The «complex probabilities» balance principle for non-Markov processes modeling

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Abstract. This paper presents the principle of “complex probabilities” balance, based on the description of the stochastic process not in the time, but in the complex domain, which allows developing models of non-stationary queuing systems with arbitrary probabilities distributions of the requests and their servicing time, taking into account random or deterministic time delays. A system of balance equations in the complex domain was compiled and solved using Laplace images. Performed an inverse Laplace transform to move from images to probabilities in the time domain. Presented models for acyclic and cyclic stochastic processes with arbitrary distributions of time points between incoming requests and their service. Given calculation examples. Recommendations for the further application of the principle of balance of "complex probabilities" are given.

1 Introduction

A significant part of the researches using analytical modeling of various purposes systems is devoted to the study of the processes of functioning in a stationary mode. Here, models and methods of stationary queuing theory have found wide application.

At the same time, it is of interest to study the functioning of queuing systems with varying workload intensity in transient and non-stationary operating modes. The solution of such problems makes it possible to evaluate the behavior of systems in the case of peak dynamic loads and transient non-stationary processes.

Models of systems with varying workload intensity functioning and methods for their calculation are considered by G. Sigalov and G. Gorov [1, 2]. However, in practical application they are complex and cumbersome. Therefore, V. Bubnov proposed [3] an engineering method for estimating the quality of functioning of systems with varying workload intensity in the transient mode. In the study conclusion it is proposed to expand the scope of the model through the use of various approximations of distributions, which make it possible to remove restrictions on the exponentiality of distributions.

In subsequent works of the described field, such mathematical structures were called models of non-stationary queuing systems. V. Bubnov studied [4] various non-stationary single-channel and multi-channel models with different types of markovization of distributions between incoming requests and the duration of their services.

A. Eremin proposed [5] to extend the properties of the non-stationary model by taking

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into account the service delay time in it, a graph corresponding to the process was constructed, and a system of differential equations was written and solved. At the same time, the proposed model is Markovian, which uses exponential distribution laws and constant time delay.

In this study the principle of "complex probabilities" balance is presented, based on the description of a stochastic process not in time, but in the complex domain, which allows developing models of non-stationary queuing systems with arbitrary distributions of the probabilities of the time of receipt of requests and their service, taking into account random or deterministic time delays.

2 Materials and methods

The term "complex probabilities" was first introduced by David Cox [6]. Later J. Riordan noted [7] that "…by applying the Cox method of "complex probabilities", we can cover the case when the service duration density for $x>0$ is given by any function of the form

$$p(x) = \sum_i C_i x^{r_i} e^{-\lambda_i x},$$

as long as it is non-negative and has an integral equal to 1". In addition, "... any laws of service duration distribution allow approximation by the sum of exponentials with polynomial factors".

This idea was applied by V. Smagin [8] when decomposing probability distributions into a sum of exponential densities with complex conjugate coefficients and parameters in order to solve the study of non-Markovian processes. Later author studied [9] the question of the probabilistic analysis of a complex variable with the introduction of the Dirac complex delta function. Eventually V. Smagin [10] given a rigorous substantiation of the Heaviside and Dirac complex functions is given and a numerical example of the application of the delta function to the study of an alternating random process with the accumulation and loss of information.

The results of the above studies form the basis of the "complex probabilities" balance principle.

In the general case, the balance of probabilities is understood as the equality of the sums of the products of the intensities and the probabilities of the states that are the requests senders (the initial vertices of the arcs of the state graph), and the sums of the products of the intensities and the probabilities of the states that are the requests recipients (the final vertices of the arcs of the state graph). In other words, the state of the process is defined as equilibrium, when the average statistical characteristics of incoming and outgoing random flows are mutually balanced. The obtained equations for the states of the process must be solved in order to find the stationary probabilities of the states. In the time domain, mathematical formalization is based on the operations of multiplying independent intensities with probabilities and adding the resulting products of incompatible events of the process states.

In the case of a "complex probabilities" balance, the stochastic process is described not in time, but in the complex domain. For this, the Laplace transform is used, which makes it possible to represent a stochastic process system of differential or integral equations in the form of a system of algebraic equations.

When using the "complex probabilities" balance, the temporal characteristics (state probabilities) must be replaced by their images in the Laplace transform. The entry of the system into a certain state is preceded by the summation of random time intervals of the trajectory of a random process. In the Laplace transform, their summation is represented by

the corresponding product of images of probability densities. As a result, to compile the state balance equations, it is necessary to use the corresponding products of images with their subsequent summation.

3 Results

To study the "complex probabilities" balance, let's depict the graph of states and transitions of a single-channel queuing system in Fig. 1.

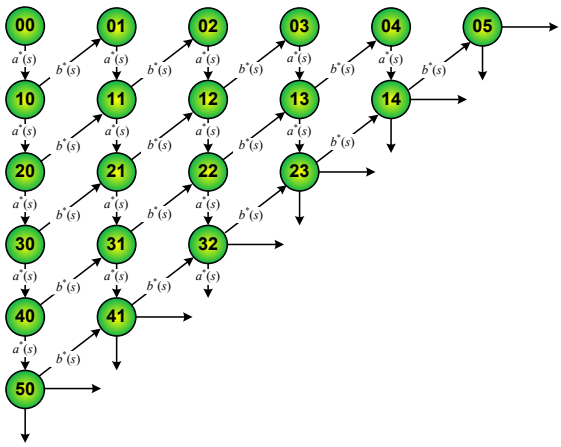

Fig. 1 Graph of states and transitions of a non-stationary queuing system

In contrast to graphs reviewed by V. Bubnov [4], this graph contains transitions - down arrows, associated not with the intensities of incoming claims λ entering the system, but with the Laplace images of the probability densities of time between claims *a*(s)*, for which **s* are notation of the transformation symbol and the complex Laplace variable. The transitions - arrows from left to right at an angle are also associated not with the intensities of service of arrived requests μ, but with the Laplace images of the probability densities of their service times - *b*(s)*. Thanks to this representation, we strive to move from Markov random processes to non-Markov ones. However, for such a transition, we will need not a model of differential equations of the "death and reproduction" type, but another mathematical model - the model of "complex representation of probabilities".

To simplify the formalization of the queuing system, here we consider only the model of the system with the same transition parameters *a*(s)*, *b*(s)* connecting all states on the graph.

First, let's consider the simplest graph to prove the validity of the "complex formalization". To do this, we draw two graphs in Fig. 2 and 3.

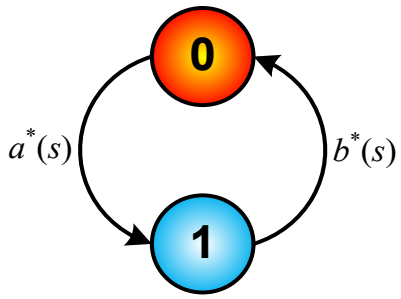

Fig. 2 Closed service graph

The graph shown in Fig. 2 has two states with possible multiple transitions between them (closed graph).

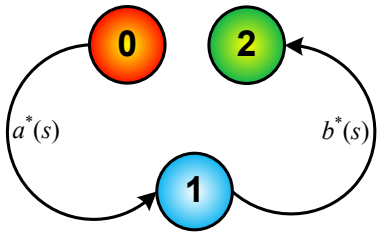

Fig. 3 Open service graph

The graph shown in Fig. 3 has three states, with only two possible transitions between them (open graph). For these graphs, the state probabilities presented in a complex form take the following forms:

$$P_0^*(s) = \frac{1-a^*(s)}{s(1-a^*(s)b^*(s))}, \qquad R_0^*(s) = \frac{1-a^*(s)}{s},$$
$$P_1^*(s) = \frac{a^*(s)}{s(1-a^*(s)b^*(s))}. \qquad R_1^*(s) = \frac{a^*(s)(1-b^*(s))}{s}, \tag{1}$$
$$R_2^*(s) = \frac{a^*(s)b^*(s)}{s}.$$

It is easy to see that

$$\sum_{i,j} P_{i,j}^*(s) = \frac{1}{s}, \quad \sum_{i,j} R_{i,j}^*(s) = \frac{1}{s}. \tag{2}$$

Formulas (2) confirm the correctness of determining the Laplace images for the probabilities of the considered events.

For clarity of representation (1), we write the formulas for exponential distributions. We will have:

$$P_0^*(s) = \frac{s+\mu}{s(s+\lambda+\mu)}, \qquad R_0^*(s) = \frac{1}{s+\lambda},$$
$$P_1^*(s) = \frac{\lambda}{s(s+\lambda+\mu)}. \qquad R_1^*(s) = \frac{\lambda}{(s+\mu)(s+\lambda)}, \tag{3}$$
$$R_2^*(s) = \frac{\mu\lambda}{s(s+\mu)(s+\lambda)}.$$

Next, from expressions (3), we proceed to their representation, taking into account the dependence on time:

$$P_0(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t}, \qquad R_0(t) = e^{-\lambda t},$$

$$P_1(t) = \frac{\lambda}{\mu+\lambda}[1 - e^{-(\mu+\lambda)t}]. \qquad R_1(t) = \frac{\lambda(e^{-\lambda t} - e^{-\mu t})}{\mu - \lambda}, \tag{4}$$

$$R_2(t) = 1 - \frac{\mu e^{-\lambda t} - \lambda e^{-\mu t}}{\mu - \lambda}.$$

To display functions (4) on the graphs, we take $\lambda = 0.3\ h^{-1}$, $\mu = 0.1\ h^{-1}$.

Dependence graphs are shown in Fig. 4 and 5. They clearly explain the difference between the two random processes for the graphs shown in Fig. 2 and 3.

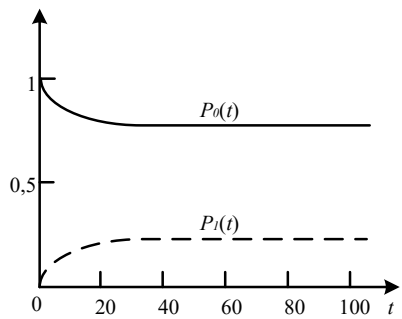

Fig. 4 Dependence of state probabilities on time for a closed graph

For further formalization of the graph model in Fig. 1 will require the introduction of two conditional transition probabilities that determine the choice of further movement of the random process from the states of the graph that have branchings.

Let the probability density *a(t)* correspond to the distribution function

$$A(t) = \int_0^t a(z)dz\ .$$

In addition, let the probability density *b(t)* correspond to the distribution function

$$B(t) = \int_0^t b(z)dz\ .$$

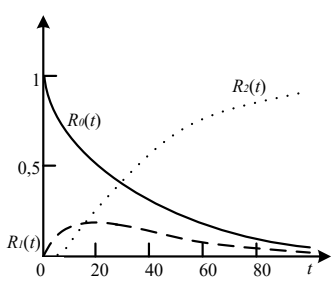

Fig. 5 Dependence of state probabilities on time for an open graph

Then the conditional probability of choosing a preference from two events, consisting in the arrival of a new customer in the system or the service of an existing customer in it, will be determined as follows: the probability of a new incoming request is

$$\alpha = \int_0^\infty \overline{B}(z)dA(z)\ ,$$

where $\overline{B}(t) = 1 - B(t)$ is the probability of a pending request processing;

$$\beta = \int_0^\infty \overline{A}(z) dB(z),$$

where $\overline{A}(t) = 1 - A(t)$. For exponential distributions we have

$$\alpha = \frac{\lambda}{\mu + \lambda}, \quad \beta = \frac{\mu}{\mu + \lambda}.$$

In order to show the need to apply the introduced conditional probabilities, consider a more complex subgraph (see Fig. 6), borrowed from the graph in Fig. 1.

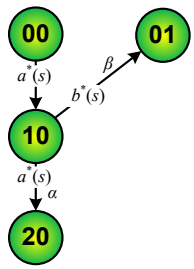

Fig. 6 Graph of states and transitions of a non-stationary queuing system

According to the graph in Fig. 6, we compose the following Laplace images of state probabilities:

$$\begin{aligned}
P_{00}^*(s) &= \frac{1 - a^*(s)}{s}, \\
P_{10}^*(s) &= \frac{a^*(s)\left(\alpha(1 - a^*(s)) + \beta(1 - b^*(s))\right)}{s}, \\
P_{01}^*(s) &= \frac{a^*(s)\beta b^*(s)}{s}, \\
P_{20}^*(s) &= \frac{\alpha(a^*(s))^2}{s}.
\end{aligned} \tag{5}$$

Summing up (5), we get:

$$\sum_{i,j} P_{i,j}^* = \frac{1}{s},$$

which confirms the correctness of the executed formulas.

Next, consider a node with four connections to other nodes and give a general formula that connects all the necessary Laplace images for adjacent probabilities. Fig. 7 shows a similar subgraph.

In accordance with the image of links on this graph, the general formula can be written in the following form:

$$\left(\alpha a^*(s) + \beta b^*(s)\right) P_{i,j}^*(s) = \alpha a^*(s) P_{i-1,j}^*(s) + \beta b^*(s) P_{i+1,j-1}^*(s). \tag{6}$$

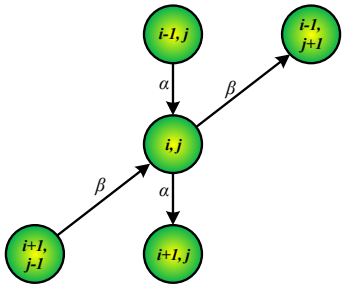

Fig. 7 Fragment of a graph with four symbols of transition probabilities

In order to use formula (6) when calculating the required indicators according to the graph in Fig. 1, it must be supplemented with formulas for those subgraphs that do not have four links. Such subgraphs are subgraphs with states *i*, *0* and subgraphs with states *0*, *j* for *i, j* = 1,2,3... For them, the corresponding link formulas are written as:

$$\left(\alpha a^*(s)+\beta b^*(s)\right)P^*_{i,0}(s)=\alpha a^*(s)P^*_{i-1,0}(s),\quad a^*(s)P^*_{0,j}(s)+\beta b^*(s)P^*_{1,j-1}(s)\,. \tag{7}$$

The Laplace image of the state probability for the initial node with index “00” is always:

$$P^*_{0,0}(s)=\frac{1-a^*(s)}{s}\,. \tag{8}$$

The rule for writing expressions (6-8) can be called the “complex probabilities” balance rule for state nodes. It is similar to the probabilistic balance rule for the nodes of the states of the "death and reproduction" Markov scheme.

It should be pointed out that in the original graph (Fig. 6), the rows determine the number of applications received by the system, and the columns determine the number of applications served in it. If we single out the node of the graph with the number *i, j*, then the probability and its image according to Laplace correspond to such a state of the system when it received *i* requests, of which *j* requests were serviced. To determine the mathematical expectation of the number of requests received by the system with the number of serviced requests equal to zero, one should apply the formula:

$$\nu^{(1)}_{0,0}=\sum_{k=0}^{\infty}kP_{k,0}\,,$$

where $P_{i,0}$ is the stationary probability of exactly *i* requests entering the system, provided that none of them has been serviced, and the superscript (1) means that the first initial moment of the number of received requests is determined.

If it *is* necessary to determine the mathematical expectation of the number of received requests *i*, provided that $j \le i$ of them were served, then you can use the formula:

$$\nu^{(1)}_{i,j}=\sum_{k=0}^{i}k\,P_{k,j}\,,$$

where $P_{k,j}$ is the stationary probability of *k* requests entering the system in the *i*-th column of the graph, or otherwise, provided that *j* requests from among the received requests were serviced.

Other necessary initial moments can be determined in a similar way.

In the case when the graph is finite and sufficiently simple in complexity, it is possible to apply the direct and inverse Laplace transform and find the representation of probabilities in the time domain. However, this should be considered as an exception to the model, which is mainly focused on obtaining stationary state probabilities.

In the proposed model, delays can be taken into account quite simply in principle. The delay can be both at the beginning of the request in the system, and at its end. Also, the delay can be at the beginning and at the end of the service time. In any of these cases, you can use

the image of the convolution of the desired probability densities in the Laplace transform. The convolute density then needs to be substituted into the graph model and proceed as described earlier. The degenerate distribution is also easily taken into account here. In addition, if necessary, the reliability of state control in the model can also be taken into account.

In this case, the proposed model fundamentally remains operational, but the process of obtaining the necessary numerical solutions becomes much more complicated.

Let for the states of the subgraph shown in Fig. 6, the time intervals for the receipt of requests and their servicing obey the normal distribution law.

Let's imagine the following initial data:

the distribution of time between incoming claims is normal with a probability density

$$a(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-m)^2}{2\sigma^2}}$$

and parameters $m=20\ h$, $\sigma=5\ h$;

the distribution of the service time of the requirements is also normal with a probability density

$$b(t) = \frac{1}{\sqrt{2\pi}\theta} e^{-\frac{(t-n)^2}{2\theta^2}}$$

and parameters $n=15\ h$, $\theta=3\ h$.

The corresponding distribution functions are equal:

$$A(t) = \int_0^t f(z)dz;$$

$$B(t) = \int_0^t g(z)dz.$$

To obtain the results, we use one of the approximate methods of hyperdelta approximation. This method has found its practical application for solving various problems, for example, the equation of nonparametric identification of a dynamical system [11].

For an approximate representation by the method of moments, we use four initial moments, including the zero moment. Then in the new notation for the normal distribution law we will have:

$$a(t) = \frac{1}{2}(\delta(t-15) + \delta(t-25)),$$
$$b(t) = \frac{1}{2}(\delta(t-12) + \delta(t-18)), \tag{9}$$

where $\delta(\bullet)$ – Dirac delta-function.

To obtain (9), the formula for the normal probability density is used [12]:

$$f(t) \approx \frac{1}{2}(\delta(t-m+\sigma) + \delta)(t-m-\sigma)) \tag{10}$$

The distribution functions for (9) will be equal to:

$$A(t) = \frac{1}{2}\big(\Phi(t-15) + \Phi(t-25)\big),$$

$$B(t) = \frac{1}{2}\big(\Phi((t-12) + \Phi(t-18)\big),$$

where $\Phi(\bullet)$ is the Heaviside function equal to $\Phi(x) = \int_{-\infty}^{x} \delta(t)dt$.

The conditional transition probabilities are found by the formulas:

$$\alpha = \int_0^\infty (1 - A(z))g(z)dz = 0,804;$$

$$\beta = \int_0^\infty (1 - B(z))dz = 0,196.$$

The Laplace images of the approximated probability densities (10) will be equal to:

$$a^*(s) = \frac{1}{2}\left(e^{-15s} + (e^{-15s})^{\frac{5}{3}} \right);$$

$$b^*(s) = \frac{1}{2}\left(e^{-12s} + (e^{-12s})^{\frac{3}{2}} \right).$$

Further, applying the inverse Laplace transform, we find all the necessary probabilities of four states (5), the graphs of which are shown in Fig. 8.

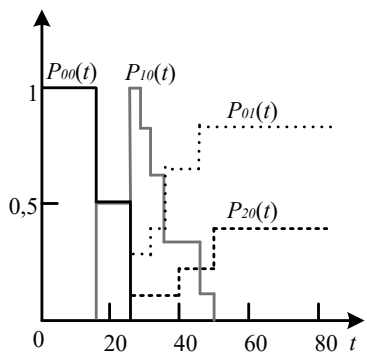

Fig. 8 State probability plots

Let it be required to find the probabilities of states of a two-phase non-Markov process, the transition graph of which is shown in Fig. 9.

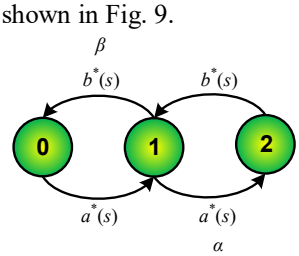

Fig. 9 Graph of states and transitions of a non-stationary queuing system

Let us compose a system of balance equations for it:

$$\begin{aligned} &a^*(s)P_0^*(s) = \beta b^*(s)P_1^*(s); \\ &(\alpha a^*(s) + \beta b^*(s))P_1^*(s) = a^*(s)P_0^*(s) + b^*(s)P_2^*(s); \\ &b^*(s)P_2^*(s) = \alpha a^*(s)P_1^*(s). \end{aligned} \tag{11}$$

Taking into account the condition of normalization of probability images

$$\sum_{i=0}^{2} P_i^*(s) = \frac{1}{s}$$

and solving the system of equations (11), we obtain:

$$P_0^*(s)=\frac{\beta(b^*(s))^2}{s\left(\beta(b^*(s))^2+a^*(s)b^*(s)+\alpha(a^*(s))^2\right)};$$

$$P_1^*(s)=\frac{a^*(s)b^*(s)}{s\left(\beta(b^*(s))^2+a^*(s)b^*(s)+\alpha(a^*(s))^2\right)}; \qquad (12)$$

$$P_2^*(s)=\frac{\alpha(a^*(s))^2}{s\left(\beta(b^*(s))^2+a^*(s)b^*(s)+\alpha(a^*(s))^2\right)}.$$

We present the initial data as in the previous case.

We denote the distribution functions corresponding to them by *A(t)* and *B(t)*. Using them, we determine the conditional probabilities of transitions when branching a random process from state "2" of the graph in Fig. 9:

$$\alpha=\int_0^\infty(1-B(z))dA(z)=0{,}196;$$

$$\beta=\int_0^\infty(1-A(z))dB(z)=0{,}804.$$

Applying formula (10), we obtain approximation formulas for probability densities:

$$a(t)\approx\frac{1}{2}(\delta(t-15)+\delta(t-25));$$

$$b(t)\approx\frac{1}{2}(\delta(t-12)+\delta(t-18)).$$

In Laplace's images, they take the form:

$$a^*(s)=\frac{1}{2}\left(e^{-15s}+(e^{-15s})^{\frac{5}{3}}\right);$$

$$b^*(s)=\frac{1}{2}\left((e^{-12s}+(e^{-12s})^{\frac{3}{2}}\right). \qquad (13)$$

Substituting formulas (13) into equations (12), we obtain them in expanded form for the given numerical data of the example.

To obtain expressions for the probabilities of states in the time domain, it is necessary to apply the inverse Laplace transform. However, it is quite difficult to achieve the expected results due to the complexity of direct Laplace images of the desired probabilities and the computational complexity of working with normal distributions. Therefore, we will make the transition from images to originals $P_0(t)$, $P_1(t)$, $P_2(t)$ using the approximate method of inverting the Laplace transform using the Alfrey formula. This formula follows from Wider's formula based on the filtering property of the Laplace transform using the delta function. It has the following form [13]:

$$f(t)\approx sf^*(s),\quad \text{for } s=\frac{1}{t}. \qquad (14)$$

As a result of applying formula (1.16), it is possible to obtain expressions for the probabilities of the process states, the numerical results of which are presented in Figs. 9.

From the obtained analytical expressions for these probabilities, one can find their stationary values. Under the conditions of the example, they are equal: $P_0(\infty)$=0.402; $P_1(\infty)$=0.5; $P_2(\infty)$=0.098.

The initial probabilities are: $P_0(0.05)$=1; $P_1(0.5)$=0; $P_2(0.05)$=0.

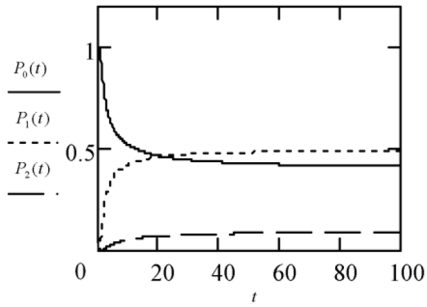

Fig. 9 State probability plots

4 Discussion

Thus, the proposed "complex probabilities" balance principle is based on the description of the stochastic process not in time, but in the complex domain. The principle is based on the Laplace transform, which makes it possible to represent a system of differential or integral equations of a stochastic process in the form of a system of algebraic equations. At the same time, to compile the state balance equations, it is necessary to use the corresponding products of state probability images with their subsequent summation.

5 Conclusion

The mathematical formalization of models using the “complex probabilities” balance principle is based not on the application of the classical model of “death and reproduction” in the time domain, but on the formal representation of the probabilities of the systems states in the Laplace transform, i.e. in a complex form.

The correctness of the models is confirmed by the correctness of the recording of the “complex state probabilities” on separate subgraphs of the system states, the fulfillment of the condition for normalizing the total sum of these indicators.

The principle of balance of "complex probabilities" found practical application in modeling and evaluating the effectiveness of a robot control system in changing environmental conditions, as well as creating various computer models of non-Markov processes [14-16].

References

1. G.G. Sigalov, G.V. Nikolaev, A.M. Lyupersolsky, Automation and Computer Science. **4**, 35-42 (1985)
2. G.V. Gorov, Ya.A. Kogan, N.V. Paradisov, Automation and telemechanics, **6**, 44-51 (1985)
3. V.P. Bubnov, V.I. Safonov, V.A. Smagin, Automation and Computer Engineering, **6**, 19-22 (1987)
4. V.P. Bubnov, V.I. Safonov, ed. "Lan", 64 (1999)
5. A.S. Eremin, Intelligent technologies in transport, **4** (4), 23-26 (2015)
6. D.R. Cox, Proc. Cambr. Phil. Soc, **51**, 313-319 (1955)
7. J. Riordan, Wiley and Sons Inc., 139 (1962)

8. V.A. Smagin, News of the Academy of Sciences of the USSR. Technical Cybernetics, **6**, 31-36 (1983)
9. V.A. Smagin, Automation and Computer Science, **5**, 3-13 (1999)
10. V.A. Smagin, Automatic Control and Computer Sciences, **1**(48), 10-16 (2014)
11. Ya.N. Gusenitsa, Journal of Machinery Manufacture and Reliability, **1**(51), 80-85 (2022)
12. V.A. Smagin, G.V. Filimonikhin, Automation and Computer Engineering, **5**, 25-31 (1990)
13. V.A. Smagin, Ministry of Defense of the USSR, 269 (1982)
14. Ya.N. Gusenitsa, O.A. Shiryamov, V.L. Rzhavitin, A.V. Khodak, A.M. Smirnov, IOP Conference Series: Earth and Environmental Science. **872** (2021)
15. Ya.N. Gusenitsa, O.A. Shiryamov, V.L. Rzhavitin, D.S. Buryj, D.I. Ljvov, AECS 2022, LNEE 971, **971**, 38-48 (2023)
16. Ya.N. Gusenitsa, O.A. Shiryamov, Scientific and Technical Journal of Information Technologies, Mechanics and Optics, **1**(23), 150-160 (2023)