The «complex probabilities» balance principle for non-Markov processes modeling

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Abstract. This paper presents the principle of “complex probabilities” balance, based on the description of the stochastic process not in the time, but in the complex domain, which allows developing models of non-stationary queuing systems with arbitrary probabilities distributions of the requests and their servicing time, taking into account random or deterministic time delays. A system of balance equations in the complex domain was compiled and solved using Laplace images. Performed an inverse Laplace transform to move from images to probabilities in the time domain. Presented models for acyclic and cyclic stochastic processes with arbitrary distributions of time points between incoming requests and their service. Given calculation examples. Recommendations for the further application of the principle of balance of “complex probabilities” are given.

1 Introduction

A significant part of the researches using analytical modeling of various purposes systems is devoted to the study of the processes of functioning in a stationary mode. Here, models and methods of stationary queuing theory have found wide application. At the same time, it is of interest to study the functioning of queuing systems with varying workload intensity in transient and non-stationary operating modes. The solution of such problems makes it possible to evaluate the behavior of systems in the case of peak dynamic loads and transient non-stationary processes.

Models of systems with varying workload intensity functioning and methods for their calculation are considered by G. Sigaev and G. Gorov [1, 2]. However, in practical application they are complex and cumbersome. Therefore, V. Bubnov proposed [3] an engineering method for estimating the quality of functioning of systems with varying workload intensity in the transient mode.

In the study conclusion it is proposed to expand the scope of the model through the use of various approximations of distributions, which make it possible to remove restrictions on the exponentiality of distributions. In subsequent works of the described field, such mathematical structures were called models of non-stationary queuing systems. V. Bubnov studied [4] various non-stationary single-channel and multi-channel models with different types of markovization of distributions between incoming requests and the duration of their services.

A. Eremin proposed [5] to extend the properties of the non-stationary model by taking...
In this study the principle of “complex probabilities” balance is presented, based on the description of a stochastic process not in time, but in the complex domain, which allows developing models of non-stationary queuing systems with arbitrary distributions of the probabilities of the time of receipt of requests and their service, taking into account random or deterministic time delays.

2 Materials and methods

The term “complex probabilities” was first introduced by David Cox [6]. Later J. Riordan noted [7] that “…by applying the Cox method of “complex probabilities”, we can cover the case when the service duration density for \( x > 0 \) is given by any function of the form

\[
p(x) = \sum_i C_i x^i e^{-\lambda_i x}
\]

as long as it is non-negative and has an integral equal to 1”. In addition, “… any laws of service duration distribution allow approximation by the sum of exponentials with polynomial factors”.

This idea was applied by V. Smagin [8] when decomposing probability distributions into a sum of exponential densities with complex conjugate coefficients and parameters in order to solve the study of non-Markovian processes. Later author studied [9] the question of the probabilistic analysis of a complex variable with the introduction of the Dirac complex delta function. Eventually V. Smagin [10] given a rigorous substantiation of the Heaviside and Dirac complex functions is given and a numerical example of the application of the delta function to the study of an alternating random process with the accumulation and loss of information.

The results of the above studies form the basis of the “complex probabilities” balance principle.

In the general case, the balance of probabilities is understood as the equality of the sums of the products of the intensities and the probabilities of the states that are the requests senders (the initial vertices of the arcs of the state graph), and the sums of the products of the intensities and the probabilities of the states that are the requests recipients (the final vertices of the arcs of the state graph). In other words, the state of the process is defined as equilibrium, when the average statistical characteristics of incoming and outgoing random flows are mutually balanced. The obtained equations for the states of the process must be solved in order to find the stationary probabilities of the states. In the time domain, mathematical formalization is based on the operations of multiplying independent intensities with probabilities and adding the resulting products of incompatible events of the process states.

In the case of a “complex probabilities” balance, the stochastic process is described not in time, but in the complex domain. For this, the Laplace transform is used, which makes it possible to represent a stochastic process system of differential or integral equations in the form of a system of algebraic equations.

When using the “complex probabilities” balance, the temporal characteristics (state probabilities) must be replaced by their images in the Laplace transform. The entry of the system into a certain state is preceded by the summation of random time intervals of the trajectory of a random process. In the Laplace transform, their summation is represented by

\[
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3 Results

To study the “complex probabilities” balance, let's depict the graph of states and transitions.

Fig. 1 Graph of states and transitions of a non-stationary queuing system

In contrast to graphs reviewed by V. Bubnov [4], this graph contains transitions—arrows associated not with the intensities of incoming claims \( \lambda \) entering the system, but with the Laplace images of the probability densities of time between claims \( a^*(s) \), for which \( *s \) are notation of the transformation symbol and the complex Laplace variable. The transitions—arrows from left to right at an angle are also associated not with the intensity of service of arrived requests \( \mu \), but with the Laplace images of the probability densities of their service times—\( b^*(s) \). Thanks to this representation, we strive to move from Markov random processes to non-Markov ones. However, for such a transition, we will need not a model of differential equations of the “death and reproduction” type, but another mathematical model—the model of “complex representation of probabilities”.

First, let's consider the simplest graph to prove the validity of the “complex formalization”. To do this, we draw two graphs in Fig. 2.
Fig. 2 Closed service graph

The graph shown in Fig. 2 has two states with possible multiple transitions between them (closed graph).

Fig. 3 Open service graph

The graph shown in Fig. 3 has three states, with only two possible transitions between them (open graph). For these graphs, the state probabilities presented in a complex form take the following forms:

\[ P_s = \frac{-a^* s}{s-\mu} \quad R_s = \frac{-a^* s}{s} \]

\[ P_s = \frac{-a^* s}{s-\mu} \quad R_s = \frac{-a^* s}{s} \]

\[ P_s = \frac{a^* s}{s-\mu} \quad R_s = \frac{a^* s}{s} \]

\[ R_s = \frac{a^* s}{s} \]

\[ \sum_{i,j} P_{i,j} = \frac{1}{s} \quad \sum_{i,j} R_{i,j} = \frac{1}{s} \]

\[ P_s = \frac{s + \mu}{s + \lambda + \mu} \quad R_s = \frac{s + \lambda}{s} \]

\[ P_s = \frac{\lambda}{s + \mu + \lambda} \quad R_s = \frac{\lambda}{s + \mu + \lambda} \]

\[ R_s = \frac{\mu \lambda}{s + \mu + \lambda} \]
To display functions (4) on the graphs, we take $\lambda = 0.3 \text{ h}^{-1}$, $\mu = 0.1 \text{ h}^{-1}$.

Dependence graphs are shown in Fig. 4 and 5. They clearly explain the difference between the two random processes for the graphs shown in Fig. 2 and 3.

For further formalization of the graph model in Fig. 1 will require the introduction of two conditional transition probabilities that determine the choice of further movement of the random process from the states of the graph that have branchings.

Let the probability density $a(t)$ correspond to the distribution function 

$$A(t) = \int a(z) \, dz.$$ 

In addition, let the probability density $b(t)$ correspond to the distribution function 

$$B(t) = \int b(z) \, dz.$$ 

Then the conditional probability of choosing a preference from two events, consisting in the arrival of a new customer in the system or the service of an existing customer in it, will be determined as follows: the probability of a new incoming request is 

$$\alpha = \int B(z) \, dA(z).$$
\[ B(t) = -B(t) \]

\[ \beta = \int \beta(z) d\beta(z) \]

\[ A(t) = -A(t) \]

\[ \alpha = \frac{\lambda}{\mu + \lambda} \quad \beta = \frac{\mu}{\mu + \lambda} \]

Fig. 6 Graph of states and transitions of a non-stationary queuing system

\[ P_{ij}^* s = \frac{-a^* s}{s} \]

\[ P_{ij}^* s = \frac{a^* s (\alpha a^* s + \beta b^* s)}{s} \]

\[ P_{ij}^* s = \frac{a^* s \beta b^* s}{s} \]

\[ P_{ij}^* s = \frac{\alpha a^* s b^* s}{s} \]

\[ \sum_{i,j} P_{ij}^* s = \frac{s}{s} \]

\[ (\alpha a^* s + \beta b^* s) P_{ij}^* s = \alpha a^* s P_{i-1,j}^* s + \beta b^* s P_{i+1,j}^* s \]
In order to use formula (6) when calculating the required indicators according to the graph in Fig. 1, it must be supplemented with formulas for those subgraphs that do not have four links. Such subgraphs are subgraphs with states \( i, 0 \) and subgraphs with states \( 0, j \) for \( i, j = 1, 2, 3, \ldots \). For them, the corresponding link formulas are written as:

\[
\begin{align*}
\alpha a^{i_0} s + \beta b^{j_0} s, \\
\alpha a^{i_1} s + \beta b^{j_1} s, \\
\alpha a^{i_2} s + \beta b^{j_2} s, \\
\alpha a^{i_3} s + \beta b^{j_3} s
\end{align*}
\]

The Laplace image of the state probability for the initial node with index “00” is always:

\[
P_{s,0} = \frac{\alpha a^{0} s}{s}
\]

(7)

The rule for writing expressions (6-8) can be called the “complex probabilities” balance rule for state nodes. It is similar to the probabilistic balance rule for the nodes of the states of the “death and reproduction” Markov scheme.

It should be pointed out that in the original graph (Fig. 6), the rows determine the number of applications received by the system, and the columns determine the number of applications served in it. If we single out the node of the graph with the number \( i, j \), then the probability and its image according to Laplace correspond to such a state of the system when it received \( i \) requests, of which \( j \) requests were serviced. To determine the mathematical expectation of the number of requests received by the system with the number of served requests equal to zero, one should apply the formula:

\[
\sum_{k=0}^{\infty} k P_{s,0} = \sum_{k=0}^{\infty} k P_{k,0}
\]

(1)

Where \( P_{i,0} \) is the stationary probability of exactly \( i \) requests entering the system, provided that none of them has been serviced, and the superscript (1) means that the first initial moment of the number of received requests is determined.

If it is necessary to determine the mathematical expectation of the number of received requests \( i \), provided that \( j \leq i \) of them were served, then you can use the formula:

\[
\sum_{k=0}^{\infty} k P_{s,0} = \sum_{k=0}^{\infty} k P_{k,0}
\]

(1)

Where \( P_{k,j} \) is the stationary probability of \( k \) requests entering the system in the \( i \)-th column of the graph, or otherwise, provided that \( j \) requests from among the received requests were serviced.

Other necessary initial moments can be determined in a similar way.

In the case when the graph is finite and sufficiently simple in complexity, it is possible to apply the direct and inverse Laplace transform and find the representation of probabilities in the time domain. However, this should be considered as an exception to the model, which is mainly focused on obtaining stationary state probabilities.

In the proposed model, delays can be taken into account quite simply in principle. The delay can be both at the beginning of the request in the system, and at its end. Also, the delay can be at the beginning and at the end of the service time. In any of these cases, you can use:

\[
\sum_{k=0}^{\infty} k P_{s,0} = \sum_{k=0}^{\infty} k P_{k,0}
\]

(2)

Where \( P_{s,0} \) is the stationary probability of exactly \( i \) requests entering the system, provided that none of them has been serviced, and the superscript (1) means that the first initial moment of the number of received requests is determined.

If it is necessary to determine the mathematical expectation of the number of received requests \( i \), provided that \( j \leq i \) of them were served, then you can use the formula:

\[
\sum_{k=0}^{\infty} k P_{s,0} = \sum_{k=0}^{\infty} k P_{k,0}
\]

(1)

Where \( P_{k,j} \) is the stationary probability of \( k \) requests entering the system in the \( i \)-th column of the graph, or otherwise, provided that \( j \) requests from among the received requests were serviced.

Other necessary initial moments can be determined in a similar way.
the image of the convolution of the desired probability densities in the Laplace transform. The convolute density then needs to be substituted into the graph model and proceed as described earlier. The degenerate distribution is also easily taken into account here. In addition, if necessary, the reliability of state control in the model can also be taken into account.

In this case, the proposed model fundamentally remains operational, but the process of obtaining the necessary numerical solutions becomes much more complicated. Let for the states of the subgraph shown in Fig. 6, the time intervals for the receipt of requests and their servicing obey the normal distribution law. Let's imagine the following initial data: the distribution of time between incoming claims is normal with a probability density

\[ a(t) = \frac{1}{\sqrt{\pi \sigma}} e^{-\frac{(t-m)^2}{2\sigma^2}} \]

and parameters \( m=20 \text{ h} \), \( \sigma=5 \text{ h} \);

the distribution of the service time of the requirements is also normal with a probability density

\[ b(t) = \frac{1}{\sqrt{\pi \theta}} e^{-\frac{(t-n)^2}{2\theta^2}} \]

and parameters \( n=15 \text{ h} \), \( \theta=3 \text{ h} \).

The corresponding distribution functions are equal:

\[
A(t) = \int f(z) \, dz \\
B(t) = \int g(z) \, dz
\]

To obtain the results, we use one of the approximate methods of hyperdelta approximation. This method has found its practical application for solving various problems, for example, the equation of nonparametric identification of a dynamical system [11]. For an approximate representation by the method of moments, we use four initial moments, including the zero moment. Then in the new notation for the normal distribution law we will have:

\[
a(t) \approx \Phi(t-m) + \Phi(t-m-\sigma) \\
b(t) \approx \Phi(t-n) + \Phi(t-n-\theta)
\]

where \( \Phi(\cdot) \) is the Heaviside function equal to

\[
\Phi(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}
\]

The distribution functions for (9) will be equal to:

\[
A(t) = \int \Phi(z) \, dz \\
B(t) = \int \Phi(z) \, dz
\]

\[
\int_{-\infty}^{\infty} \Phi(t) \, dt = 1
\]
The conditional transition probabilities are found by the formulas:

\[
\alpha = \int_0^\infty - A(z) g(z) \, dz = \ldots
\]

\[
\beta = \int_0^\infty - B(z) \, dz = \ldots
\]

\[
a^* (s) = \frac{1}{2} \left( e^{-\alpha s} + e^{-\beta s} \right)
\]

\[
b^* (s) = \frac{1}{2} \left( e^{-\alpha s} + e^{-\beta s} \right)
\]

The Laplace images of the approximated probability densities (10) will be equal to:

\[
5 \quad 15 \quad 3 \quad 3 \quad 12 \quad 12
\]

Further, applying the inverse Laplace transform, we find all the necessary probabilities of four states (5), the graphs of which are shown in Fig. 8.

Let it be required to find the probabilities of states of a two-phase non-Markov process, the transition graph of which is shown in Fig. 9.

Let us compose a system of balance equations for it:

\[
a \quad 1 
\]

\[
b^* (s) + \alpha a^* (s) = \beta b^* (s)
\]

\[
b^* (s) = \alpha a^* (s)
\]

\[
\sum_{i=0}^{\infty} P_i^* (s) = \ldots
\]
We present the initial data as in the previous case. We denote the distribution functions corresponding to them by \( A(t) \) and \( B(t) \). Using them, we determine the conditional probabilities of transitions when branching a random process from state “2” of the graph in Fig. 9:

\[
P_0(t) = \frac{\beta b^* s}{s \left( \beta b^* s + \alpha^* a^* s + \alpha \right)}
\]

\[
P_1(t) = \frac{a^* s b^* s}{s \left( \beta b^* s + \alpha^* a^* s + \alpha \right)}
\]

\[
P_2(t) = \frac{\alpha^* a^* s}{s \left( \beta b^* s + \alpha^* a^* s + \alpha \right)}
\]

\[
\alpha = \int_0^\infty -B(z) dA(z) = \int_0^\infty -A(z) dB(z)
\]

\[
a(t) \approx -\delta t - \delta t - \delta t - \delta t
\]

\[
b(t) \approx -\delta t - \delta t - \delta t - \delta t
\]

\[
a^* s = e^{-\delta t} + e^{-\delta t}
\]

\[
b^* s = e^{-\delta t} + e^{-\delta t}
\]

Applying formula (10), we obtain approximation formulas for probability densities:

\[
f(t) \approx sf^* s s t = \int f(t)
\]

From the obtained analytical expressions for these probabilities, one can find their stationary values. Under the conditions of the example, they are equal:

\[
P_0(\infty) = 0.402;
\]

\[
P_1(\infty) = 0.5;
\]

\[
P_2(\infty) = 0.098.
\]
4 Discussion

Thus, the proposed “complex probabilities” balance principle is based on the description of the stochastic process not in time, but in the complex domain. The principle is based on the Laplace transform, which makes it possible to represent a system of differential or integral equations of a stochastic process in the form of a system of algebraic equations. At the same time, to compile the state balance equations, it is necessary to use the corresponding products of state probability images with their subsequent summation.

5 Conclusion

The mathematical formalization of models using the “complex probabilities” balance principle is based not on the application of the classical model of “death and reproduction” in the time domain, but on the formal representation of the probabilities of the systems states in the Laplace transform, i.e. in a complex form.

“complex state probabilities” on separate subgraphs of the system states, the fulfillment of the condition for normalizing the total sum of these indicators.

The principle of balance of “complex probabilities” found practical application in modeling and evaluating the effectiveness of a robot control system in changing environmental conditions, as well as creating various computer models of non-Markov processes [14-16].

References

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