quad \forall \bar{v} \in V. \tag{2}$$

For $V_0 = \{\bar{v}: \bar{v} \in V, v_3(z) = 0\}$, find $\lambda = \lambda^{(0)} \in \mathbb{R}$, $\bar{u} = \bar{u}^{(0)} \in V_0\backslash\{0\}$, such that

$$a(\bar{u}, \bar{v}) = \lambda b(\bar{u}, \bar{v}) \quad \forall \bar{v} \in V_0. \tag{3}$$

Find $\bar{u} = \bar{u}_0 \in V\backslash\{0\}$ such that

$$a(\bar{u}, \bar{v}) = v_3(z) \quad \forall \bar{v} \in V. \tag{4}$$

Problem (1) has the solutions $\lambda_i = \lambda_i(\xi, \mu)$, $\bar{u}_i = \bar{u}_i^{\xi,\mu}$, $a(\xi, \bar{u}_i, \bar{u}_j) = \lambda_i \delta_{ij}$, $b(\mu, \bar{u}_i, \bar{u}_j) = \delta_{ij}$, $i, j = 0,1, \ldots$

Problem (3) has the solutions $\lambda_i = \lambda_i^{(0)}$, $\bar{u}_i = \bar{u}_i^{(0)}$, $a(\bar{u}_i, \bar{u}_j) = \lambda_i \delta_{ij}$, $b(\bar{u}_i, \bar{u}_j) = \delta_{ij}$, $i, j = 1,2, \ldots$

Problem (4) has a unique solution $\bar{u}_0$. Problem (2) has one simple eigenvalue and a unique corresponding eigenfunction defined by

$$\eta(\xi, \mu) = \frac{\xi}{\mu} + \frac{1}{\mu u_{03}(z)}, \quad \bar{u}^{(\xi,\mu)}(x) = \frac{\bar{u}_0(x)}{\sqrt{\mu} u_{03}(z)}, \quad x \in \bar{\Omega}, \quad u_3^{(\xi,\mu)}(z) = \frac{1}{\sqrt{\mu}}.$$

For brevity, set $\lambda_i(\mu) = \lambda_i(0, \mu)$, $\lambda_i(\xi) = \lambda_i(\xi, 0)$, $\bar{u}_i^{\mu} = \bar{u}_i^{0,\mu}$, $\bar{u}_i^{\xi} = \bar{u}_i^{\xi,0}$, $i = 0,1, \ldots$

The following asymptotic properties hold.

For fixed $\xi \in \overline{\Lambda}$, $0 < \eta(\xi, \mu) - \lambda_0(\xi, \mu) \to 0$, $\| \bar{u}^{(\xi,\mu)} - \bar{u}_0^{\xi,\mu} \|_V \to 0$, $\mu \to \infty$, $u_{03}^{\xi,\mu}(z) > 0$. Hence fundamental natural frequency of the shell-load-spring system can be comput as natural frequency of the spring-load-spring system by the formula $\omega = \sqrt{\eta} = \sqrt{\xi + \varkappa}/\sqrt{\mu}$, $\varkappa = (u_{03}(z))^{-1}$, when μ is a large value.

For fixed $\xi \in \overline{\Lambda}$, $0 \le \lambda_i(\xi, \mu) - \lambda_i^{(0)} \to 0$, $\| \bar{u}_i^{\xi,\mu} - \bar{u}_i^{(0)} \|_V \to 0$, $\mu \to \infty$, $b(\bar{u}_i^{(0)}, \bar{u}_i^{\xi,\mu}) > 0$, $i = 1,2, \ldots$

For fixed $\xi \in \overline{\Lambda}$, $0 \le \lambda_i(\xi) - \lambda_i(\xi, \mu) \to 0$, $\| \bar{u}_i^{\xi,\mu} - \bar{u}_i^{\xi} \|_V \to 0$, $\mu \to 0$, $b(\bar{u}_i^{\xi,\mu}, \bar{u}_i^{\xi}) > 0$, $i = 0,1, \ldots$

For fixed $\mu \in \overline{\Lambda}$, $0 \le \lambda_{i+1}^{(0)} - \lambda_i(\xi, \mu) \to 0$, $\| \bar{u}_i^{\xi,\mu} - \bar{u}_{i+1}^{(0)} \|_V \to 0$, $\xi \to \infty$, $b(\bar{u}_{i+1}^{(0)}, \bar{u}_i^{\xi,\mu}) > 0$, $i = 0,1, \ldots$

For fixed $\mu \in \overline{\Lambda}$, $0 \le \lambda_i(\xi, \mu) - \lambda_i(\mu) \to 0$, $\| \bar{u}_i^{\xi,\mu} - \bar{u}_i^{\mu} \|_V \to 0$, $\mu \to 0$, $b(\bar{u}_i^{\xi,\mu}, \bar{u}_i^{\mu}) > 0$, $i = 0,1, \ldots$

This asymptotic properties for eigenvectors are valid for a subsequence, when the corresponding limit eigenvalue is multiple.

The following estimates hold $|\lambda_i(\xi, \mu) - \lambda_i(\zeta, \eta)| \le c(|\xi - \zeta| + |\mu - \eta|)$, $\| \bar{u}_i^{\xi,\mu} - \bar{u}_i^{\zeta,\eta} \|_V \le c(|\xi - \zeta| + |\mu - \eta|)$, $b(\mu, \bar{u}_i^{\xi,\mu}, \bar{u}_i^{\zeta,\eta}) > 0$.

3 Finite element approximation

Define nodes $x_{ij} = (t_i, t_j)$, $t_i = ih$, $i, j = 0,1, \dots, n$, $h = 1/n$, and the partition of the set $\overline{\Omega}$ into elements $e_{ij} = [t_{i-1}, t_i] \times [t_{j-1}, t_j]$, $i, j = 1,2, \dots, n$, $h = 1/n$. Denote by $V_h = V_{1h} \times V_{1h} \times V_{2h}$ the Bogner-Fox-Schmit finite element subspace of the space V. Denote by N the dimension of the space V_h. Assume that $z = x_{k_1 k_2}$, $0 < k_1 < n$, $0 < k_2 < n$.

Introduce the following finite element approximations for problems (1)–(4).

For $\xi \in \overline{\Lambda}$ and $\mu \in \overline{\Lambda}$, find $\lambda^h = \lambda^h(\xi, \mu) \in \mathbb{R}$, $\bar{u}^h = \bar{u}^{\xi,\mu,h} \in V_h \backslash \{0\}$, such that

$$a(\xi, \bar{u}^h, \bar{v}^h) = \lambda^h b(\mu, \bar{u}^h, \bar{v}^h) \quad \forall \bar{v}^h \in V_h. \tag{5}$$

For $\xi \in \overline{\Lambda}$ and $\mu \in \Lambda$, find $\eta^h = \eta^h(\xi, \mu) \in \mathbb{R}$, $\bar{u}^h = \bar{u}^{(\xi,\mu,h)} \in V \backslash \{0\}$, such that

$$a(\xi, \bar{u}^h, \bar{v}^h) = \eta^h \mu c(\bar{u}^h, \bar{v}^h) \quad \forall \bar{v}^h \in V_h. \tag{6}$$

For $V_{0h} = \{\bar{v}^h : \bar{v}^h \in V_h, v_3^h(z) = 0\}$, find $\lambda^h = \lambda^{(0,h)} \in \mathbb{R}$, $\bar{u}^h = \bar{u}^{(0,h)} \in V_{0h} \backslash \{0\}$, such that

$$a(\bar{u}^h, \bar{v}^h) = \lambda^h b(\bar{u}^h, \bar{v}^h) \quad \forall \bar{v}^h \in V_{0,h}. \tag{7}$$

Find $\bar{u}^h = \bar{u}_0^h \in V_h \backslash \{0\}$ such that

$$a(\bar{u}^h, \bar{v}^h) = v_3^h(z) \quad \forall \bar{v}^h \in V_h. \tag{8}$$

Problem (5) has the solutions $\lambda_i^h = \lambda_i^h(\xi, \mu)$, $\bar{u}_i^h = \bar{u}_i^{\xi,\mu,h}$, $a(\xi, \bar{u}_i^h, \bar{u}_j^h) = \lambda_i^h \delta_{ij}$, $b(\mu, \bar{u}_i^h, \bar{u}_j^h) = \delta_{ij}$, $i, j = 0,1, \dots, N-1$.

Problem (7) has the solutions $\lambda_i^h = \lambda_i^{(0,h)}$, $\bar{u}_i^h = \bar{u}_i^{(0,h)}$, $a(\bar{u}_i^h, \bar{u}_j^h) = \lambda_i^h \delta_{ij}$, $b(\bar{u}_i^h, \bar{u}_j^h) = \delta_{ij}$, $i, j = 1,2, \dots, N-1$.

Problem (8) has a unique solution $\bar{u}_0^h$. Problem (6) has one simple eigenvalue and a unique corresponding eigenfunction defined by

$$\eta(\xi, \mu) = \frac{\xi}{\mu} + \frac{1}{\mu u_{03}^h(z)}, \quad \bar{u}^{(\xi,\mu,h)}(x) = \frac{\bar{u}_0^h(x)}{\sqrt{\mu} u_{03}^h(z)}, \quad x \in \bar{\Omega}, \quad u_3^{(\xi,\mu,h)}(z) = \frac{1}{\sqrt{\mu}}.$$

If the exact eigenfunctions belong to the space $H^{1+\alpha}(\Omega) \times H^{1+\alpha}(\Omega) \times H^{2+\alpha}(\Omega)$, $\alpha \in (0,2]$, then the following error estimates are valid.

For problems (5) and (7), $0 \le \lambda_i^h - \lambda_i \le ch^{2\alpha}$, $\| \bar{u}_i^h - \bar{u}_i \|_V \le ch^{\alpha}$.

For problem (6), $0 \le \eta^h - \eta \le ch^{2\alpha}$, $\| \bar{u}^h - \bar{u} \|_V \le ch^{\alpha}$.

4 Matrix problem

Formulate the following algorithms for solving problems (5) and (6) with constant coefficients ρ, d, E, ν, R_1, and R_2. Asume that $\varkappa = R_1^{-1} = R_2^{-1}$. The symbol $\otimes$ denotes the Kronecker product of matrices. Define the continuously differentiable functions $f_i(t)$, $t \in [0,1]$, $i = 0,1, \dots, 2n-1$, such that $f_i(t)$, $t \in [t_{j-1}, t_j]$, is a cubic polynomial for $j = 1,2, \dots, n$, $f_{2i-1}(t_j) = \delta_{ij}$, $f'_{2i-1}(t_j) = 0$, $f_{2i}(t_j) = 0$, $f'_{2i}(t_j) = \delta_{ij}$, $i, j = 1,2, \dots, n-1$, $f_0(t_0) = 0$, $f'_0(t_0) = 1$, $f_{2n-1}(t_n) = 0$, $f'_{2n-1}(t_n) = 1$.

Algorithm for problem (5).

Step 1. Define n, k_1, k_2, ξ, μ, ρ, d, E, ν, $\varkappa$. Compute $G = E/(2(1+\nu))$, $S = E/(1-\nu^2)$, $D = d^3S/12$, $\alpha_{11} = \alpha_{22} = dS$, $\alpha_{12} = \nu dS$, $\alpha_{66} = dG$, $m = 2(n-1)$.

Step 2. Compute the matrices G^{rs}, $\bar{G}^{rs}$, $\overleftarrow{G}^{rs}$, $\vec{G}^{rs}$, $r, s = 0,1,2$,

$$G_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j, i = 1,2, \dots, m,$$

$$\bar{G}_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j, i = 0,1, \dots, m+1,$$

$$\overleftarrow{G}_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j = 1,2, \dots, m, \quad i = 0,1, \dots, m+1,$$

$$\vec{G}_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j = 0,1, \dots, m+1, \quad i = 1,2, \dots, m.$$

Step 3. Compute the matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}, C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix},$$

with non-zero blocks

$$\begin{gathered}
A_{11} = \alpha_{11}\bar{G}^{11} \otimes \bar{G}^{00} + \alpha_{66}\bar{G}^{00} \otimes \bar{G}^{11}, \\
A_{12} = \alpha_{12}\bar{G}^{01} \otimes \bar{G}^{10} + \alpha_{66}\bar{G}^{10} \otimes \bar{G}^{01}, \\
A_{13} = \varkappa\alpha_{11}\vec{G}^{01} \otimes \vec{G}^{00} + \varkappa\alpha_{12}\vec{G}^{01} \otimes \vec{G}^{00}, \\
A_{21} = \alpha_{11}\bar{G}^{10} \otimes \bar{G}^{01} + \alpha_{66}\bar{G}^{01} \otimes \bar{G}^{10}, \\
A_{22} = \alpha_{66}\bar{G}^{11} \otimes \bar{G}^{00} + \alpha_{22}\bar{G}^{00} \otimes \bar{G}^{11}, \\
A_{23} = \varkappa\alpha_{12}\vec{G}^{00} \otimes \vec{G}^{01} + \varkappa\alpha_{22}\vec{G}^{00} \otimes \vec{G}^{01}, \\
A_{31} = \varkappa\alpha_{11}\overleftarrow{G}^{10} \otimes \overleftarrow{G}^{00} + \varkappa\alpha_{12}\overleftarrow{G}^{10} \otimes \overleftarrow{G}^{00}, \\
A_{32} = \varkappa\alpha_{22}\overleftarrow{G}^{00} \otimes \overleftarrow{G}^{10} + \varkappa\alpha_{12}\overleftarrow{G}^{00} \otimes \overleftarrow{G}^{10}, \\
A_{33} = D(G^{22} \otimes G^{00} + 2G^{11} \otimes G^{11} + G^{00} \otimes G^{22}) + \\
+\varkappa^2(\alpha_{11} + 2\alpha_{12} + \alpha_{22})G^{00} \otimes G^{00}, \\
B_{11} = \rho d\bar{G}^{00} \otimes \bar{G}^{00}, B_{22} = \rho d\bar{G}^{00} \otimes \bar{G}^{00}, B_{33} = \rho dG^{00} \otimes G^{00}, \\
C_{33} = (e_{2k_{1-1}}(e_{2k_{1-1}})^{\mathrm{T}}) \otimes (e_{2k_{2-1}}(e_{2k_{2-1}})^{\mathrm{T}}), \\
e_j = (\delta_{1j}, \delta_{2j}, \dots, \delta_{mj})^{\mathrm{T}}.
\end{gathered}$$

Step 4. Solve the eigenvalue problem $(A + \xi C)y = \lambda(B + \mu C)y$.

Algorithm for problem (6).

Step 1. Define n, k_1, k_2, ξ, μ, d, E, ν, $\varkappa$. Compute $G = E/(2(1+\nu))$, $S = E/(1-\nu^2)$, $D = d^3S/12$, $\alpha_{11} = \alpha_{22} = dS$, $\alpha_{12} = \nu dS$, $\alpha_{66} = dG$, $m = 2(n-1)$.

Step 2. Compute the matrices G^{rs}, $\bar{G}^{rs}$, $\overleftarrow{G}^{rs}$, $\vec{G}^{rs}$, $r, s = 0,1,2$,

$$G_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j, i = 1,2, \dots, m,$$

$$\bar{G}_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j, i = 0,1, \dots, m+1,$$

$$\overleftarrow{G}_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j = 1,2, \dots, m, \quad i = 0,1, \dots, m+1,$$

$$\vec{G}_{ji}^{rs} = \int_0^1 f_i^{(r)}(t) f_i^{(s)}(t)dt, \quad j = 0,1, \dots, m+1, \quad i = 1,2, \dots, m.$$

Step 3. Compute the matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix},$$

with non-zero blocks

$$\begin{gathered} A_{11} = \alpha_{11}\bar{G}^{11}\otimes\bar{G}^{00} + \alpha_{66}\bar{G}^{00}\otimes\bar{G}^{11}, \\ A_{12} = \alpha_{12}\bar{G}^{01}\otimes\bar{G}^{10} + \alpha_{66}\bar{G}^{10}\otimes\bar{G}^{01}, \\ A_{13} = \varkappa\alpha_{11}\vec{G}^{01}\otimes\vec{G}^{00} + \varkappa\alpha_{12}\vec{G}^{01}\otimes\vec{G}^{00}, \\ A_{21} = \alpha_{11}\bar{G}^{10}\otimes\bar{G}^{01} + \alpha_{66}\bar{G}^{01}\otimes\bar{G}^{10}, \\ A_{22} = \alpha_{66}\bar{G}^{11}\otimes\bar{G}^{00} + \alpha_{22}\bar{G}^{00}\otimes\bar{G}^{11}, \\ A_{23} = \varkappa\alpha_{12}\vec{G}^{00}\otimes\vec{G}^{01} + \varkappa\alpha_{22}\vec{G}^{00}\otimes\vec{G}^{01}, \\ A_{31} = \varkappa\alpha_{11}\overleftarrow{G}^{10}\otimes\overleftarrow{G}^{00} + \varkappa\alpha_{12}\overleftarrow{G}^{10}\otimes\overleftarrow{G}^{00}, \\ A_{32} = \varkappa\alpha_{22}\overleftarrow{G}^{00}\otimes\overleftarrow{G}^{10} + \varkappa\alpha_{12}\overleftarrow{G}^{00}\otimes\overleftarrow{G}^{10}, \\ A_{33} = D(G^{22}\otimes G^{00} + 2G^{11}\otimes G^{11} + G^{00}\otimes G^{22}) + \\ +\varkappa^2(\alpha_{11} + 2\alpha_{12} + \alpha_{22})G^{00}\otimes G^{00}, \\ C_{33} = (e_{2k_{1-1}}(e_{2k_{1-1}})^{\mathrm{T}})\otimes(e_{2k_{2-1}}(e_{2k_{2-1}})^{\mathrm{T}}), \\ e_j = (\delta_{1j}, \delta_{2j}, \dots, \delta_{mj})^{\mathrm{T}}, F = \operatorname{diag} C. \end{gathered}$$

Step 4. Solve the system of linear algebraic equations $Ay = F$.

Step 5. Compute $\eta = \frac{\xi}{\mu} + \frac{1}{\mu(F,y)}$, $Y = \frac{y}{\sqrt{\mu}(F,y)}$.

Algorithm for problem (6) with large value μ is more efficient than algorithm for problem (5), since algorithm for problem (6) requires solving a system of linear algebraic equations instead of an eigenvalue problem.

5 Classification of resonance curves

Resonance curves are the functions $\lambda(\xi, \mu)$ in problem (1) for a fixed first or second argument. Theoretical studies of the properties of the resonant frequencies have led to the division of the class of mathematical models defined by problem (1) into the following three groups. Group 1 is characterized by strictly monotonic behavior of all resonance curves without intersection points. This group includes one-dimensional problems with loading at a boundary point. In group 2, there may additionally occur separate resonant curves described by constant functions. This group includes one-dimensional problems with loading at an interior point. In group 3, different resonant curves may have common points of intersection and consist of sections with strictly monotonic behavior and sections with constant values of resonant frequencies. This group includes two-dimensional problems with loading at an interior point. To illustrate this classification of resonance curves, consider the following examples.

Example 1. Set $\Omega = (0,1)$, $z = 1$. Define the Hilbert spaces $H = L_2(\Omega)$ and $V = \{v: v \in H^1(\Omega), v(0) = 0\}$ with the norms $\|\bar{v}\|_H^2 = |v|_0^2$, $\|\bar{v}\|_V^2 = |v|_1^2$, $\bar{v} = v$. For gived functions ρ, d, and $\mathcal{A}$ from the space $L_\infty(\Omega)$, assume that the following inequalities hold $\rho_1 \le \rho(x) \le \rho_2$, $d_1 \le d(x) \le d_2$, $\mathcal{A}_1 \le \mathcal{A}(x) \le \mathcal{A}_2$, for almost all $x \in \Omega$ and for some positive constants ρ_1, ρ_2, d_1, d_2, $\mathcal{A}_1$, $\mathcal{A}_2$. Put $J\bar{u} = v'$. For these data, the problem of the form (1) simulates the natural vibrations of a bar fixed at $x = 0$, elastically supported at $x = 1$, with load attached at the point $x = 1$. Figure 1 shows the resonance curves $\lambda_i(\xi, \mu)$, $i = 0,1, \dots, 5$,

for $\mathcal{A}(x) = e^{2x}$, $\rho(x) = \cos x, d(x) = 1, \mu \in [0,2], \xi = 50$. The resonance curves $\lambda_i(\xi, \mu)$, $i = 0,1, \dots, 5$, $\mu \in [0,2]$, are strictly decreasing functions.

Example 2. Set $\Omega = (0,1)$, $z = 0.5$. Define the Hilbert spaces $H = L_2(\Omega)$ and $V = H_0^1(\Omega)$ with the norms $\|\bar{v}\|_H^2 = |v|_0^2$, $\|\bar{v}\|_V^2 = |v|_1^2$, $\bar{v} = v$. For gived functions ρ, d, and $\mathcal{A}$ from the space $L_\infty(\Omega)$, assume that the following inequalities hold $\rho_1 \le \rho(x) \le \rho_2$, $d_1 \le d(x) \le d_2$, $\mathcal{A}_1 \le \mathcal{A}(x) \le \mathcal{A}_2$, for almost all $x \in \Omega$ and for some positive constants ρ_1, ρ_2, d_1, d_2, $\mathcal{A}_1$, $\mathcal{A}_2$. Put $J\bar{u} = v'$. For these data, the problem of the form (1) simulates the natural vibrations of a string fixed at $x = 0$ and $x = 1$ with load attached at the point $x = 0.5$ and with elastic support of stiffness ξ at the point z. Figure 2 shows the resonance curves $\lambda_i(\xi, \mu)$, $i = 0,1, \dots, 5$, for $\mathcal{A}(x) = 1$, $\rho(x) = 1$, $d(x) = 1$, $\mu \in [0,2]$, $\xi = 50$. The resonance curves $\lambda_i(\xi, \mu)$, $i = 0,2,4$, $\mu \in [0,2]$, are strictly decreasing functions. The resonance curves $\lambda_i(\xi, \mu)$, $i = 1,3,5$, $\mu \in [0,2]$, are constant functions.

Example 3. Set $\Omega = (0,1) \times (0,1)$, $z = (0.2,0.7)$. Define the Hilbert spaces $H = L_2(\Omega)$ and $V = H_0^2(\Omega)$ with the norms $\|\bar{v}\|_H^2 = |v|_0^2$, $\|\bar{v}\|_V^2 = |v|_2^2$, $\bar{v} = v$. Introduce positive constant functions ρ, d, and $\mathcal{A}$. Put $J\bar{v} = \partial_{11}v + \partial_{22}v$. For these data, the problem of the form (1) simulates the natural vibrations of a clamped plate with a load of mass μ attached at the point z and with elastic support of stiffness ξ at the point z. Figure 3 shows the resonance curves $\lambda_i(\xi, \mu)$, $i = 0,1, \dots, 5$, for $\mathcal{A} = 1$, $\rho = 1$, $d = 1$, $\mu \in [0,2]$, $\xi = 5000$. Figure 3 demonstrates the following properties: 1) $\lambda_i(\xi, \mu)$, $i = 0,3$, are strictly decreasing functions, 2) there exists a point $\mu_1 \in [0,2]$ such that $\lambda_1(\xi, \mu_1) = \lambda_2(\xi, \mu_1)$, $\lambda_1(\xi, \mu)$, $\mu \in [\mu_1, 2]$, and $\lambda_2(\xi, \mu)$, $\mu \in [0, \mu_1]$ are strictly decreasing functions, $\lambda_2(\xi, \mu)$, $\mu \in [\mu_1, 2]$, and $\lambda_1(\xi, \mu)$, $\mu \in [0, \mu_1]$ are constant functions, 3) there exists a point $\mu_2 \in [0,2]$ such that $\lambda_4(\xi, \mu_2) = \lambda_5(\xi, \mu_2)$, $\lambda_4(\xi, \mu)$, $\mu \in [\mu_2, 2]$, and $\lambda_5(\xi, \mu)$, $\mu \in [0, \mu_2]$ are strictly decreasing functions, $\lambda_5(\xi, \mu)$, $\mu \in [\mu_2, 2]$, and $\lambda_4(\xi, \mu)$, $\mu \in [0, \mu_2]$ are constant functions.

Example 4. Set $\Omega = (0,1) \times (0,1)$, $z = (0.2,0.7)$. The problem (1) simulates the natural vibrations of a clamped shallow shell with a load of mass μ attached at the point z and with elastic support of stiffness ξ at the point z. Figure 4 shows the resonance curves $\lambda_i(\xi, \mu)$, $i = 0,1, \dots, 5$, for $E = 210000$, $\rho = 7800$, $d = 0.06$, $\nu = 0.167$, $R_1 = 5$, $R_2 = 5$, $\mu \in [0,500]$, $\xi = 10000$. Figure 4 illustrates the following properties: 1) $\lambda_i(\xi, \mu)$, $i = 0,3$, are strictly decreasing functions, 2) there exists a point $\mu_1 \in [0,500]$ such that $\lambda_1(\xi, \mu_1) = \lambda_2(\xi, \mu_1)$, $\lambda_1(\xi, \mu)$, $\mu \in [\mu_1, 500]$, and $\lambda_2(\xi, \mu)$, $\mu \in [0, \mu_1]$ are strictly decreasing functions, $\lambda_2(\xi, \mu)$, $\mu \in [\mu_1, 500]$, and $\lambda_1(\xi, \mu)$, $\mu \in [0, \mu_1]$ are constant functions, 3) there exists a point $\mu_2 \in [0,500]$ such that $\lambda_4(\xi, \mu_2) = \lambda_5(\xi, \mu_2)$, $\lambda_4(\xi, \mu)$, $\mu \in [\mu_2, 500]$, and $\lambda_5(\xi, \mu)$, $\mu \in [0, \mu_2]$ are strictly decreasing functions, $\lambda_5(\xi, \mu)$, $\mu \in [\mu_2, 500]$, and $\lambda_4(\xi, \mu)$, $\mu \in [0, \mu_2]$ are constant functions.

6 Conclusions

In the present paper, the parametric symmetric variational eigenvalue problems in the Hilbert space modeling of the eigenvibrations of the shallow shell with attached load are studied. The asymptotic properties of the eigenvalues and eigenvectors are investigated in dependence on the load parameters. The original parametric variational eigenvalue problem is approximated by the finite element method with Bogner-Fox-Schmit finite elements in the square domain. New error estimates of the approximate eigenvalues and the approximate eigenvectors are established. The new efficient algorithms for solving finite element eigenvalue problems are also developed and investigated. In addition, the new fast algorithm for calculating the main resonance frequency and the corresponding resonance form for large mass values is proposed. This algorithm is based on solving the boundary value problem instead of the eigenvalue problem. As a consequence of the research conducted, the new classification of resonance curves for one-dimensional and two-dimensional thin-walled

structures with attached load described by the parametric symmetric variational eigenvalue problem is obtained.

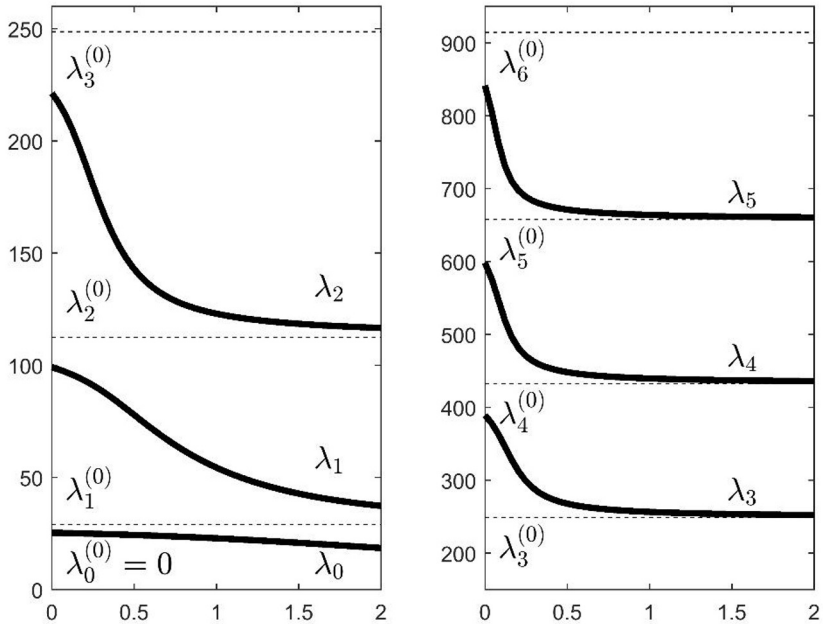

Fig. 1. Resonance curves of the bar-load-spring system.

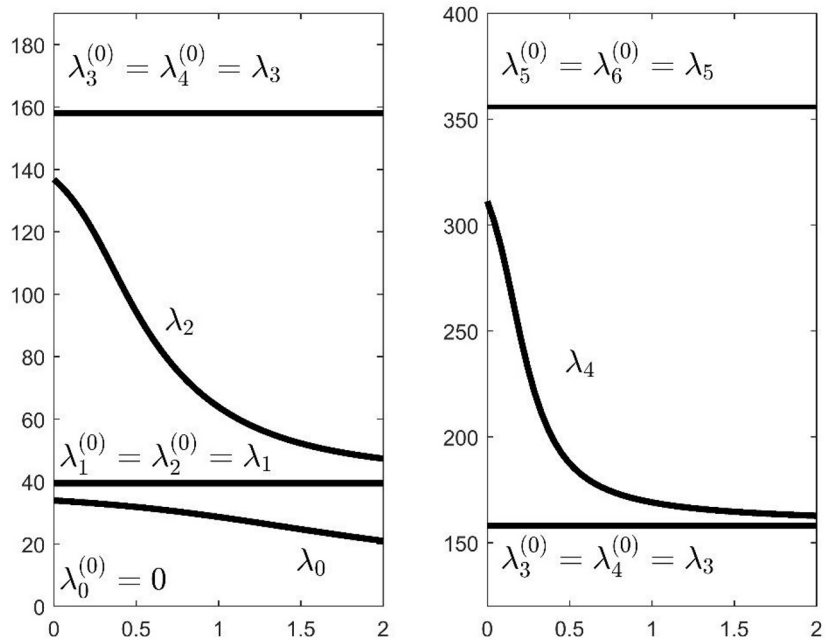

Fig. 2. Resonance curves of the string-load-spring system.

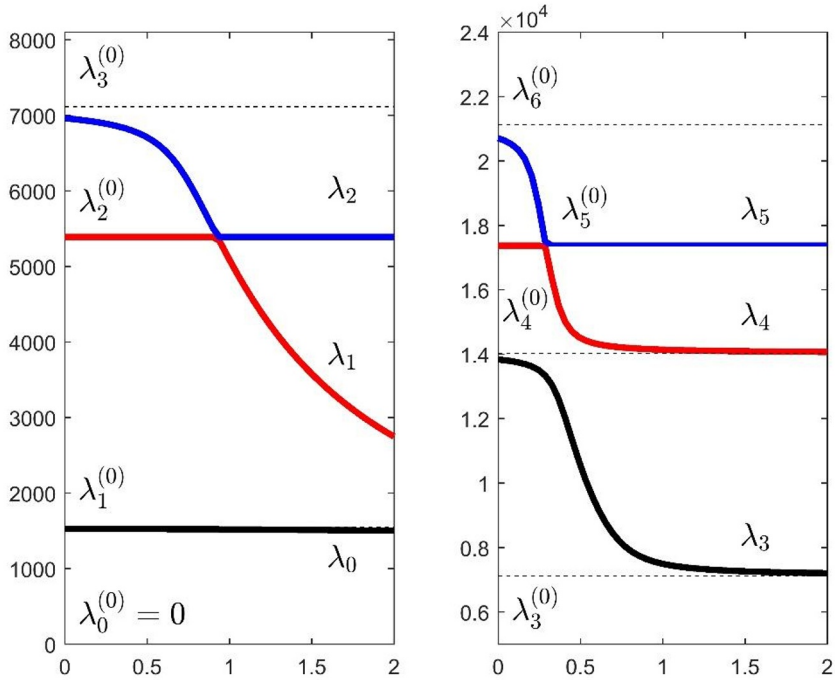

Fig. 3. Resonance curves of the plate-load-spring system.

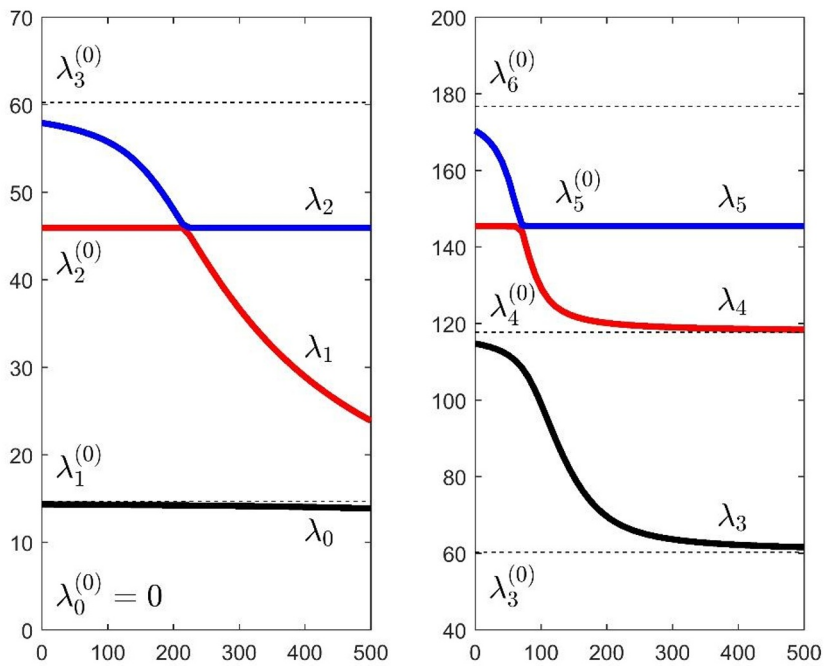

Fig. 4. Resonance curves of the shell-load-spring system.

Acknowledgements

This paper has been supported by the Kazan Federal University Strategic Academic Leadership Program ("PRIORITY-2030").

References

1. X. Pang, J. Sun, Z. Zhang, J. Comput. Appl. Math. **410**, 114156 (2022) DOI: 10.1016/j.cam.2022.114156
2. O. Barry, D.C.D. Oguamanam, J.W. Zu, Shock and Vibration, 485630 (2014) DOI: 10.1155/2014/485630
3. S.A.M. Rozlan, I. Zaman, B. Manshoor, A. Khalid, S.W. Chan, M.S.M. Sani, MATEC Web of Conf., **90**, 01030 (2017) DOI: 10.1051/ matecconf/20179001030
4. E.B. Koreneva, Internat. J. Comput. Civil Struct. Engin., **17** (1), 48–54 (2021) DOI:10.22337/2587-9618-2021-17-1-48-54
5. G.S. Leizerovich, S.V. Seregin, J. Appl. Mech. Tech. Phy. **57**, 841–846 (2016) DOI: 10.1134/S0021894416050102
6. S.V. Seregin, Dinamics of thin cylindrical shells with added mass (Komsomolsk-na-Amure State Technical University, Komsomolsk-on-Amur, 2016)
7. S.I. Solov'ev, Differ. Equat. **53** (3), 409–423 (2017) DOI: 10.1134/S0012266117030119
8. A.A. Samsonov, S.I. Solov'ev, D.M. Korosteleva, Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki, **162** (1), 52–65 (2020) DOI: 10.26907/2541-7746.2020.1.52-65
9. D.M. Korosteleva, A.A. Samsonov, P.S. Solov'ev, S.I. Solov'ev, Lobachevskii J. Math. **42** (7), 1697–1705 (2021) DOI: 10.1134/S1995080221070131
10. D.M. Korosteleva, L.N. Koronova, A.A. Samsonov, S.I. Solov'ev, Lobachevskii J. Math. **43** (4), 996–1005 (2022) DOI: 10.1134/S1995080222070150
11. W. Soedel, Vibrations of shells and plates (Marcel Dekker, New York, 2004)
12. P.G. Ciarlet, The finite element method for elliptic problems (SIAM, Philadelphia, 2002)
13. S.C. Brenner, L.R. Scott, The mathematical theory of finite element methods (Springer, New York, 2008)
14. S.C. Brenner, L.Y. Sung, J. Sci. Comput. **22**, **23**, 83–118 (2005) DOI: 10.1007/s10915-004-4135-7
15. J. Meng, L. Mei, J. Comput. Appl. Math. **402**, 113783 (2022) DOI: 10.1016/ j.cam.2021.113783
16. J. Valdman, Lect. Notes Comput. Sci., **12044**, 256–266 (2020) DOI: 10.1007/978-3-030-43222-5_22
17. E. Burman, P. Hansbo, M.G. Larson, Adv. Model. Simul. Eng. Sci. **7**, 27 (2020) DOI: 10.1186/s40323-020-00164-3