

Using penalty function method in identification stiffness of elastic spring of the underground structure

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Abstract. This paper studies a method to identify the spring stiffness of the frame - soil structure. The model of the problem is a two-dimensional structure, linear elastic deformation, model of soil using the Winkler model (with normal, tangential spring). The problem will be solved by the penalty function method - the minimum of the objective function - combined with the finite element method. The numerical calculations show that the model, algorithm and calculation program are reliable. The program can be used to identify the spring stiffness of the frame - soil structure in two dimensions, serving to determine the actual working state of the structure, to propose solutions for reinforcement, repairing, improving bearing capacity, prolonging the life of the structure.

1 Introduction

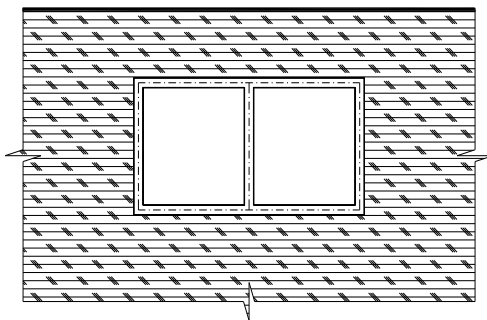


Fig. 1. Underground structure

For underground structure or frame - soil structure, parameter of connection between the structure and the foundation is very important. In the calculation, the Winkler foundation model is widely used. An important factor to determine the stress and displacement of the structure is the spring stiffness between the structure and soil. However, in practice, it is difficult to exactly determine the spring stiffness (due to the large amplitude coefficients for each type of soil), or under the effect of external loads, the above links degraded and changed

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over time. Determining the stress and displacement of the structure to propose technical measures to restore, maintain and improve the subsequent working capacity of the structure, first, it is necessary to evaluate the real state of the connections at any time during the use of the work. That is, the stiffness of the elastic spring must be identified based on measured data (displacement, stress, frequency, ...).

Structural identification has been focused on by many scientists. In particular in the aerospace and oil - shore there are many authors interested in research such as Farrar and Doebling [1], Ghoshal et al. [2], Friswell and Penny [3], Lee and Shin [4], Chan Ghee Koh, Lin Ming See, Thambirajah, Balendra [5] evaluated the hardness index of each floor to diagnose damage of frame structure. Narkis Y. [6] locates cracks in the beam structure. Hassiotis S. and Jeong G.D. [7] use the global planning method with finite element method to solve the problem of structural identification. J.K.Sinha, M.I. Friswell, S. Edwards [8], J.K.Sinha, P.M.Mujumdar, R.I.K.Moorthy [9], M.I.Friswell, J.E.T.Penny, S.D.Garvey [10], M.I. Friswell [11], J.K.Sinha, M.I. Friswell [12,13] studied the problem of diagnosing structural damage, stiffness link by the penalty function method. Petersen [14], Davide Balatti [15], Gardner [16] studied the problem of diagnosing structural damage, stiffness linked by the penalty function method. Mousa Rezaee studied damage detection and structural health monitoring with the autoregressive moving average (ARMA) model and fuzzy classification [17]. N.X.Bang and D.V.Thanh [18] determined the Equivalent Fixed Depth of 3D frame, N.X.Bang and N.D.Duan [19] identified the elastic fixed stiffness of the frame structure by the penalty function method. Mingqiang Xu using Horseshoe and Bayesian lasso methods to identify the damage of structural [20].

In this paper, a method of identifying Winkler elastic stiffness (normal, tangent spring) is proposed on the basis of measured data, which is the natural vibration frequency of the structure, which is the method of the penalty function.

2 Methods

2.1. The equations of motion of the frame structural system in 2-dimensional frame

Investigation of frame structural system in the form of 2-dimensional frame under dynamic load effect (Figure 2) in the coordinates Oxz.

Recognize the following assumptions:

- Frame-soil link is replaced by Winkler spring, with elastic normal and tangential spring stiffness.

- The strain of the frame structural system is linear and small.

The analysis model of the structure is shown in Figure 3.

To build the equation of the motion of the frame structural system, the finite element method (FEM) will be used.

The equations of motion of the frame - soil structural system according to FEM method [21, 23], after applying boundary conditions to the system, can be formulated as follows:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + (\mathbf{K}_e + \mathbf{K}_s)\mathbf{U}(t) = \mathbf{P}(t) \quad (1)$$

Where $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$, $\ddot{\mathbf{U}}(t)$ respectively are the displacement vector, velocity and node acceleration of the structural system.

$\mathbf{M}$, $\mathbf{K}_e$, $\mathbf{C}$ respectively are mass, stiffness, and damping matrices of the structural system.

$\mathbf{K}_s$ is stiffness matrices of the soil structural system.

$\mathbf{P}(t)$ is the nodal load vector of the structural system.

The damping matrix of structural system can be calculated according to the mass

matrix and stiffness matrix as:

$$\mathbf{C} = \alpha \mathbf{M} + \beta (\mathbf{K}_c + \mathbf{K}_s) \tag{2}$$

Where α, β are the factors depend on the specific vibration frequencies of the system and the viscous damping factors of the material.

α, β are *Rayleigh* damping coefficient, determined by the lowest individual frequencies of the structure ω_1, ω_2 ; and the corresponding damping ratios ζ_1, ζ_2 according to the formula:

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} \frac{2(\zeta_1 \omega_2 - \zeta_2 \omega_1) \omega_1 \omega_2}{\omega_2^2 - \omega_1^2} \\ \frac{2(\zeta_2 \omega_2 - \zeta_1 \omega_1)}{\omega_2^2 - \omega_1^2} \end{Bmatrix} \tag{3}$$

ζ_1, ζ_2 are damping ratios depend on the structural material also as nature of work of the system.

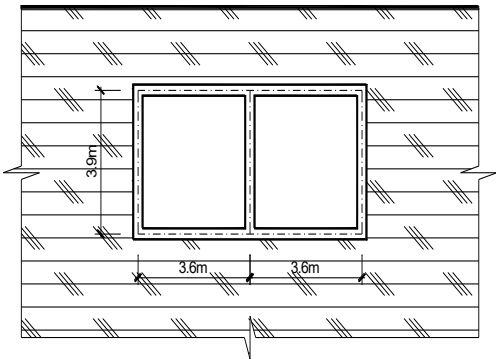


Fig. 2. Real model of frame structural system.

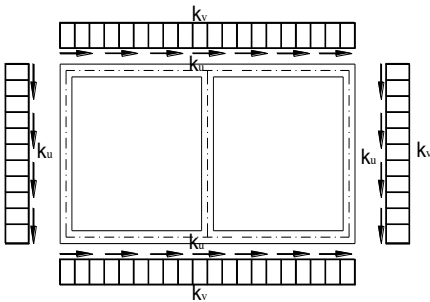


Fig. 3. The analysis model of underground structure.

The matrices of the whole system in equation (3) can be built from the matrices of FEM in the system by the "direct stiffness" method [21]. The following are the matrices of FEM for the 2 dimensional frame structural system.

In order to establish the overall matrix of structural system $\mathbf{M}, \mathbf{K}_c$ and nodal load vector $\mathbf{P}$, it is necessary to define mass matrices $\mathbf{m}$, stiffness matrices $\mathbf{k}$, and nodal load vector $\mathbf{p}$

of element in local coordinate system. Mass matrices $\mathbf{m}$, stiffness matrices $\mathbf{k}$, and nodal load vectors $\mathbf{p}$ of elements in the local coordinate system have the following form:

$$\mathbf{m} = ma \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{13}{35} & \frac{11}{210}a & 0 & \frac{9}{70} & -\frac{13}{420}a \\ 0 & \frac{11}{210}a & \frac{a^2}{105} & 0 & \frac{13}{420}a & -\frac{a^2}{140} \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{9}{70} & \frac{13}{420}a & 0 & \frac{13}{35} & -\frac{11}{210}a \\ 0 & -\frac{13}{420}a & -\frac{a^2}{140} & 0 & -\frac{11}{210}a & \frac{a^2}{105} \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} \frac{EF}{a} & 0 & 0 & -\frac{EF}{a} & 0 & 0 \\ 0 & \frac{12EI_z}{a^3} & \frac{6EI_z}{a^2} & 0 & -\frac{12EI_z}{a^3} & \frac{6EI_z}{a^2} \\ 0 & \frac{6EI_z}{a^2} & \frac{4EI_z}{a} & 0 & -\frac{6EI_z}{a^2} & \frac{2EI_z}{a} \\ -\frac{EF}{a} & 0 & 0 & \frac{EF}{a} & 0 & 0 \\ 0 & -\frac{12EI_z}{a^3} & -\frac{6EI_z}{a^2} & 0 & \frac{12EI_z}{a^3} & -\frac{6EI_z}{a^2} \\ 0 & \frac{6EI_z}{a^2} & \frac{2EI_z}{a} & 0 & -\frac{6EI_z}{a^2} & \frac{4EI_z}{a} \end{bmatrix}$$

$$\mathbf{k}_s = \begin{bmatrix} \frac{1}{3}k_u a & 0 & 0 & \frac{1}{6}k_u a & 0 & 0 \\ 0 & \frac{13}{35}k_v a & \frac{11}{210}k_v a^2 & 0 & \frac{9}{70}k_v a & -\frac{13}{420}k_v a^2 \\ 0 & \frac{11}{210}k_v a^2 & \frac{1}{105}k_v a^3 & 0 & \frac{13}{420}k_v a^2 & -\frac{1}{140}k_v a^3 \\ \frac{1}{6}k_u a & 0 & 0 & \frac{1}{3}k_u a & 0 & 0 \\ 0 & \frac{9}{70}k_v a & \frac{13}{420}k_v a^2 & 0 & \frac{13}{35}k_v a & -\frac{11}{210}k_v a^2 \\ 0 & -\frac{13}{420}k_v a^2 & -\frac{1}{140}k_v a^3 & 0 & -\frac{11}{210}k_v a^2 & \frac{1}{105}k_v a^3 \end{bmatrix}$$

$$\mathbf{p} = \frac{a}{2} \begin{bmatrix} p_u & p_v & \frac{1}{6}p_v a & p_u & p_v & -\frac{1}{6}p_v a \end{bmatrix}^T \quad (4)$$

Where I_z are moment of inertia for the axis z ;

F is area section of the element;

a is length of the element;

m is mass distributed on the long unit of element;

p_u, p_v are vertical load and distributed horizontal;

G is elastic shear modulus of materials;

k_u, k_v - normal, tangential elastic stiffness spring.

2.2. The problem of identification of stiffness spring and the solution method

Investigate the structural system in the form of an existing 2D frame (for example, figure 2). It is necessary to identify the spring stiffness (normal, tangential) on the basis of the vibration frequency measured by dynamic testing of the structure at the site.

To solve the problem, we will apply the penalty function method of the FEM update model in structural dynamics [22], [23], whereby the identification parameters of the problem are determined on the basis of minimizing the penalty function - is the sum of squares of errors between measured individual values and calculated values.

Symbols: $\theta = [\theta_1, \theta_2, \dots, \theta_j, \dots, \theta_{2p}]^T = [k_{u1}, k_{v1}, k_{u2}, k_{v2}, \dots, k_{uj}, k_{vj}, \dots, k_{up}, k_{vp}]^T$ is the vector of the identification parameters has an unknown value; $\mathbf{z}_e = [\lambda_{e1}, \lambda_{e2}, \dots, \lambda_{ei}, \dots, \lambda_{eN}]^T$ is the vector of the first N values obtains from measurement when dynamically testing the structure at the site; $\mathbf{z}_c = [\lambda_{c1}, \lambda_{c2}, \dots, \lambda_{ci}, \dots, \lambda_{cN}]^T$ is the vector of the first N values receives from the analysis, depending on the identification parameters, $\mathbf{z}_c = \mathbf{z}_c(\theta)$.

$\boldsymbol{\varepsilon} = (\mathbf{z}_e - \mathbf{z}_c(\theta)) = \boldsymbol{\varepsilon}(\theta)$ is errors vector between measured individual values and calculated values.

The penalty function $J(\theta)$ has the form

$$J(\theta) = \|\boldsymbol{\varepsilon}(\theta)\|^2 = (\mathbf{z}_e - \mathbf{z}_c(\theta))^T \mathbf{W}_e (\mathbf{z}_e - \mathbf{z}_c(\theta)) = \sum_{i=1}^N W_{ei} (\lambda_{ei} - \lambda_{ci}(\theta))^2 \quad (5)$$

where $\mathbf{W}_e = \text{diag}(W_{e1}, W_{e2}, \dots, W_{ei}, \dots, W_{eN})$ is diagonal matrix is positive and is usually the inverse matrix of the variance of the eigenvalues measurement data.

Develop Taylor error vector $\boldsymbol{\varepsilon}(\theta)$ according to certain identification parameters in a given vector $\theta = \theta_k$, obtaining

$$\boldsymbol{\varepsilon}_k(\theta) = (\mathbf{z}_e - \mathbf{z}_c(\theta_k)) + \left[\frac{\partial(\mathbf{z}_e - \mathbf{z}_c(\theta))}{\partial \theta} \right]_{\theta=\theta_k} \delta \theta_k + \text{higher order components} \quad (6)$$

If only the first two elements of the string (6) are retained, resulting in

$$\boldsymbol{\varepsilon}_k(\theta) \approx (\mathbf{z}_e - \mathbf{z}_c(\theta_k)) + \left[\frac{\partial(\mathbf{z}_e - \mathbf{z}_c(\theta))}{\partial \theta} \right]_{\theta=\theta_k} \delta \theta_k = \delta \mathbf{z}_k - \mathbf{S}_k \delta \theta_k = \boldsymbol{\varepsilon}_k(\delta \theta_k) \quad (7)$$

Where $\delta \theta_k$ is the increment of the identification parameter; $\delta \mathbf{z}_k$ is errors vector between eigenvalues measured and eigenvalues calculated when $\theta = \theta_k$.

$$\delta \mathbf{z}_k = \mathbf{z}_e - \mathbf{z}_{c,k} \quad (8)$$

$$\mathbf{z}_{c,k} = \mathbf{z}_c(\theta_k) = [\lambda_{c1,k}, \lambda_{c2,k}, \dots, \lambda_{ci,k}, \dots, \lambda_{cN,k}]^T \quad (9)$$

$\mathbf{S}_k$ is sensitivity matrix - the first derivative of eigenvalues calculated according to the identification parameters at $\theta = \theta_k$.

$$\mathbf{S}_k = \left[\frac{\partial \mathbf{z}_c(\theta)}{\partial \theta} \right]_{\theta=\theta_k} = \begin{bmatrix} S_{11,k} & S_{12,k} & \dots & S_{1j,k} & \dots & S_{1p,k} \\ S_{21,k} & S_{22,k} & \dots & S_{2j,k} & \dots & S_{2p,k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ S_{i1,k} & S_{i2,k} & \dots & S_{ij,k} & \dots & S_{ip,k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ S_{N1,k} & S_{N2,k} & \dots & S_{Nj,k} & \dots & S_{Np,k} \end{bmatrix}_k \quad (10)$$

with:

$$S_{ij,k} = \frac{\partial \lambda_{ci,k}}{\partial \theta_j} \quad (11)$$

The penalty function $J(\theta)$ is in this form

$$J_k(\theta) = \varepsilon_k^T (\delta \theta_k) W_\varepsilon \varepsilon_k (\delta \theta_k) = (\delta z_k - S_k \delta \theta_k)^T W_\varepsilon (\delta z_k - S_k \delta \theta_k) = J_k(\delta \theta_k) \quad (12)$$

The solution of equation (12) is obtained by minimizing the function $J_k(\delta \theta_k)$ follow $\delta \theta_k$, whereby

$$\frac{\partial J_k(\delta \theta_k)}{\partial \delta \theta_k} = 0 \quad (13)$$

Replace $J_k(\delta \theta_k)$ from (12) to (13), we get

As of (7), minimize (5) according to $\delta \theta_k$ have the result:

$$\delta \theta_k = [S_k^T W_\varepsilon S_k]^{-1} S_k^T W_\varepsilon \delta z_k \quad (14)$$

Because the function (7) is a linear approximation function θ , to get as close to the exact value of the problem as iterative. If performed:

$$\delta \theta_k = \theta_{k+1} - \theta_k \quad (15)$$

from (14) may write:

$$\theta_{k+1} = \theta_k + [S_k^T W_\varepsilon S_k]^{-1} S_k^T W_\varepsilon \delta z_k \quad (16)$$

or:

$$\theta_k = \theta_{k-1} + [S_{k-1}^T W_\varepsilon S_{k-1}]^{-1} S_{k-1}^T W_\varepsilon \delta z_{k-1} \quad (17)$$

Here (k-1), k, (k+1) index indicate the iteration steps.

The looping process ends when the solution of the problem converges with the required accuracy.

The elements of a sensitive matrix S can be obtained from partial vibration differential equation of the structure:

$$[K - \lambda M] \varphi = 0 \quad (18)$$

where: λ and φ are the normalized eigenvalues and eigenvectors of the structure.

$$K = K_c + K_s$$

The result we have:

$$S_{ij} = \frac{\partial \lambda_{ci}}{\partial \theta_j} = \varphi_i^T \left[\frac{\partial K}{\partial \theta_j} - \lambda_i \frac{\partial M}{\partial \theta_j} \right] \varphi_i = \varphi_i^T \frac{\partial K_s}{\partial \theta_j} \varphi_i \quad (19)$$

With: λ_i, φ_i are Normalized eigenvalues and eigenvectors i of the structure.

At the iteration k the above quantity has the form:

$$S_{ij,k} = \frac{\partial \lambda_{ci,k}}{\partial \theta_j} = \varphi_{i,k}^T \left[\frac{\partial K_s}{\partial \theta_j} \right]_{\theta=\theta_k} \varphi_{i,k} \quad (20)$$

Where: $\varphi_{i,k}$ is Normalized vector of the structure corresponding to the i value at the k iteration (or at $\theta = \theta_k$).

According to the iterative algorithms established above, the content of calculating the updated parameters is conducted in the following order:

** The original analysis*

Select vector of the parameters need to be identified θ of the structure model (here is elastic stiffness spring vector).

$$\theta = [\theta_1, \theta_2, \dots, \theta_j, \dots, \theta_{2p}]^T = [k_{u1}, k_{v1}, k_{u2}, k_{v2}, \dots, k_{uj}, k_{vj}, \dots, k_{up}, k_{vp}]^T$$

- Set up the stiffness matrix K_s and $\frac{\partial K_s}{\partial \theta_j}$ ($j = 1, 2, \dots, p$) according to the identification

parameters θ .

- Select a vector of initial identification parameters θ_0 .

$$\theta = [\theta_{1,0}, \theta_{2,0}, \dots, \theta_{j,0}, \dots, \theta_{2p,0}]^T = [k_{u1,0}, k_{v1,0}, k_{u2,0}, k_{v2,0}, \dots, k_{uj,0}, k_{vj,0}, \dots, k_{up,0}, k_{vp,0}]^T$$

- Calculate the matrix M , K_s when $\theta = \theta_0$.

- Calculate eigenvalues $\lambda_{ci,0}$ ($i=1, 2, \dots, N$), eigenvectors $\varphi_{i,0}$ ($i=1, 2, \dots, N$) and $\left[\frac{\partial K_s}{\partial \theta_j} \right]$ of

the structural system when ...

- Calculate the sensitivity $S_{ij,0}$ according to formula (20) (with index $k=0$; $i=1, 2, \dots, N$; $j=1, 2, \dots, p$) and form the sensitivity S_0 by the formula (10) with $k=0$.

- Forming eigenvalue measurement vector z_e

$$z_e = [\lambda_{e1}, \lambda_{e2}, \dots, \lambda_{ei}, \dots, \lambda_{eN}]^T$$

- Calculate the vector of eigenvalue error δz_0

$$\delta z_0 = z_e - z_{e,0}$$

$$\text{With: } z_{e,0} = [\lambda_{e1,0}, \lambda_{e2,0}, \dots, \lambda_{ei,0}, \dots, \lambda_{eN,0}]^T$$

** Repeatability*

At the iteration k ($k=1, 2, 3, \dots$), calculate identification parameter vector θ_k , eigenvalues

$\lambda_{ci,k}$, eigenvector $\varphi_{i,k}$, error matrix $z_{e,k}$, eigenvalue error vectors δz_k , $\left[\frac{\partial K_s}{\partial \theta_j} \right]_{\theta=\theta_k}$, sensitivity

$S_{ij,k}$, sensitivity matrix S_k , parameter vector θ_{k+1} according to the formulas set above.

- Compare errors between θ_{k+1} and θ_k with allowed errors ε .

- Check $\frac{\theta_{k+1} - \theta_k}{\theta_{k+1}} < \varepsilon$? If not satisfied: repeat calculation; If satisfied: stop calculating.

Based on the received algorithms, the author has built the **UFEM** program to solve the problem of identifying the elastic stiffness spring of the 2 dimensional frame structure in MATLAB language [24]. **UFEM** has been tested for reliability [23].

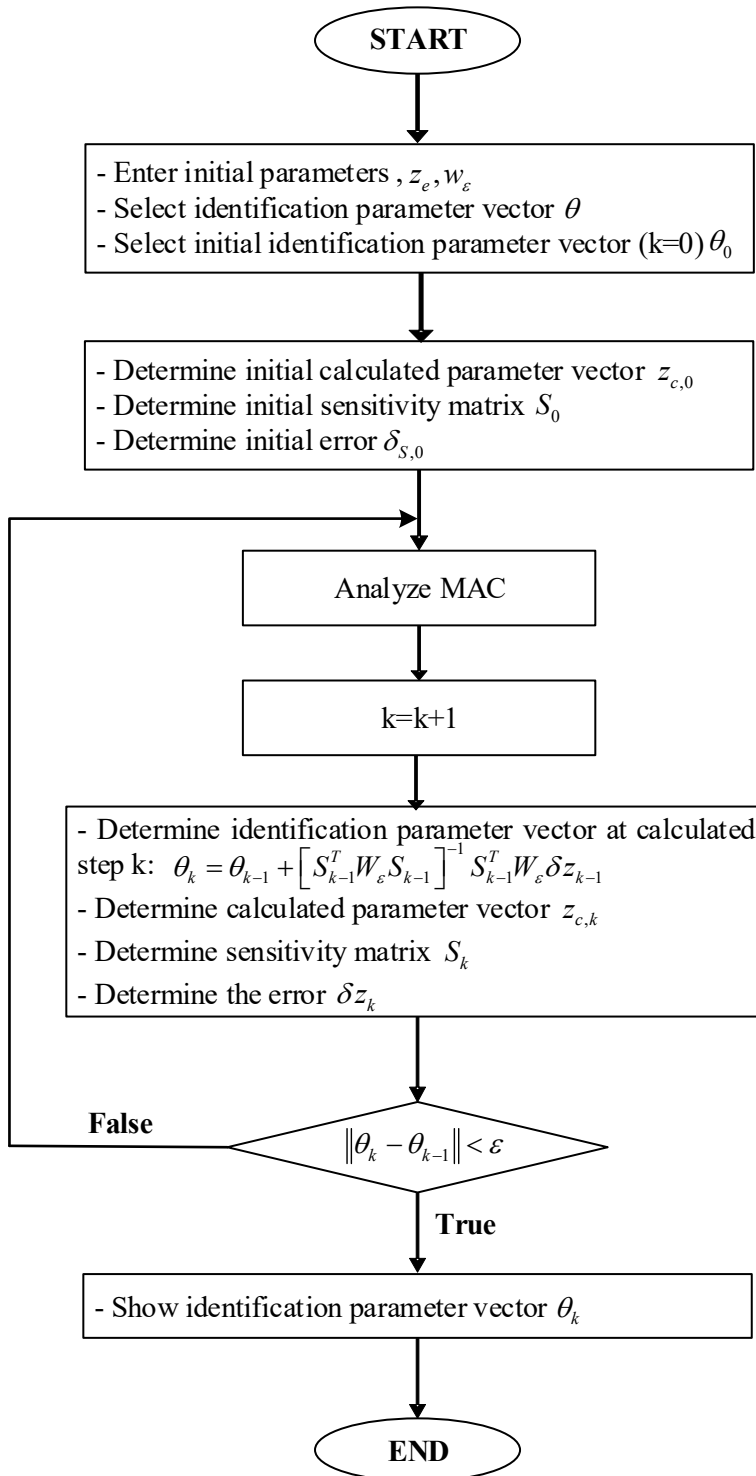


Fig. 4. UFEM program block diagram.

3 Results

The numerical calculations below are performed to check the reliability of established algorithms and programs. Identify elastic stiffness spring of the structure (Figure 2, Figure 3), with 02 elastic stiffness (k_u , k_v), therefore, there are 02 identification parameters).

- * Starting data:
 - The size of structure shown in figure 2; elastic modulus $E=2.25.10^7$ KN/m², density $\rho = 2.5$ T/m³, beam cross-section 0.3x1.0 m.

- Measurement vector (assumed):

$$\lambda_e = [\lambda_{e1} \ \lambda_{e2} \ \lambda_{e3} \ \lambda_{e4} \ \lambda_{e5} \ \lambda_{e6} \ \lambda_{e7} \ \lambda_{e8}]$$
$$= [37.258 \ 4032.649 \ 4814.176 \ 5277.911 \ 5560.007 \ 9242.009 \ 10904.999 \ 20061.865] \tag{21}$$

- The assumed eigenvalues here are not actually true measurements, it is eigenvalues calculate for the structure with a elastic stiffness spring

$$k_{tt} = [k_u \ k_v]^T = [10000 \ 100]^T \text{ (kN/m}^3\text{)} \tag{22}$$

- Choose the allowable error: $\varepsilon = 0.5\%$.
- * Calculation results:
 - Select the identification parameters as the elastic stiffness spring: $k_{tt} = [k_u \ k_v]^T$.
 - Discrete FEM for structure.
 - Select initial parameters: $\theta_0 = k_{tt,0} = [k_{u,0} \ k_{v,0}]^T = [5000 \ 50]^T \text{ (kN/m}^3\text{)}.$
 - Solution of the problem: The solution of the problem when calculating iteration needs to focus on the values of vector (22) with allowed errors $\varepsilon = 0.5\%$. The results of calculating the value of the identification parameters according to the iterative calculation steps are shown in figure 5.

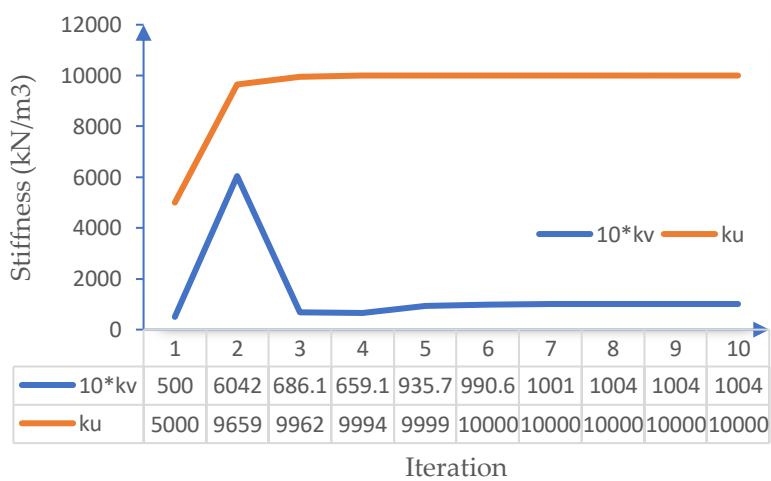


Fig. 5. Identification parameter graph by calculation step.

The results of the iterative calculation step by step gradually converge to the initial parameters, solution of convergence problems after 10 steps calculated with the results :

$$\theta = k_{tt} = [k_u \ k_v]^T = [10000 \ 100.41]^T \text{ (kN.m)} \tag{23}$$

4 Conclusions

Replacing the structure - soil link by Winkler spring, will simplify the calculation model, and thus will reduce the time for calculation, but still ensure accuracy and close to the actual working structure. When the connection parameters of the structure-soil change (due to incorrect design assumptions or during of using the structure), here is the stiffness of the spring, which will change the state of the structure. Therefore identification of spring stiffness is very important and necessary.

The solutions to the above problem converge on the values to be sought, and the numerical analytical data demonstrate the reliability of studied results, proving that the established algorithms and programs (UFEM) can be used to identify the stiffness spring (normal, tangential) of the structure in a two-dimensional model.

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