Mathematical model of non-stationary diffusion – advection of radon in the soil – atmosphere system

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Abstract. A one-dimensional mathematical model of non-stationary diffusion - advection of radon in the soil - atmosphere system is considered. Using the integral Laplace transform, an analytical solution of the mathematical model was obtained and, on its basis, distribution curves of radon concentration in the soil and atmosphere were constructed. It has been shown that an increase in the rate of advection in homogeneous porous soil will lead to the accumulation of radon near the earth's surface.

1 Introduction

The study of radon transfer processes in the soil-atmosphere system is of certain scientific interest. This is due to the fact that radon is an indicator of the stress-strain state of the geological environment and affects some geophysical fields [1]. For example, radon participates in the formation of the electric field of the surface layer of the atmosphere and is one of the precursors of earthquakes, which in a sense determines the relevance of the study of radon in various systems and environments [1-5].

The study of radon propagation in various environments is carried out using mathematical models that take into account various transport mechanisms, mainly advection and diffusion [6]. However, theoretical estimates of these parameters, which were obtained using classical models, do not explain the anomalously high values of volumetric radon activity near the earth's surface [6]. Consequently, there is a need to develop a mathematical model that would include mechanisms that accelerate diffusion and advection. For example, this may be a model of anomalous diffusion and advection of radon in a fractal porous medium [7,8] or a model of ordinary diffusion and advection that takes into account acoustic disturbances in the soil.

However, in the first stage of modeling it is first necessary to obtain some understanding of the transport of radon in the soil-atmosphere system in time and space. For this purpose, it is necessary to develop a model of non-stationary radon transfer in the soil-atmosphere system and, if possible, obtain its analytical solution.

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Let us also note that in article [9] the problem of unsteady diffusion in a layered pore medium was studied by numerical methods, and in article [10] - in a pore medium with inclusions.

2 Formulation of the problem

According to the theory of the emanation method in radiometric reconnaissance, the transfer of radon from porous homogeneous soil to the earth's surface is carried out using the mechanisms of diffusion and convection [6, 11, 12].

In this work we will consider advection as transport, which can include either convection or filtration. We will assume that the transfer characteristics are constant given values. Then the problem of non-stationary radon transfer in the soil-atmosphere system can be presented as:

$$\frac{\partial A(z,t)}{\partial t} = D_g \frac{\partial^2 A(z,t)}{\partial z^2} + v_g \frac{\partial A(z,t)}{\partial z} - \lambda\left(A(z,t) - A_\infty\right), z > 0$$

$$\frac{\partial A(z,t)}{\partial t} = D_a \frac{\partial^2 A(z,t)}{\partial z^2} + v_a \frac{\partial A(z,t)}{\partial z} - \lambda A(z,t), z > 0$$

$$D_g \frac{\partial A(z,t)}{\partial z}\bigg|_{z=0+0} + v_g A(z,t)\big|_{z=0+0} = D_a \frac{\partial A(z,t)}{\partial z}\bigg|_{z=0-0} + v_a A(z,t)\big|_{z=0-0},$$

$$A(z,t)\big|_{z=0+0} = A(z,t)\big|_{z=0-0}, \lim_{z\to\infty} A(z,t) = A_\infty, \lim_{z\to-\infty} A(z,t) = 0,$$

(1)

where D_a, D_g – radon diffusion coefficients in the soil and in the surface atmosphere, m^2/s; v_a, v_g – radon advection speed in the soil and in the surface atmosphere, respectively, m/s; λ – radon decay constant, 1/s; A_∞ – volumetric activity of radon, which is in radioactive equilibrium with radium (^{226}Ra) at a given depth in the soil ($A_\infty = K_{em} A_{Ra} \rho_s (1-\eta)$), where K_{em} is the radon emanation coefficient, rel. units; A_{Ra} – specific activity of ^{226}Ra, Bq/kg; ρ_s – density of solid soil particles, kg/m^3; η – soil porosity, rel. units; $A(z,t)$ – volumetric activity of radon (^{222}Rn) in soil, Bq/m^3.

It should be noted that the boundary conditions at the soil-atmosphere boundary, equality of fluxes and radon concentrations, are necessary to obtain a continuous solution in the soil-atmosphere system. The latter conditions indicate that the concentration of radon in the atmosphere decreases due to turbulent diffusion, and at a certain depth of homogeneous porous soil there is a source of radon emanation.

3 Solution method

Let's simplify problem (1) by making the following transformation - replacement of the form:

$$\begin{cases} A(z,t) = e^{-\lambda t} u(x_g, t), x_g = z + v_g t, x_g > v_g t, \\ A(z,t) = e^{-\lambda t} u(x_a, t), x_a = z + v_a t, x_a < v_a t. \end{cases}$$

(2)

Substituting expressions (2) into the model equation and boundary conditions (1), we obtain the following problem:

$$\frac{\partial u\left(x_g,t\right)}{\partial t} = D_g \frac{\partial^2 u\left(x_g,t\right)}{\partial x_g^2} + \lambda A_\infty e^{\lambda t}, x_g > v_g t,$$

(3)

$$\frac{\partial u\left(x_a,t\right)}{\partial t} = D_a \frac{\partial^2 u\left(x_a,t\right)}{\partial x_a^2}, x_a < v_a t.$$

$$D_g \frac{\partial u\left(x_g,t\right)}{\partial x_g}\Bigg|_{x_g=v_g t+0} + v_g u\left(x_g,t\right)\Big|_{x_g=v_g t+0} = D_a \frac{\partial u\left(x_a,t\right)}{\partial x_a}\Bigg|_{x_a=v_a t-0} + v_a u\left(x_a,t\right)\Big|_{x_a=v_a t-0},$$

$$u\left(x_g,t\right)\Big|_{x_g=v_g t+0} = u\left(x_a,t\right)\Big|_{x_a=v_a t-0}, \lim_{x_g \to \infty} u\left(x_g,t\right) = A_\infty e^{-\lambda t}, \lim_{x_a \to -\infty} u\left(x_a,t\right) = 0,$$

Let's do the Laplace transform with respect to time variable t for problem (3). We arrive at the following problem for the image:

$$D_g \frac{d^2 F\left(x_g,p\right)}{dx_g^2} - pF\left(x_g,p\right) + \frac{\lambda A_\infty}{p-\lambda} = 0, D_a \frac{d^2 F\left(x_a,p\right)}{dx_a^2} - pF\left(x_a,p\right) = 0,$$

(4)

$$D_g \frac{dF\left(x_g,p\right)}{dx_g}\Bigg|_{x_g=v_g t+0} + v_g F\left(x_g,p\right)\Big|_{x_g=v_g t+0} = D_a \frac{dF\left(x_a,p\right)}{dx_a}\Bigg|_{x_a=v_a t-0} + v_a F\left(x_a,p\right)\Big|_{x_a=v_a t-0},$$

$$F\left(x_g,p\right)\Big|_{x_g=v_g t+0} = F\left(x_a,p\right)\Big|_{x_a=v_a t-0}, \lim_{x_g \to \infty} F\left(x_g,p\right) = \frac{A_\infty}{p-\lambda}, \lim_{x_a \to -\infty} F\left(x_a,p\right) = 0.$$

The solutions to the differential equations in equation (4) are known, and taking into account the boundary conditions on the external boundaries, they can be written as follows:

Equations should be centred and should be numbered with the number on the right-hand side.

$$F\left(x_g,p\right) = C_1 e^{-x_g \sqrt{\frac{p}{D_g}}} + \frac{A_\infty}{p-\lambda}, F\left(x_a,p\right) = C_2 e^{x_a \sqrt{\frac{p}{D_a}}}.$$

(5)

Let's find the integration constants C_1 and C_2. For this purpose, we use the boundary conditions at the internal soil–atmosphere interface of problem (4). We obtain the following algebraic system of equations:

$$\begin{cases} C_1 e^{-v_g t \sqrt{\frac{p}{D_g}}} = C_2 e^{v_a t \sqrt{\frac{p}{D_a}}} - \frac{A_\infty}{p-\lambda}, \\ C_1 e^{-v_g t \sqrt{\frac{p}{D_g}}} \left(v_g - \sqrt{D_g p}\right) = C_2 e^{-v_a t \sqrt{\frac{p}{D_a}}} \left(v_a + \sqrt{D_a p}\right) - \frac{v_g A_\infty}{p-\lambda}. \end{cases}$$

(6)

The solution for system (6) has the form:

$$C_1 = -\frac{A_\infty\left(a+b\sqrt{p}\right)e^{\frac{v_g t\sqrt{p}}{\sqrt{D_g}}}}{\left(p-\lambda\right)\left(a+c\sqrt{p}\right)}, C_2 = \frac{A_\infty e^{-\frac{v_a t\sqrt{p}}{\sqrt{D_a}}}}{p-\lambda}\left(1-\frac{a+b\sqrt{p}}{a+c\sqrt{p}}\right), \quad (7)$$

$$a = v_g - v_a, b = -\sqrt{D_a}, c = -\sqrt{D_a}-\sqrt{D_g}.$$

Substituting the found constants (7) into the solutions for image (5), we obtain:

$$F\left(x_g,p\right) = \frac{A_\infty}{p-\lambda}\left(1-\frac{\left(a+b\sqrt{p}\right)e^{-\tau_g\sqrt{p}}}{a+c\sqrt{p}}\right)+\frac{A_\infty}{p-\lambda}, \quad F\left(x_a,p\right) = \frac{A_\infty e^{-\tau_a\sqrt{p}}}{p-\lambda}\left(1-\frac{a+b\sqrt{p}}{a+c\sqrt{p}}\right), \quad (8)$$

$$\tau_g = \frac{x_g - v_g t}{\sqrt{D_g}}, \tau_a = \frac{x_a - v_a t}{\sqrt{D_a}}.$$

All that remains is to go to the original. Note that the author's article [13] also considered a method for numerically finding the original. For this purpose, we use the transition table from the reference book [14] and find:

$$\frac{A_\infty}{p-\lambda} \Rightarrow A_\infty e^{\lambda t},$$

$$\frac{A_\infty e^{-\tau_a\sqrt{p}}}{p-\lambda} \Rightarrow \frac{A_\infty e^{\lambda t}}{2}\left(e^{-\tau_a\sqrt{\lambda}}\operatorname{erfc}\left(\frac{\tau_a}{2\sqrt{t}}-\sqrt{\lambda t}\right)+e^{\tau_a\sqrt{\lambda}}\operatorname{erfc}\left(\frac{\tau_a}{2\sqrt{t}}+\sqrt{\lambda t}\right)\right),$$

where $erfc\left(z\right) = 1-erf\left(z\right), erf\left(z\right) = 2\pi^{-1/2}\int\limits_0^z e^{-x^2}dx$ is a probability integral.

Let us introduce the following notation:

$$E_+ = \operatorname{erfc}\left(\frac{z}{2\sqrt{D_g t}}+\sqrt{\lambda t}\right), E_- = \operatorname{erfc}\left(\frac{z}{2\sqrt{D_g t}}-\sqrt{\lambda t}\right), E_0 = \operatorname{erfc}\left(\frac{z}{2\sqrt{D_g t}}+\gamma\sqrt{t}\right).$$

Let's find the original for the expression

$$\frac{\left(a+b\sqrt{p}\right)e^{-\tau_a\sqrt{p}}}{\left(p-\lambda\right)\left(a+c\sqrt{p}\right)} = \frac{\gamma e^{-\tau_a\sqrt{p}}}{\left(p-\lambda\right)\left(\gamma+\sqrt{p}\right)}+\frac{\xi\sqrt{p}e^{-\tau_a\sqrt{p}}}{\left(p-\lambda\right)\left(\gamma+\sqrt{p}\right)}, \gamma = a/c, \xi = b/c.$$

For the first term, the original will take the following form:

$$\frac{\gamma e^{-\tau_a\sqrt{p}}}{\left(p-\lambda\right)\left(\gamma+\sqrt{p}\right)} \Rightarrow \frac{\gamma e^{\lambda t}}{2}\left(\frac{e^{-\tau_a\sqrt{\lambda}}E_-}{\gamma+\sqrt{\lambda}}+\frac{e^{-\tau_a\sqrt{\lambda}}E_+}{\gamma-\sqrt{\lambda}}\right)-\frac{\gamma^2 e^{\gamma\tau_a+\gamma^2 t}E_0}{\gamma^2-\lambda}.$$

For the second term - the corresponding form:

$$\frac{\xi\sqrt{p}e^{-\tau_a\sqrt{p}}}{(p-\lambda)(\gamma+\sqrt{p})} \Rightarrow \frac{\xi\sqrt{\lambda}e^{\lambda t}}{2}\left(\frac{e^{-\tau_a\sqrt{\lambda}}E_-}{\gamma+\sqrt{\lambda}} - \frac{e^{-\tau_a\sqrt{\lambda}}E_+}{\gamma-\sqrt{\lambda}}\right) + \frac{\gamma^2 e^{\gamma\tau_a+\gamma^2 t}E_0}{\gamma^2-\lambda}.$$

Let's find the original for the first image from equation (8):

$$\frac{A_\infty e^{-\tau_a\sqrt{p}}}{p-\lambda}\left(1-\frac{\gamma+\xi\sqrt{p}}{\gamma+\sqrt{p}}\right) \Rightarrow \frac{A_\infty e^{\lambda t}(1-\xi)\sqrt{\lambda}e^{-\tau_a\sqrt{\lambda}}E_-}{2(\gamma+\sqrt{\lambda})} - \frac{A_\infty e^{\lambda t}(1-\xi)\sqrt{\lambda}e^{-\tau_a\sqrt{\lambda}}E_+}{2(\gamma-\sqrt{\lambda})} + \frac{A_\infty(1-\xi)\gamma^2 e^{\gamma\tau_a+\gamma^2 t}E_0}{\gamma^2-\lambda}$$

Solutions taking into account transformation (2) will finally be written as follows:

$$\begin{cases} A(z,t) = A_\infty(1-R_1), z > 0, \\ A(z,t) = A_\infty R_2, z < 0, \end{cases} \tag{9}$$

$$R_1 = \frac{(\gamma+\xi\sqrt{\lambda})e^{-z\sqrt{\frac{\lambda}{D_g}}}E_-}{2(\gamma+\sqrt{\lambda})} - \frac{(\gamma-\xi\sqrt{\lambda})e^{z\sqrt{\frac{\lambda}{D_g}}}E_+}{2(\gamma-\sqrt{\lambda})} + \frac{(\xi-1)\gamma^2 e^{\frac{\gamma z}{\sqrt{D_g}}+\gamma^2 t-\lambda t}E_0}{\gamma^2-\lambda}.$$

$$R_2 = \frac{(1-\xi)\sqrt{\lambda}e^{z\sqrt{\frac{\lambda}{D_a}}}E_-}{2(\gamma+\sqrt{\lambda})} - \frac{(1-\xi)\sqrt{\lambda}e^{-z\sqrt{\frac{\lambda}{D_a}}}E_+}{2(\gamma-\sqrt{\lambda})} + \frac{(\xi-1)\gamma^2 e^{-\frac{\gamma z}{\sqrt{D_a}}+\gamma^2 t-\lambda t}E_0}{\gamma^2-\lambda}.$$

Solution (9) has an exponential character, i.e. the concentration of radon in the soil and in the atmosphere decreases according to an exponential law. It is somewhat similar to the solution for a stationary atmosphere, previously obtained by the author in [7]. Let us carry out a numerical study of solution (9).
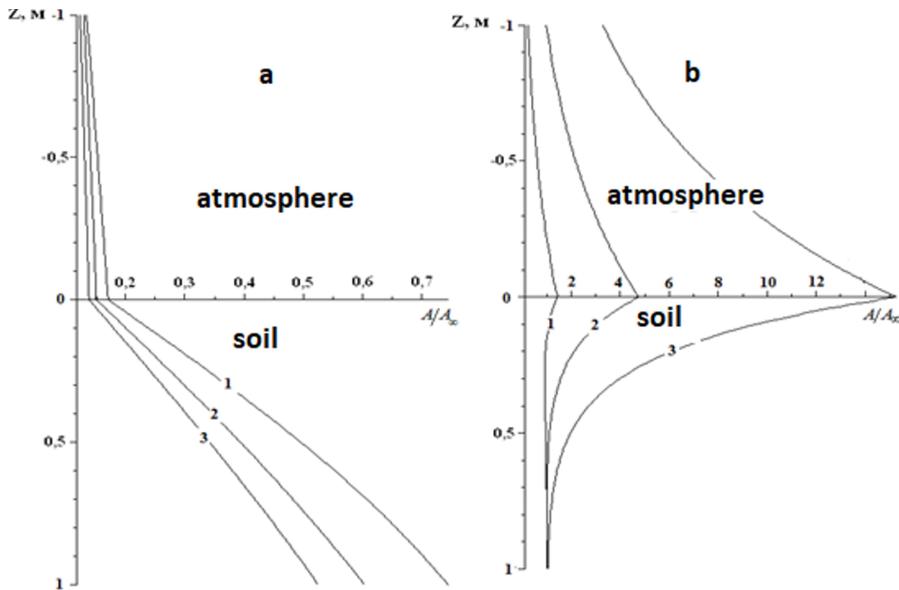
4 Numerical modelling

In numerical modeling, in order to avoid large gradients at the soil-atmosphere interface, the problem parameters were taken to be commensurate: $D_a = 5\cdot10^{-3}$ m²/s, $D_g = 5\cdot10^{-3}$ m²/s, $v_a = 10^{-3}$ m/s, $v_g = 10^{-4}$ m/s.

The calculated distribution curves of radon concentration, constructed for soil and atmosphere, are shown in the figure 1. According to the graph shown in figure (a), the concentration of radon decreases over time towards the earth's surface. The values of the relative volumetric activity at the interface fluctuate in the range of $0.05 \div 0.15$, which is about 0.1 from the background value of the volumetric activity of radon in the soil.

In the atmosphere, due to diffusion and convection, the radon concentration tends to zero. The calculated distribution curves of radon concentration in the soil-atmosphere system (b) are constructed according to the assumption that there is no radon convection in the

atmosphere, i.e., $v_a = 0$ m/s, and the radon advection speed for soil takes the value of $v_g = 5 \cdot 10^{-2}$ m/s.

Note that the distribution curves of radon concentration in the soil and atmosphere are similar in shape to the distribution curves obtained in [7], which considered the problem of radon mass transfer from fractal porous soil to the surface layer of the atmosphere. The mechanisms of radon transfer were superdiffusion and anomalous advection. The shape of the calculated curves is similar to the shape of the calculated curves characterizing anomalous advection. Therefore, we can conclude that an increase in the radon advection rate in homogeneous porous soil may correspond to the occurrence of anomalous advection in fractal porous soil.

Fig. 1. Distribution curves of radon concentration in the soil-atmosphere system at different times (1 – 1000 s; 2 – 2000 s; 3 – 3000 s): a – at $v_a = 10^{-3}$ m/s, $v_g = 10^{-4}$ m/s; b – at $v_a = 0$ m/s, $v_g = 10^{-2}$ m/s.

5 Conclusion

An increase in advection velocity in homogeneous porous soil leads to the accumulation of radon near the earth's surface. This effect can be caused by the destruction of the porous structure of the soil as a result of deformation disturbances.

An increase in the rate of radon transfer to the earth's surface can also be caused by acoustic disturbances in the soil as a result, for example, of the formation of cracks. It is known that acoustic and electric fields are interconnected, and radon, in turn, influences the formation of the electric field of the surface atmosphere.

Therefore, taking into account the influence of acoustic signals on the process of radon transfer in soil is the next stage in the development of the mathematical model (1) with its subsequent generalization to the case of a fractal porous medium according to work [7].

Also of interest is the study of the influence of atmospheric radon on the flux of radon from the earth's surface [15].

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