Towards the substantiation of methods for optimizing the transportation work of locomotives under railway operating conditions

O. Ablyalimov[1*]

[1]Tashkent state transport university (Tashkent, Uzbekistan)

Abstract. An analysis of existing methods for optimizing the transportation work of locomotives in operation is presented, based on a refined formulation and criteria for the problem of optimal control of train movement by locomotives of different types of traction. The advantage of the dynamic step-by-step programming method proposed by the author, based on the principle of parametric optimization, is substantiated, which makes it possible to calculate optimal train driving modes in areas of locomotive circulation and locomotive crew work with savings in natural diesel fuel consumption of up to eight to ten percent per trip. The analytical dependence of the stepwise optimization of the process, which determines the sequence of solving the optimization problem, is recommended to be used in the development of optimized train driving modes maps on sections of the Uzbek railways of varying difficulty. Key words: Dynamic programming, freight train, diesel locomotive, calculus of variations, maximum principle, optimal control, optimization task, train driving mode.

1 Introduction

Currently, optimization, as a matter of course, seems to be a powerful tool in the future development of any complex engineering systems associated with the operation of all types of transport, including, especially, railway.

The constantly growing demand for high precision in manufacturing and increasing the operational reliability of structures, taking into account the search for promising optimal technological solutions, creates additional difficulties that can only be realized through the development of special optimization models.

For railway transport, one of these models are models of optimal control of train movements, which take into account all artificial and natural restrictions imposed on the rolling stock by operating conditions.

At the Department of Locomotives and Locomotive Facilities of TSTU (Tashkent State Transport University), research is very actively being carried out related to solving problems of choosing the optimal mode for controlling train traffic on sections of railways of varying degrees of difficulty (complexity), including of Uzbek ones.

* Corresponding author: o.ablyalimov@gmail.com

2 Objects and methods of research

In this regard, for many years, issues related to solving problems of optimal control of train traffic with various types of locomotive traction have been an active topic of research by many foreign scientists [1-32] and concern a wide range of cargo and passenger transportation, not only mainline [1-18, 30-32], suburban [19] and high-speed [20, 21] railway transport, but also using the metro [22-25] and involving public urban rail transport [26-29].

As a result of the analysis of a considerable part of these studies, carried out by the author of the work [33], a classification of methods for calculating the energy-optimal trajectory of a train was justified and proposed, which is shown in fig. 1. And, in essence, he comes to the conclusion that all the optimization methods he analyzes can be grouped into two main categories - these are analytical and numerical methods for solving optimization problems. Basically, research [33] is devoted to energy-optimal calculation methods for locomotives of electric traction and they do not at all consider specific optimization issues for locomotives of diesel traction, the influence on optimal control of the characteristics of the track profile of railway sections and of the organization of freight and passenger types of movement in real operating conditions.

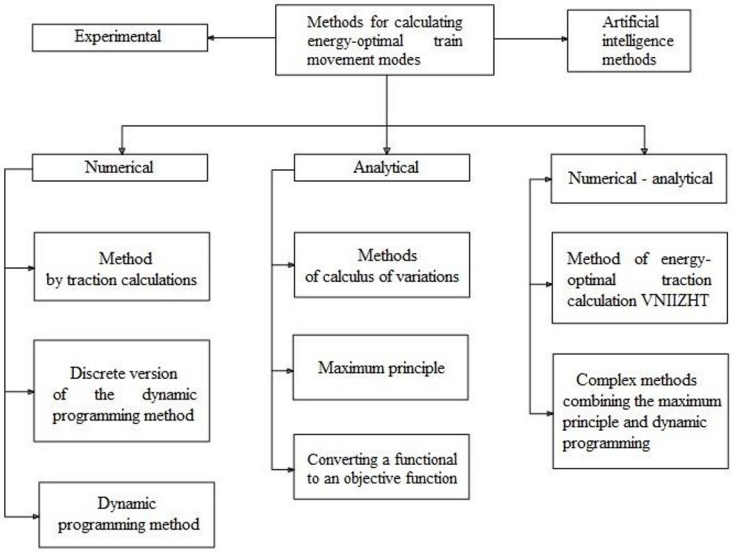

Fig. 1. Classification of energy optimal calculation methods Source: «Compiled by the author».

The purpose of the study is to analyze existing optimization methods for selecting optimal train movement control modes on railway sections with different types of track profiles in real and virtual operating conditions of railways, including Uzbek ones.

The object of study is the tasks of optimizing the transportation work of locomotives and mathematical methods of the theory of optimal control of motion processes in space.

The subject of the study is the criteria for setting and methods for solving the problem of optimal control of train movement using the example of locomotives of diesel tractions.

The starting point of the modern theory of optimal control is a system of equations of the state of an object that describe the behavior of a dynamic system in space.

In a continuous process, the equation of state of an object can be a system of ordinary differential equations of the first order, namely:

$$\begin{aligned} \dot{x}^1 &= f^1(x^1, x^2, \dots, x^n, u^1, u^2, \dots, u^n) \\ \dot{x}^2 &= f^2(x^1, x^2, \dots, x^n, u^1, u^2, \dots, u^n) \\ &= = = = = = = = = = = = = = = = = = = \} \\ \dot{x}^n &= f^n(x^1, x^2, \dots, x^n, u^1, u^2, \dots, u^n) \end{aligned} \quad (1)$$

or in vector form

$$\dot{x} = f(x, u) \quad (2)$$

where x – object state vector including phase coordinates $x^1, x^2, \dots, x^n$, that is x = $(x^1, x^2, \dots, x^n)$;

u – control vector including control parameters $u^1, u^2, \dots, u^n$, that is u = $u^1, u^2, \dots, u^n$;

f – vector is a function whose coordinates are the right-hand sides of the system of equations (1).

The mathematical formulation of the optimal control problem is as follows. The equations of state of the object are ask; boundary conditions imposed on object state variables; restrictions placed on state and control variables; quality indicator (optimality criterion) of the control object. It is necessary to determine such admissible control under which the optimality criterion (quality indicator) will be minimal or maximum.

Let us specifically describe the formulation of the optimization problem for the technological process of the transportation work of a locomotive, based on Chapter II of the monograph [38], and introduce a conditional concept about the signs of the optimization problem.

1th sign. The process of a moving train is described by a system of differential equations that characterize the feature of the mathematical description (MD), that is, the patterns of changes in state coordinates:

$$\frac{dx}{dS} = f(x, u) \quad (3)$$

where x = V – the state vector of a moving train, including the phase coordinate of the speed V km/h; the remaining integral coordinates - time t min, energy consumption E kg and so on, which are determined uniquely;

u - control vector containing the i th positions of the driver controller:

$$\begin{aligned} u &= 1,2,\dots,n_m \text{ – working stroke} \\ u &= 0 \text{ – idling} \quad \} \\ u &= -1 \text{ - braking} \end{aligned} \quad (4)$$

2th sign. Boundary conditions of the state of the process of a moving train (set by the phase coordinates of the path *S* km and speed *V* km/h) and restrictions imposed on the state and control variables characterizing the sign of boundary conditions (*BC*) and restrictions (*R*), that is

$$\begin{aligned} &(S_{\text{н}}, V_{\text{н}}) \text{ – initial state} \\ &\qquad\qquad\qquad\qquad \} \\ &(S_{\text{к}} V_{\text{к}},) \text{ – final state} \end{aligned} \quad (5)$$

$$V \le V^{\text{ог}}$$

$$F_к \leq F_{ксц} \} \text{ – coordinate limitation} \quad (6)$$
$$0 \leq N \leq N_{max}$$

where F - locomotive traction force;

N - locomotive power.

3th sign. Optimality criterion (OC), characterizing the effect of the process (can be in the type min of reduced costs Э rub., max profit Pr rub. or min energy consumption E kg):

$$Э\,(x.\,u) \rightarrow min \quad (7)$$

$$E\,(x.\,u) \rightarrow min \quad (8)$$

The problem in the general case is formulated as follows: for conditions characterized by equations (3) - (6), find such an optimal control u(S) for which the optimality criterion (7), (8) is the smallest (minimal).

4th sign. An additional sign of optimization is a sign that characterizes the goal of the process.

Comparison of problem statements based on the specified optimization criteria allows us to highlight the general properties of the characteristics and their features (table 1), which were not sufficiently taken into account by a number of researchers [4, 6, 16, 30-32 and others] and were noted in work [39].

When solving problems of optimal control of train movement, the basic methods of the mathematical theory of optimal processes are used, which include dynamic programming (DP) [34, 35], the maximum principle (MP) [36, 37] and methods of the calculus of variations (CA) [40, 41].

Table 1. Statement of the problem of optimal control of train movement

Signs of setting an optimization task	General properties and characteristics
1	2
First	The required functions are that the trajectory of the process is continuous throughout the entire section. The independent variable is the path (*S*) or time (*t*).
Second	In principle, it corresponds to the formulation of the problem in the general case.
Third	Different in view: a) *E→min* energy consumption b) *E→min* transportation costs.
Fourth	The overall goal of the process is set.

Signs of setting an optimization task	Peculiarities
1	3
First	Function of the trajectory has the form piecewise smooth, and its derivatives are discontinuous, that is, the trajectory of the process does not have continuous differentiability.
Second	It is necessary to divide the controls u into stages.

Third	The value E is approximate and reflects only transportation costs.
Fourth	It is necessary to identify local goals at individual stages of the process.

Source: «Compiled by the author».

Methods of the calculus of variations are applicable to problems in which there are no restrictions on state and control variables, optimality criteria are presented in the form of functionals, and unknown functions serve as solutions. The method of calculus of variations allows in such cases to reduce the solution of the optimization problem to the integration of Euler differential equations, each of which is a second-order nonlinear differential equation with boundary conditions specified at both ends of the integration. This leads to a two-point boundary value problem, the analytical solution of which, with the exception of some very simple cases, is associated with great difficulties. Obtaining a numerical solution to a two-point boundary value problem is also quite difficult.

Dynamic programming is the most universal method for solving multi-stage optimization problems, for which the general optimality criterion is described by an additive function of the optimality criteria of individual stages. In principle, this method is an algorithm for selecting optimal control at all stages of the process.

The peculiarity of the method is that in order to find the optimal control (train movement mode), the section is divided into a number of successive segments, called variation steps. Optimal control at each step is selected based on R. Bellman's principle of optimality [34], which is formulated as follows: the optimal strategy has the property that, whatever the initial state of the multi-stage process and the initial decision made, subsequent decisions must constitute an optimal strategy with respect to condition resulting from the original decision.

The main disadvantage of dynamic programming is the huge volume of the computational procedure and the excessive requirements for the machine's memory to store promising calculation options.

The maximum principle is used to solve problems of optimization of processes described by systems of differential equations, and restrictions can be imposed on the range of changes in variables. Finding the optimal solution comes down to the problem of integrating the system of differential equations of the process and the adjoint system for auxiliary variables under boundary conditions specified at both ends of the integration interval.

The maximum principle is valid for a class of problems in which the mathematical dependencies describing the process have continuous derivatives of the first and second order.

The task of optimal control of train movement was solved by a number of authors using dynamic programming of DP [34, 35], variational calculus of VC [40, 41], and the principle of maximum PM [36, 37].

The shortcomings and simplifications identified during the analysis of methods for solving optimization problems, admitted and accepted in the studies [4, 6, 16, 18-21, 30-32 and others], are presented in table. 2.

Based on a comparison of formulations of optimization problems and analysis of methods for solving them, the following intermediate conclusions can be drawn:

- the problem of optimal control of train movement has a number of features of the process of a moving train, which were not taken into account by researchers and led to an insufficiently convenient and accurate formulation of the problem of optimizing the transportation work of locomotives and her formulations;

- when solving the problem, only various expressions of optimality conditions were given, and the solutions themselves were carried out in separate, small areas using well-known methods of optimal control theory;
- the calculation results could not have practical application;
- it is obvious that there is a need to clarify the formulation of the problem of optimal control of train movement and to develop a method for solving it.

Table 2. Results of analysis and comparison of methods for solving an optimization problem transportation work of locomotives

Researches	Disadvantages and simplifications
1	2
[4, 6, 16, 18-21, 30-32 and others]	1. The problem is solved without taking into account the hroughput factor. 2. The length of the track profile element is taken as the variation step. 3. Varying locomotive control modes only at points of change in the steepness of the track profile. 4. Difficulty in choosing the place to start braking, that is, it is necessary to enter a sign for selecting the place of braking. 5. The optimality criterion is usually expressed by the approximate relation. 6. The algorithm does not provide for varying modes during “start-up”. 7. Investigation are only the “running” positions of the locomotive. 8. The optimality condition is satisfied by choosing λ – the Lagrange multiplier, which leads to multivariate calculations. 9. The state of the object, in some cases, is considered without restrictions imposed on the phase coordinates of the speed V of the object (train), the path S. 10. Efficiency locomotive is taken to be a constant value. 11. Fuel consumption is approximated by a linear dependence on diesel power. 12. In some cases, the braking force of the train is not taken into account. 13. A graphical method is proposed for solving the problem of joining optimal “pieces” of a trajectory, which is of little use for mass calculations. 14. The route section is divided into hauls characterized by “accelerating” and “decelerating” ascents. 15. Calculations for areas of short length (usually S = 15 - 20 km).

Source: «Compiled by the author».

The last intermediate conclusion was implemented in the monograph [38], where the author of this article proposed a refined formulation and formulated the problem of optimal control of train movement, based on the method of dynamic step-by-step programming of its solution, which based on the principle of parametric optimization, according to which the value of the gain parameter in a given section of the account is expressed by the formula for stepwise process optimization, which determines the sequence of solving the optimization problem, namely:

$$\bar{E}(S_k) = \min_{\substack{\{u_i\} \\ \to Пс}} \sum_{i=1}^{N} \min_{\substack{\{u_j\} \\ \to ПΔj \\ \to П_н}} \sum_{\{j\}}^{\Lambda_i} \min_{\substack{\{u\} \\ \to Пj}} \int_{Sjн}^{Sjk} \varphi_j\left(s, v, u_j\right) ds \qquad (9)$$

where j = а, б, в,… - process stages with corresponding controls and continuously differentiable functions fj and φj;

Λ i - number of j – x stages at the i – th optimization step;

N - the number of optimization steps in a given area Sн - Sк, which is revealed during the calculation process;

i – another optimization step;

$\{u_j\}_{\substack{\to ПΔj \\ \to П_н}}$ - the entry means that the optimization is carried out according to the sequence →ПΔ controls {uj} at the corresponding i -th optimization step, while the choice of the sequence →Пн controls {uj} is based on the parameters ПΔj or Пн (a similar notation for other parameters Пj and Пс).

3 Results and their discussion

Optimal train driving modes on some sections of the Central Asian Railway [38], obtained by the method of dynamic step-by-step programming, were tested of experimental trips, which showed that their practical application is possible, both from the point of view of train safety movement and from taking into account the recommended positions of the controller of driver's, since the optimal mode ensures accurate fulfillment of the travel times movement for the hauls and for the entire section, and the amount of consumed diesel fuel is within the calculated values, which indicates possible savings in natural diesel fuel.

So, depending on the selected movement mode of the train on individual trips, deviations in energy consumption from average values, under the same conditions of transportation work of diesel locomotives on railway sections, are approximately 10 - 15 percent, which indicates huge reserves in savings energy due to the using optimal modes.

Available works on the issue of driving trains with locomotives [2-19, 31-33] do not provide a sufficiently substantiated theoretical basis for choosing optimal control modes for diesel locomotives, which makes their practical use difficult. Some authors have made attempts to use methods of the mathematical theory of optimal control for this purpose. However, the use of the dynamic programming method based on R. Bellman's optimality principle [5, 33 and others], as well as of the maximum principle [7, 31 and others], did not give practically satisfactory results.

As a result of the research carried out by the author of this article in this area [38], a method of dynamic step-by-step programming (DSP) has been proposed, the use of which allows one to obtain satisfactory results a relatively at small amount of time.

To implement the above, first, the values of the "desirable" speeds of movement along the axes of separate points are identified, with two at the end of the stage, of which (fig. 2) one $V^{п}_{нi+1}$ provides the lowest energy consumption when moving on the considered i - m stage, and the second $V^{0}_{нi+1}$ - the lowest consumption on the next i+1 - m stage, as well as two at the beginning of the stage, which, by analogy with the above, are the "desirable" speeds at the end of the previous stage.

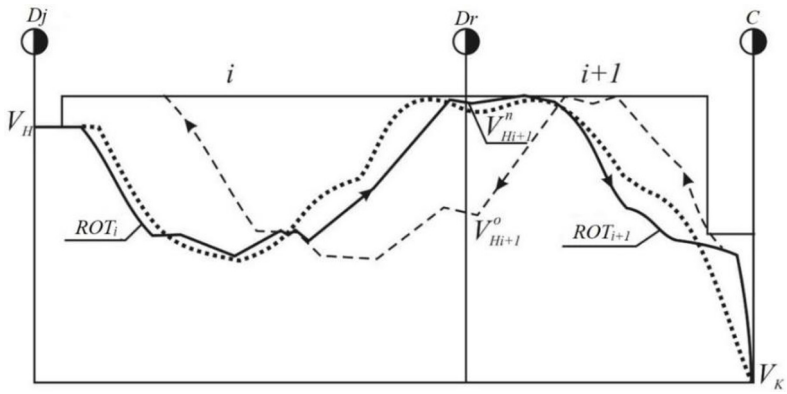

Fig.2. Actual and calculated modes of driving a train on the section *Dj - C*:
——— - calculated mode, ° ° ° ° ° ° - actual mode according to the speed measure tape, – – – – – - by mode idle movement

Source: «Compiled by the author».

Thus, calculations lead from the two indicated "desirable" conditionally optimal speeds of movement in the direction to two final, also "desirable" speeds of movement, which allows us to identify four total locally optimal trajectories (TLOT) on the stretch, of which two are left for further calculations relatively - optimal trajectories (ROT) with the best performance - one in the direction of the "desired" speed of movement $V^{п}_{нi+1}$, and the other - $V^{0}_{нi+1}$.

If the initial or final speeds on the stretch have been specified (for example, there is a train stop), then respectively, two total locally optimal trajectories (TLOT) are identified, which will also be relatively optimal trajectories (ROT) on this stretch. Throughout the entire section of the account, the total sums relative to the optimal trajectories of ROT in the direction of $V^{п}_{нi+1}$ and $V^{0}_{нi+1}$ are compared and the best, optimal option for the entire section of the account, but according to the minimum value of the total gain.

At each stage, the identification of the locally optimal trajectory (LOT) and the corresponding mode of driving the train in operation is carried out according to the condition of realizing the highest efficiency of power transformations (power circuit) [38]. By changing the value of the smallest and largest positions of the controller of the working stroke driver (range of positions being sorted out), the coordinates of the transition points from the working stroke to the idle stroke are adjusted. The mode of driving the train at idle, as well as during braking, is adopted as single-position.

For a given train travel time along the stretch, which is the final value of the time coordinate, in the general case, one or another mode of driving the train nк (S) can be adopted based on the appropriate method for selecting control mode of the rolling stock.

4 Conclusion

1. The above characterizes the possible practical application of the dynamic step-by-step programming method for solving the complex and important problem of optimal control of train movement on various, including elongated, traction shoulders of circulation of locomotive.

2. The use of the principle of parametric optimization and the method of dynamic step-by-step programming eliminates uncertainty in the choice of the number and length of calculation steps, the need to enumerate options for all possible states at the boundaries of these steps, sharply reduces the amount of computational work, and makes it possible to

calculate optimal train driving modes for real operating conditions on areas of work of the locomotive crew.

3. The calculated optimal modes of movement trains, according to a preliminary comparative analysis, can provide (for certain areas) savings in natural diesel fuel on average up to eight to ten percent.

4. To solve the problem of optimizing the transportation operation of locomotives by comparing the method of dynamic step-by-step programming proposed by the author of this article with classical methods of optimal control theory, based on the optimality principle of R. Bellman (Dynamic programming) and the maximum principle (of L. Pontryagin).

References

1. A. Grünig, Efficient Generation of Train Speed Profiles, Bachelor's Thesis, Institute for Operations Research, ETH Zurich, 2009.
2. C. Chang and S. Sim, "Optimising train movements through coast control using genetic algorithms," IEE Proceedings - Electric Power Applications, vol. **144**, no. 1, pp. 65–73, Jan. 1997.
3. Golovitcher, I. M. (2001). "Energy efficient control of rail vehicles. Systems, man, and cybernetics." Proc., IEEE Int. Conf., **Vol. 1**, Tucson, AZ, pp. 658–663.
4. H. Ko, T. Koseki, and M. Miyatake, "Application of dynamic programming to optimization of running profile of a train," in Computers in Railways IX, WIT Press, vol. 15, Southampton, Boston, Sept. 2004, pp. 103–112.
5. K. Ichikawa, "Application of optimization theory for bounded state variable problems to the operation of a train," Bulletin of Japanese Society of Mechanical Engineering, vol. **11**, no. 47, pp. 857–865, Nov. 1968.
6. M. Miyatake and H. Ko, "Optimization of train speed profile for minimum energy consumption," IEEE Transactions on Electrical and Electronic Engineering, vol. 5, no. 3, pp. 263–269, May 2010.
7. Nilam R. Dongre1. "Optimization of Energy Consumption In Electric Traction System By Using Interior Point Method." IOSR Journal of Electrical and Electronics Engineering (IOSRJEEE**) 13.2** (2018): pp. 9–15.
8. P. G. Howlett, "Existence of an Optimal Strategy for the Control of a Train," South Australian Inst. Technol., School Math. Rep., Adelaide, SA, Australia, 1987.
9. P. G. Howlett, "Necessary conditions on an optimal strategy for the control of a train," South Australian Inst. Technol., School Math. Rep., Adelaide, SA, Australia, 1987.
10. P. Howlett, "The optimal control of a train," Annals of Operations Research, vol. **98**, no. 1–4, pp. 65–87, Dec. 2000.
11. P. Kokotovic and G. Singh, "Minimum-energy control of a traction motor," IEEE Trans. Autom. Control, vol. **17**, no. 1, pp. 92–95, Feb. 1972.
12. Peter Horn. Uber die Anwendung des MaximumPrinzips von Pontrjagin zur Ermittlung von Algorithmen fur eine energieoptimale Zugsteuerung. Wassenschaftliche Zeitschrift der Hochschule fur Verkehrswesen "Fhedrich List" in Dresden, 18(4), 1971.
13. R. Franke, M. Meyer, and P. Terwiesch, "Optimal control of the driving of trains," Automatisierungstechnik, vol. **50**, no. 12, pp. 606–614, Dec.2002.
14. R. Liu and I. M. Golovicher, "Energy-efficient operation of rail vehicles," Transportation Research Part A: Policy and Practice, vol. **37**, no. 10, pp. 917–931, Oct. 2003.

15. S. H. Han, Y. S. Byen, J. H. Baek, T. K. An, S. G. Lee, and H. J. Park, "An optimal automatic train operation (ATO) control using genetic algorithms (GA)," in Proceedings of the IEEE Region 10 Conference (TENCON 99), vol. 1, Korea, Aug. 1999, pp. 360–362.
16. S. Su, T. Tang, C. Roberts, and L. Huang, "Cooperative train control for energy-saving," in Proc. IEEE Int. Conf. Intell. Rail Transp., Beijing, China, Aug. 2013, pp. 7–12.
17. S. Yasunobu, S. Miyamoto, and H. Ihara, "Fuzzy control for automatic train operation system," in Proceedings of 4th IFAC/IFIP/IFORS International Conference on Control in Transportation Systems, Baden, Germany, June 1983, pp. 39–45.
18. Y. V. Bocharnikov, A. M. Tobias, and C. Roberts, "Reduction of train and net energy consumption using genetic algorithms for trajectory optimization," in Proc. IET Conf. Railway Traction Syst., Birmingham, U.K., Apr. 2010, pp. 32–36.
19. Y. Bocharnikov, A. Tobias, C. Roberts, S. Hillmansen, and C. Goodman, "Optimal driving strategy for traction energy saving on dc suburban railways," IEE Proceedings - Electric Power Applications, vol. **1,** no. 5, pp. 675–682, Sept. 2007.
20. R. Chen, L. Liu and J. Guo, "Optimization of High-Speed Train Control Strategy for Traction Energy Saving Using an Improved Genetic Algorithm", Journal of Traffic and Transportation Engineering. **1**, 12 (2012).
21. S. Sun, Y. Li and H. Xu, "Energy Consumption Optimization for High-speed Railway based on Particle Swarm Algorithm", Proceedings of 4th International Conference on Computational Intelligence and Communication Networks, (2012) November 3–5; Mathura, India.
22. S. Su, X. Li, T. Tang, and Z. Gao, "A subway train timetable optimization approach based on energy-efficient operation strategy," IEEE Trans. Intell. Transp. Syst., vol. 14, no. 2, pp. 883–893, Jun. 2013.
23. X. Li and H. K. Lo, "An energy-efficient scheduling and speed control approach for metro rail operations," Transp. Res. Part B.: Methodol., vol. 64, pp. 73–89, Jun. 2014.
24. X. Yang, X. Li, B. Ning, and T. Tang, "An optimization method for train scheduling with minimum energy consumption and travel Time in metro rail systems," Transportmetrica B: Transp. Dyn., vol. 3, no. 2, pp. 79–98, 2015.
25. X. Yang, X. Li, Z. Gao, H. Wang, and T. Tang, "A cooperative scheduling model for timetable optimization in subway systems," IEEE Trans. Intell. Transp. Syst., vol. 14, no. 1, pp. 438–447, Mar. 2013.
26. C. Chang and D. Xu, "Differential evolution based tuning of fuzzy automatic train operation for mass rapid transit system," IEE Proceedings – Electric Power Applications, vol. 147, no. 3, pp. 206–212, May 2000.
27. K. K. Wong and T. K. Ho, "Dwell-time and run-time control for DC mass rapid transit railways," Proc. Inst. Elect. Eng.—Elect. Power Appl., vol. 6, no. 1, pp. 956–966, Nov. 2007.
28. S. Acikbas and M. Soylemez, "Coasting point optimization for mass rail transit lines using artifical neural networks and genetic algorithms," Proceedings of the IEE Proceedings - Electric Power Applications, vol. 2, no. 3, pp. 172–182, May 2008.
29. Y. Ding, H. Liu, Y. Bai, and F. Zhou, "A two-level optimization model and algorithm for energy-efficient urban train operation," J. Transp. Syst. Eng. Inf. Technol., vol. 11, no. 1, pp. 96–101, Feb. 2011.

30. Kostromin A. M. Optimization of locomotive management, Moscow, Transport, 1979, 119 p (In Russian).
31. Klimovich A. V. Optimization of train movement control to minimize the cost of energy resources for traction, Moscow, Sputnik+ company, 2008, 263 p. (In Russian).
32. Yurenko K. I. Calculation of energy-optimal modes of movement of promising rolling stock by dynamic programming method. Bulletin of universities. electromechanics, 2013, no.3, pp.78-82 (In Russian).
33. Lesov A. T. Analysis of methods for solving problems of optimal control of train movement. // Electronics and electrical equipment of transport. – Tomilinsky Electronic Plant, 2022, issue. 3. – pp. 23–29.
34. Bellman R. Dynamic programming / R. Bellman - M.: Foreign literature, 1960, 400 p.
35. Bellman R. Applied problems of dynamic programming (translation from English Lurie K. A.) / R. Bellman, S. Dreyfus. - M.: Nauka, 1965, 460 p.
36. Pontryagin L. S., Boltyanskij V. G., Gamkrelidze R. V., Mishchenko E. F. Mathematical theory of optimal processes, Moscow, Nauka Publ., 1983, 392 p (In Russian).
37. Pontryagin L. S. The principle of maximum in optimal control, Moscow, Nauka Publ., 1989, 62 p (In Russian).
38. Ablyalimov, O. S. Optimization of locomotive transportation work: theoretical issues, methods, calculations, results. / O. S. Ablyalimov. – Text: direct // Monograph / Tashkent Institute of Railway Transport Engineers. – Tashkent: “Complex Print” nashriyoti, 2020. – 488 p.
39. Ablyalimov O. S. On methods for studying the transportation work of locomotives [Text] / O. S. Ablyalimov // Republican scientific and technical conference with the participation of foreign scientists “Resource-saving technologies in railway transport” / Tashkent Institute. Eng. railway transport. – Tashkent, 2011. – pp. 30 – 33.
40. Petrov Yu. P. Variational methods of optimal control theory / Yu. P. Petrov // Monograph. – M.-L.: Energy, 1965, 220 p.
41. Petrov Yu. P. Variational methods of optimal control theory / Yu. P. Petrov // Monograph, 2nd ed. reworked and additional – M.-L.: Energy, 1977, 280 p.