On stability of motion of satellite

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Abstract. In this paper we consider the translational and rotational motion of satellites around the Earth. First, we study the motion of an arrow-type satellite. We consider the system of equations of motion in the first approximation. We consider the linear Hamiltonian systems of differential equations. The Hamiltonian systems arise in transportation problems. We consider the normalization of Hamiltonian matrix. We solve the system of matrix equations to find the generating function of the canonical transformation. We obtain stability criterion in several cases. Further, we study the motion of a spoke-type satellite. We consider eigenvalues of characteristic equation corresponding to the motion equations. We get the normal form of the Hamiltonian matrix. We obtain the stability criterion.

1 Introduction

The motion stability issue of satellites is considered. We study the translational and rotational motion of satellites of the arrow and spoke type around the Earth. We explore the system of equations of motion in the first approximation. We consider the linear Hamiltonian system of differential equations. We use the properties of the Hamiltonian matrix. Satellites are used for navigation and communication.

2 The translational and rotational motion of an arrow-type satellite around the Earth

Let us consider the translational and rotational motion of an arrow-type satellite around the Earth. The canonical system of differential equations of motion in the first approximation has the following form [1]:

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$$\begin{cases} \dfrac{d^2x_1}{dt^2} = (3-b^2)x_1 + 2\dfrac{dx_2}{dt} \\ \dfrac{d^2x_2}{dt^2} = -2\dfrac{dx_1}{dt} + b^2(x_2 - x_4) \\ \dfrac{d^2x_4}{dt^2} = -3m(1-b^2)(x_2 - x_4) \\ \dfrac{d^2x_3}{dt^2} = -x_3 \\ \dfrac{d^2x_5}{dt^2} = -x_5 \end{cases}$$

The system of equations is divided into two subsystems. Let us consider the subsystem of the first three equations. The second subsystem has an obvious solution. The first subsystem can be written as follows:

$$\frac{d^2u}{dt^2} + P\frac{du}{dt} - D_1 u = O,$$

where

$$u = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix},\ P = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -P^T,\ D_1 = \begin{pmatrix} 3-b^2 & 0 & 0 \\ 0 & b^2 & -b^2 \\ 0 & -3m(1-b^2) & 3m(1-b^2) \end{pmatrix}$$

P is a skew-symmetric matrix, and D_1 is not a symmetric matrix. Therefore, let's change the variables:

$$u = \Lambda w,\ \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix},\ a = \frac{\sqrt{3m(1-b^2)}}{b}$$

Then the equation takes the following form:

$$\frac{d^2w}{dt^2} + P\frac{dw}{dt} - Dw = O \tag{1}$$

where

$$D = \begin{pmatrix} 3-b^2 & 0 & 0 \\ 0 & b^2 & -b\sqrt{3m(1-b^2)} \\ 0 & -b\sqrt{3m(1-b^2)} & 3m(1-b^2) \end{pmatrix} = D^T$$

We denote

$$d_1 = 3-b^2,\ d_2 = -b\sqrt{3m(1-b^2)},\ k = -\frac{b}{\sqrt{3m(1-b^2)}}$$

Then we get

$$D=\begin{pmatrix} d_1 & 0 & 0 \\ 0 & kd_2 & d_2 \\ 0 & d_2 & k^{-1}d_2 \end{pmatrix},\ d_1 \neq 0,\ d_2 \neq 0$$

Now we move from the matrix differential equation (1) of the second order to a system of differential equations of the first order.

After changing the variables:

$$\begin{cases} x = w \\ y = \dfrac{dw}{dt} + \dfrac{1}{2}Pw \end{cases}$$

we get the Hamiltonian system of differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = V\begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

where

$$V=\begin{pmatrix} B^T & E \\ -A & -B \end{pmatrix} \tag{3}$$

$$B=\frac{1}{2}P,\ A=BB^T-D=-\frac{1}{4}P^2-D$$

The matrix V is a Hamiltonian matrix.

The set of eigenvalues of a Hamiltonian matrix consists of pairs (λ_i, λ_i') of opposite numbers [2-6]: $\lambda_i' = -\lambda_i$. The non-zero eigenvalues have the same multiplicity. The zero eigenvalue has an even multiplicity [2-6]. The eigenvalues of the Hamiltonian matrix V we find by solving the characteristic equation (E_6 is the identity matrix of order 6):

$$\det(V-\lambda E_6)=0$$

It is easy to verify that [7-9]:

$$V-\lambda E_6 \sim \begin{pmatrix} E & O \\ O & Q(\lambda) \end{pmatrix},$$

$$Q(\lambda)=\lambda^2 E+\lambda P-D \tag{4}$$

where E, O are identity and zero matrices of order 3.

Then characteristic equation looks like

$$\det Q(\lambda)=\det(\lambda^2 E+\lambda P-D)=0$$

The matrix V has zero eigenvalues since

$$\det V=\det D=0$$

Zero eigenvalue forms $m = defQ(0)$ Jordan blocks. Then we have

$$rgQ(0) = rgD = 2$$

Therefore, $m = defQ(0) = 3 - 2 = 1$. If the zero eigenvalue has multiplicity 2, it forms the Jordan block of order 2. If the zero eigenvalue has multiplicity 4, it forms the Jordan block of order 4. It is easy to verify that the zero eigenvalue has multiplicity 2 if $d_1(k^2+1) \neq 4$. The zero eigenvalue has multiplicity 4 if $d_1(k^2+1) = 4$.

Case 1. Let $d_1(k^2+1) \neq 4$.

Let us consider the system (2).

Let $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0, \lambda_4 = -\lambda_1, \lambda_5 = -\lambda_2, \lambda_6 = 0$ be eigenvalues of the matrix *V*. Let $\lambda_1 \neq \pm\lambda_2$. We want to reduce the matrix *V* with the help of canonical transformation to the following normal form [2,7]:

$$\Phi = \begin{pmatrix} U & I \\ O & -U^T \end{pmatrix}$$

where

$$U = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l \end{pmatrix}, \; l \neq 0$$

Let *q*, *p* be 3-dimensional column-vectors of the new Hamiltonian variables. We make the canonical transformation with the help of the generating function:

$$S(x, p) = 1/2\, p^T Kp + p^T Lx + 1/2\, x^T Mx\,,$$

where *K, L, M* are matrices of order *n*, *K* and *M* are symmetric, and *L* is a non-singular matrix. New and old variables are related by relations:

$$\frac{\partial S}{\partial p} = q, \; \frac{\partial S}{\partial x} = y$$

The matrices $K = K^T$, L, $M = M^T$ of the generating function we find from the following system of matrix equations [7,10-13]:

$$\begin{cases} M^2 + MB^T + BM + A = O, \\ L(B^T + M)L^{-1} = U, \\ KU^T + UK + I = LL^T \end{cases} \tag{5}$$

The first equation of the system (5) is called algebraic matrix Riccati equation [11-16]. Let the matrix $T = \begin{pmatrix} F & H \\ G & W \end{pmatrix}$ be such that $T^{-1}VT = \Phi$, $\det F \neq 0$. Then we get the following system of matrix equations:

$$\begin{cases} B^T F + G = FU \\ -AF - BG = GU \end{cases}$$

Therefore

$$\begin{cases} G = -B^T F + FU \\ -DF + PFU + FU^2 = O \end{cases}$$

We find from this system

$$F = \begin{pmatrix} 2\lambda_1(\lambda_1^2 - k^{-1}d_2) & 2\lambda_2(\lambda_2^2 - k^{-1}d_2) & 0 \\ (\lambda_1^2 - d_1)(\lambda_1^2 - k^{-1}d_2) & (\lambda_2^2 - d_1)(\lambda_2^2 - k^{-1}d_2) & 1 \\ d_2(\lambda_1^2 - d_1) & d_2(\lambda_2^2 - d_1) & -k \end{pmatrix}$$

$$G = -B^T F + FU$$

Then we find the determinant:

$$\det F = \begin{pmatrix} 2\lambda_1(\lambda_1^2 - k^{-1}d_2) & 2\lambda_2(\lambda_2^2 - k^{-1}d_2) & 0 \\ (\lambda_1^2 - d_1)(\lambda_1^2 - k^{-1}d_2) & (\lambda_2^2 - d_1)(\lambda_2^2 - k^{-1}d_2) & 1 \\ d_2(\lambda_1^2 - d_1) & d_2(\lambda_2^2 - d_1) & -k \end{pmatrix}$$

Using the third column we transform the first and second columns of the determinant:

$$\det F = \det \begin{pmatrix} 2\lambda_1(\lambda_1^2 - k^{-1}d_2) & 2\lambda_2(\lambda_2^2 - k^{-1}d_2) & 0 \\ 0 & 0 & 1 \\ (\lambda_1^2 - d_1)k\lambda_1^2 & (\lambda_2^2 - d_1)k\lambda_2^2 & -k \end{pmatrix} =$$

$$= 2k\lambda_1\lambda_2 \det \begin{pmatrix} (\lambda_1^2 - k^{-1}d_2) & (\lambda_2^2 - k^{-1}d_2) & 0 \\ 0 & 0 & 1 \\ (\lambda_1^2 - d_1)\lambda_1 & (\lambda_2^2 - d_1)\lambda_2 & 0 \end{pmatrix}$$

The determinant is equal to zero if and only if

$$\lambda_1(\lambda_1^2 - d_1)(\lambda_2^2 - k^{-1}d_2) = \lambda_2(\lambda_2^2 - d_1)(\lambda_1^2 - k^{-1}d_2)$$

However, we can always choose the signs of the eigenvalues so that this equality does not hold. Therefore, the system of matrix equations (5) has a solution. From the first and second equations we find the matrices of the generating function [7,9-13]:

$$M = GF^{-1},\ L = F^{-1}$$

Let us introduce the following matrix $(F^T F)^{-1} = (f_{ij})$. From the third equation of the system we find the entries of the matrix $K = (k_{ij})$:

$$k_{ij} = \frac{f_{ij}}{\lambda_i + \lambda_j}, \; i + j < 5$$
$$k_{33} = 0, \; l = f_{33}$$

In the new variables we get the following system of differential equations:

$$\begin{cases} \dfrac{dq_1}{dt} = \lambda_1 q_1 \\ \dfrac{dq_2}{dt} = \lambda_2 q_2 \\ \dfrac{dq_3}{dt} = l p_3 \end{cases} \quad \begin{cases} \dfrac{dp_1}{dt} = -\lambda_1 p_1 \\ \dfrac{dp_2}{dt} = -\lambda_2 p_2 \\ \dfrac{dp_3}{dt} = 0 \end{cases}$$

The system has the general solution:

$$\begin{cases} q_1 = c_1 e^{\lambda_1 t} \\ q_2 = c_2 e^{\lambda_2 t} \\ q_3 = c_3 + c_6 t \end{cases} \quad \begin{cases} p_1 = c_4 e^{-\lambda_1 t} \\ p_2 = c_5 e^{-\lambda_2 t} \\ p_3 = \dfrac{c_6}{l} \end{cases}$$

In the old variables the general solution has the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F & -FK \\ G & (F^{-1})^T - GK \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

In this case the motion of the satellite in the first approximation is unstable.

Case 2. Let $d_1(k^2 + 1) = 4$.

Let us consider the system (2).

Let $\lambda_1 \neq 0, \lambda_4 = -\lambda_1, \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = 0$ be eigenvalues of the matrix *V*. The zero eigenvalue forms the Jordan block of order 4. Therefore, the matrix *V* has the following normal form [2,7]:

$$\Phi = \begin{pmatrix} U & I \\ O & -U^T \end{pmatrix}$$

where

$$U = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this case the motion of the satellite in the first approximation is also unstable. The results are shown in the table.

Table 1. The arrow-type satellite stability investigation

$d_1(k^2+1)$	**Eigenvalues of Hamiltonian matrix**	**Motion of satellite**
$\frac{(3-b^2)(b^2+3m-3mb^2)}{3m(1-b^2)} \neq 4$	$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0, \lambda_4 = -\lambda_1, \lambda_5 = -\lambda_2, \lambda_6 = 0$	unstable
$\frac{(3-b^2)(b^2+3m-3mb^2)}{3m(1-b^2)} = 4$	$\lambda_1 \neq 0,\ \lambda_4 = -\lambda_1,\ \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = 0$	unstable

3 The translational and rotational motion of a spoke-type satellite around the Earth

Let us consider the translational and rotational motion of spoke-type satellite around the Earth. The canonical system of differential equations of motion in the first approximation has the following form [1]:

$$\begin{cases} \frac{d^2x_1}{dt^2} = (3+2b^2)x_1 + 2\frac{dx_2}{dt} \\ \frac{d^2x_2}{dt^2} = -2\frac{dx_1}{dt} + b^2(x_4 - x_2) \\ \frac{d^2x_3}{dt^2} = -(1+b^2)x_3 \pm b^2 x_5 \\ \frac{d^2x_4}{dt^2} = -3m(1+b^2)(x_4 - x_2) \\ \frac{d^2x_5}{dt^2} = -(1+3m(1+b^2))x_5 \pm 3m(1+b^2)x_3 \end{cases}$$

We introduce the following designations:

$$u = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix},\ P = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -P^T,\ D_2 = \begin{pmatrix} 3+2b^2 & 0 & 0 \\ 0 & -b^2 & b^2 \\ 0 & 3m(1+b^2) & -3m(1+b^2) \end{pmatrix}$$

The first, second and forth equations of the system can be written in the form of a matrix equation:

$$\frac{d^2u}{dt^2} + P\frac{du}{dt} - D_2 u = O$$

Further let's change the variables:

$$u = \Lambda z,\ \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e \end{pmatrix},\ e = \frac{\sqrt{3m(1+b^2)}}{b}$$

Then the equation takes the following form:

$$\frac{d^2z}{dt^2}+P\frac{dz}{dt}-D_3z=O$$

where

$$D_3=\begin{pmatrix}3+2b^2 & 0 & 0\\ 0 & -b^2 & b\sqrt{3m(1+b^2)}\\ 0 & b\sqrt{3m(1+b^2)} & -3m(1+b^2)\end{pmatrix}=D_3^T$$

After changing the variables:

$$\begin{cases}x=z\\ y=\dfrac{dz}{dt}+\dfrac{1}{2}Pz\end{cases}$$

we get the Hamiltonian system of differential equations:

$$\frac{d}{dt}\begin{pmatrix}x\\ y\end{pmatrix}=V\begin{pmatrix}x\\ y\end{pmatrix},$$

where

$$V=\begin{pmatrix}B^T & E\\ -A & -B\end{pmatrix}$$

$$B=\frac{1}{2}P,\ A=BB^T-D_3=-\frac{1}{4}P^2-D_3$$

Then characteristic equation looks like

$$\det Q(\lambda)=\det(\lambda^2E+\lambda P-D_3)=0$$

The matrix V has zero eigenvalues since

$$\det V=\det D_3=0$$

Zero eigenvalue forms $m=defQ(0)$ Jordan blocks. Then we have

$$rgQ(0)=rgD_3=2$$

Therefore, $m=defQ(0)=3-2=1$. The zero eigenvalue forms one Jordan block. In this case the motion of the satellite in the first approximation is unstable.

4 Conclusions

Thus, we can draw the following conclusion. We consider the translational and rotational motion of satellites of the arrow and spoke type around the Earth. We explore the system of equations of motion in the first approximation. We consider the linear Hamiltonian system

of differential equations. The Hamiltonian systems have an important role in multivariable and large-scale systems, scattering theory, detection and transportation [17-20]. We use the properties of the Hamiltonian matrix. The normalization of the Hamiltonian matrix is considered. We solve the system of matrix equations to find the generating function of the canonical transformation. We get the analytical solution of the system of differential equations of motion of the arrow-type satellite in the first approximation. Satellites are used for navigation and communication. We find the solution of the nonlinear algebraic matrix Riccati equation in one case. We obtain stability criterion based on finding the eigenvalues. The results can be used to study the stability of the motion of aircrafts [18-22].

The author would like to thank associate professor Matseevich T.A. for helpful discussions and valuable comments concerning this work.

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