Blind channel estimation in ZP-OFDM system using new precoding matrix over channels with memory

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Abstract. Using a new precoding matrix, a new method for blind channel estimation of OFDM systems with zero padding has been investigated. A single vector of the correlation matrix is employed to obtain the normalized estimated channel vector, and one pilot signal is used to resolve the scalar ambiguity. Two models of OFDM systems have been investigated, and their NMSE and BER results have been compared in various scenarios. In particular conditions, the new precoding matrix offers an advantage over the conventional circulant. To aid in the creation of novel precoding matrices, a proof of the full rank property is stated under some conditions.

1 Introduction

Due to their various benefits, parallel systems with multiple carriers are used in modern wireless communication systems. One such breakthrough is the "Orthogonal Frequency Division with Multiplexing" technology. Multi-carrier transmission utilized in high-speed communication systems is also referred to as OFDM. The idea of OFDM is the transmission of high-speed data over a lot of low-speed carriers.

A frequency-selective channel is transformed into a group of sub-bands channels in OFDM by splitting the overall channel into numerous small subchannels. Moreover, the usage of cyclic prefix (CP), which is accomplished by extending an OFDM signal with a piece of its head or tail, prevents inter-symbol interference (ISI)[1]. The CP is selected to be more than or equal to the length of the channel impulse response. The true OFDM symbol will be received undistorted because ISI will only influence the redundant portion. Zero-padded OFDM (ZP-OFDM), which inserts zeros in each OFDM symbol rather than duplicating the last samples, has recently been proposed [2]. Together with all of CP-benefits, ZP-OFDM also provides symbol recovery and finite impulse response (FIR) equalization. Nevertheless, implementing a ZP-OFDM system requires changing the transmitter, which makes the equalizer more difficult [4].

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There are two types of channel estimate methods for OFDM systems: blind and non-blind. The non-blind channel estimation method makes use of some or all of the transmitted signals, i.e., pilot tones or training sequences, which are available to the receiver and can be used for the channel estimation. The blind channel estimation method makes use of the statistical behavior of the received signals. The receiver has access to some or all of the transmitted signals, such as pilot tones or training sequences, which can be used for channel estimation.

The key benefit of blind channel estimation is the potential removal of training sequences, which reduce the effectiveness of the system's bandwidth [3,4]. Also, the training sequence must be delivered on a regular basis, which further reduces channel throughput, because the channel in some wireless applications is time-varying. These factors make cutting the quantity of training symbols a top priority, and blind channel estimation techniques have drawn a lot of attention [3]. Blind channel estimation strategies come in a variety of forms. For instance, subspace-based algorithms using redundant linear precoding are taken into account in [5,6] and nonredundant linear precoding in [6,7]. In these methods, a linear block precoder is used at the transmitter, and the covariance matrix of the received signals is used to extract the channel information.

In this paper we propose a blind channel estimation for OFDM system with zero padding as a guard interval using precoding matrix. A new design of the precoding matrix is been investigated with the proof of full rank property. Due to the special design of the precoding matrix will affect the structure of correlation matrix (autocorrelation in first model and cross-correlation in second model) of the received data vectors and so the estimation of the final channel vector is obtained by normalizing one column of this matrix. Two different precoding models were been studied with a comparison of BER and NMSE at different scenarios.

2 Methods

The proposed zero-padding OFDM system is shown in Fig. 1.

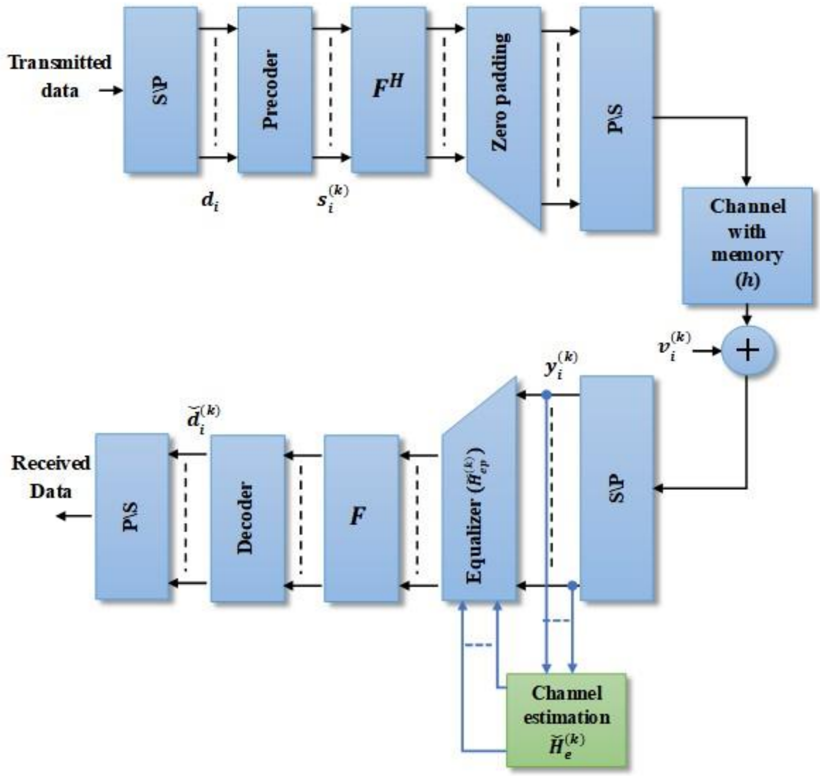

Fig. 1. Block diagram of blind channel estimation.

The modulated signal is turned into a series of blocks that take the form $\boldsymbol{d}_i = [d_{i,0},...,d_{i,N-1}]^T$, where $d_{i,m}, m = 0,1,...,N-1$ is a zero mean (Independent identically distributed random variables), temporarily white, unit variance, spatially uncorrelated , and N is the number of subcarriers in the OFDM symbol, $i = 1,2,....,$ OFDM symbols , then the precoded vector is obtained as follows:

$$\boldsymbol{s}_i^{(k)} = \boldsymbol{A}^{(k)}\boldsymbol{d}_i \tag{1}$$

Where $\boldsymbol{s}_i^{(k)} = [s_{i,0}^{(k)},...,s_{i,N-1}^{(k)}]^T$, k =1,2,3 is the precoded data symbols using the precoders $A^{(1)},A^{(2)}$ and $\boldsymbol{A}^{(3)}$. In this system, we propose a non-redundant precoding matrix of size $N \times N$.

Each block $\boldsymbol{s}_i^{(k)}$ is modulated using the Inverse Fourier Transform, then a sequence of zeros (ZP) of length L is inserted at the end of each block as follows:

$$\boldsymbol{x}_i^{(k)} = \begin{bmatrix} \boldsymbol{F}^H \\ \boldsymbol{0}_{L\times N} \end{bmatrix} \boldsymbol{s}_i^{(k)} \tag{2}$$

where $\boldsymbol{F}$ is a normalized Discrete Fourier Transform matrix of the form in Eq. 3:

$$\boldsymbol{F} = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \quad (3)$$

where $W_N = e^{-j\frac{2p}{N}}$, $\boldsymbol{0}_{L\times N}$ is the matrix of the zero element of dimensions $L\times N$. The block $\boldsymbol{x}_i^{(k)}$ is transformed into a serial data sequence for transmission. The data is transmitted through a multipath channel $\boldsymbol{h} = [h_0, ..., h_L]^T$ of order $L+1$ and $h_i, i = 0,...,L$ is the samples of channel impulse response.

The matrix representation of the received signal $\boldsymbol{y}_i^{(k)}$ after convolution with the channel coefficients is as follows:

$$\boldsymbol{y}_i^{(k)} = \boldsymbol{H}\boldsymbol{F}^{\boldsymbol{H}}\boldsymbol{A}^{(k)}\boldsymbol{d}_i + \boldsymbol{v}_i^{(k)} \quad (4)$$

where $\boldsymbol{y}_i^{(k)}$ is of length $N+L$, $\boldsymbol{v}_i^{(k)} = [v_{i,0}^{(k)},...,v_{i,N+L-1}^{(k)}]^T$ is additive white gaussian noise, $\boldsymbol{H}$ is the channel convolution matrix of size $(N+L)\times N$ as in Eq. 5:

$$\boldsymbol{H} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \ddots & \vdots \\ \vdots & h_1 & \ddots & \ddots \\ h_L & \vdots & \ddots & h_0 \\ 0 & h_L & \ddots & h_1 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & h_L \end{bmatrix} \quad (5)$$

We assume that the noise is a complex Gaussian, zero mean with variance of $s_{v^{(k)}}^2$. One of the advantages of using zero padding is that the receiver does not require additional processes to remove the portion of the ZP, as in systems with cyclic prefix.

In order to estimate the channel, we propose several steps in which a subspace approach is implemented in order to finally estimate the impulse response of the channel as follows:

Considering the autocorrelation matrix of the received signal $\boldsymbol{y}_i^{(k)}$ as shown below.

$$\boldsymbol{R}^{(k)} = \boldsymbol{E}\left[\boldsymbol{y}_i^{(k)}\boldsymbol{y}_i^{(k)\boldsymbol{H}}\right] = \boldsymbol{E}\left[\left(\boldsymbol{H}\boldsymbol{F}^{\boldsymbol{H}}\boldsymbol{A}^{(k)}\boldsymbol{d}_i + \boldsymbol{v}_i^{(k)}\right)\left(\boldsymbol{d}_i^{\boldsymbol{H}}\boldsymbol{A}^{(k)\boldsymbol{H}}\boldsymbol{F}\boldsymbol{H}^{\boldsymbol{H}} + \boldsymbol{v}_i^{(k)\boldsymbol{H}}\right)\right]$$

$$\boldsymbol{R}^{(k)} = \boldsymbol{HF}^{\boldsymbol{H}}\boldsymbol{A}^{(k)}\boldsymbol{E}\left[\boldsymbol{d}_i\boldsymbol{d}_i^{\boldsymbol{H}}\right]\boldsymbol{A}^{(k)\boldsymbol{H}}\boldsymbol{FH}^{\boldsymbol{H}} + \boldsymbol{E}\left[\boldsymbol{v}_i^{(k)}\boldsymbol{v}_i^{(k)\boldsymbol{H}}\right] \tag{6}$$

We propose that the data symbols has an average power equal to one, then the data and noise auto-correlation matrices has the following form: $\boldsymbol{E}\left[\boldsymbol{d}_i\boldsymbol{d}_i^{\boldsymbol{H}}\right] = s_s^2\boldsymbol{I} = \boldsymbol{I}$, $\boldsymbol{E}\left[\boldsymbol{v}_i^{(k)}\boldsymbol{v}_i^{(k)\boldsymbol{H}}\right] = s_{v^{(k)}}^2\boldsymbol{I}$ For all k cases of precoding matrices, we implement a matrix that reduces Eq. 6 to the following:

$$\boldsymbol{R}^{(k)} = \boldsymbol{HP}^{(k)}\boldsymbol{H}^{\boldsymbol{H}} + s_{v^{(k)}}^2\boldsymbol{I} \tag{7}$$

where $\boldsymbol{P}^{(k)}$ is a diagonal matrix with the elements of the main diagonal $\left(P_0^{(k)}, P_1^{(k)}, \ldots, P_{N-1}^{(k)}\right)$. The matrix $\boldsymbol{P}^{(k)}$ is represented by the following equation:

$$\boldsymbol{P}^{(k)} = \boldsymbol{F}^{\boldsymbol{H}}\boldsymbol{A}^{(k)}\boldsymbol{A}^{(k)\boldsymbol{H}}\boldsymbol{F} \tag{8}$$

So we can find that the first term of Eq. 7 has a structure similar to Eq. **Ошибка! Источник ссылки не найден.**:

$$\boldsymbol{HP}^{(k)}\boldsymbol{H}^{\boldsymbol{H}} = \begin{pmatrix} P_0^{(k)}\left|h_0\right|^2 & P_0^{(k)}h_0h_1^* & \cdots & P_0^{(k)}h_0h_L^* & \cdots & 0 \\ P_0^{(k)}h_1h_0^* & \sum\limits_{n=0}^{1}P_{1-n}^{(k)}\left|h_n\right|^2 & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \sum\limits_{n=0}^{2}P_{2-n}^{(k)}\left|h_n\right|^2 & \cdots & \cdots & 0 \\ P_0^{(k)}h_Lh_0^* & \vdots & \cdots & \ddots & \cdots & P_{N-1}^{(k)}h_0h_L^* \\ 0 & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \ddots & P_{N-1}^{(k)}h_{L-1}h_L^* \\ 0 & 0 & P_{N-1}^{(k)}h_Lh_0^* & \cdots & P_{N-1}^{(k)}h_Lh_{L-1}^* & P_{N-1}^{(k)}\left|h_L\right|^2 \end{pmatrix} \tag{9}$$

We pay attention to the following:

1- The matrix has a complex structure, particularly for the elements in the positions of indexes $1 < i, j < N + L$ where i, j are the indexes of row and column respectively.

2- Vectors outside the zone stated above have a clearly defined structure, where all of the channel impulse response coefficients are scaled by one matrix element of the matrix ($P_0^{(k)}$ or $P_{N-1}^{(k)}$).

3- Since the receiver may not always be able to determine the channel length L, the last row and column of the matrix in Eq. **Ошибка! Источник ссылки не найден.** for channels with a length less than L would contain zero elements.

As a result, in this study, we've proposed that the channel can be estimated using just one vector from the auto-correlation matrix:

If $\breve{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)}$ is the vector representing the first column of the matrix $\boldsymbol{R}^{(k)}$ of length $L+1$. This vector can be viewed as follows:

$$\breve{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)} = \begin{bmatrix} P_0^{(k)} \left|h_0\right|^2 + s_{v^{(k)}}^2 \\ P_0^{(k)} h_1 h_0^* \\ \vdots \\ P_0^{(k)} h_L h_0^* \end{bmatrix} \tag{10}$$

The vector $\breve{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)}$ is composed of scaled channel vector $\boldsymbol{h}$ and unit vector $\boldsymbol{e}_1$ scaled by noise power as in (9):

$$\breve{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)} = \begin{bmatrix} P_0^{(k)} \left|h_0\right|^2 + s_{v^{(k)}}^2 \\ P_0^{(k)} h_1 h_0^* \\ \vdots \\ P_0^{(k)} h_L h_0^* \end{bmatrix} = P_0^{(k)} h_0^* \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_L \end{bmatrix} + s_{v^{(k)}}^2 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = P_0^{(k)} h_0^* \boldsymbol{h} + s_{v^{(k)}}^2 \boldsymbol{e}_1 \tag{9}$$

The effect of using precoding is obvious in the previous equation where the channel vector is scaled by $P_0^{(k)}$. The normalized estimated channel vector $\breve{\boldsymbol{h}}_{\boldsymbol{n}}^{(k)}$ can be derived directly by normalizing $\breve{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)}$ as in:

$$\breve{\boldsymbol{h}}_{\boldsymbol{n}}^{(k)} = \frac{\breve{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)}}{\left\|\breve{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)}\right\|} \tag{10}$$

This method can be grouped with subspace methods. A common problem with estimating a channel blindly by subspace approach is known as scalar ambiguity, in which a scalar complex value a_k is required for the final estimation of the channel coefficients. To estimate a_k, we use Eq.11, the proof of this equation is mentioned in Appendix A.

$$a_k = \frac{E\left(s_{i,\mathrm{m}}^{(k)*}\left(\boldsymbol{j}_{m,N+L}\boldsymbol{y}_i^{(k)}\right)\right)}{\sqrt{N}\left(\boldsymbol{j}_{m,L}\boldsymbol{h}_{\boldsymbol{n}}^{(k)}\right)E\left(\left|s_{i,\mathrm{m}}^{(k)}\right|^2\right)} \tag{11}$$

Hence, m denotes the location of the pilot signal. The channel pulse vector's final estimation is determined by Eq. 12:

$$\tilde{\boldsymbol{h}}_{\boldsymbol{e}}^{(k)} = a_k \breve{\boldsymbol{h}}_{\boldsymbol{n}}^{(k)} \tag{12}$$

To estimate the transmitted signal, we first create an estimated convolution matrix $\breve{\boldsymbol{H}}_{e}^{(k)}$ using the vector $\tilde{\boldsymbol{h}}_{e}^{(k)}$, as in Eq. 5. Next, we implement the signal estimation using a straightforward equalizer with zero forcing:

$$\breve{\boldsymbol{d}}_{i}^{(k)}=\left(\boldsymbol{A}^{(k)}\right)^{-1}\boldsymbol{F}\breve{\boldsymbol{H}}_{ep}^{(k)}\boldsymbol{y}_{i}^{(k)} \tag{13}$$

where $\breve{\boldsymbol{d}}_{i}^{(k)}$ is the estimated symbols vector, $\breve{\boldsymbol{H}}_{ep}^{(k)}$ is the pseudo−inverse matrix $\breve{\boldsymbol{H}}_{e}^{(k)}$.

In this model we suggest to precode data vectors using block of precoding matrix. This is accomplished by constructing a block matrix $\boldsymbol{A}_{b}^{(k)}$ of precoding matrices by utilizing the following method:

Assuming that $\boldsymbol{d}_{2i}$ and $\boldsymbol{d}_{2i+1}$ are even and odd data vectors, respectively, we define the block version of the data vectors as $\boldsymbol{D}_{i}=\left[\boldsymbol{d}_{2i} \quad \boldsymbol{d}_{2i+1}\right]^{T}$, and the transmitted symbols are then stated as:

$$\boldsymbol{S}_{i}^{(k)}=\begin{bmatrix}\boldsymbol{s}_{2i}^{(k)}\\ \boldsymbol{s}_{2i+1}^{(k)}\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}_{11}^{(k)} & \boldsymbol{A}_{12}^{(k)}\\ \boldsymbol{A}_{21}^{(k)} & \boldsymbol{A}_{22}^{(k)}\end{bmatrix}\begin{bmatrix}\boldsymbol{d}_{2i}\\ \boldsymbol{d}_{2i+1}\end{bmatrix}=\boldsymbol{A}_{b}^{(k)}\boldsymbol{D}_{i} \tag{14}$$

where $\boldsymbol{S}_{i}^{(k)}=\left[\boldsymbol{s}_{2i}^{(k)} \quad \boldsymbol{s}_{2i+1}^{(k)}\right]^{T}, k=1,2,3$ is a matrix of precoded blocks, and $\boldsymbol{A}_{b}^{(k)}$ of size $2N\times 2N$ is a block precoding matrix of the matrices $(\boldsymbol{A}_{11}^{(k)},\boldsymbol{A}_{12}^{(k)},\boldsymbol{A}_{21}^{(k)},\boldsymbol{A}_{22}^{(k)})$. The signal then through the same processes as in the prior model. The final block vector is as follows:

$$\boldsymbol{Y}_{i}^{(k)}=\begin{bmatrix}\boldsymbol{y}_{2i}^{(k)}\\ \boldsymbol{y}_{2i+1}^{(k)}\end{bmatrix}=\begin{bmatrix}\boldsymbol{HF}^{\boldsymbol{H}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{HF}^{\boldsymbol{H}}\end{bmatrix}\begin{bmatrix}\boldsymbol{A}_{11}^{(k)} & \boldsymbol{A}_{12}^{(k)}\\ \boldsymbol{A}_{21}^{(k)} & \boldsymbol{A}_{22}^{(k)}\end{bmatrix}\begin{bmatrix}\boldsymbol{d}_{2i}\\ \boldsymbol{d}_{2i+1}\end{bmatrix}+\begin{bmatrix}\boldsymbol{v}_{2i}^{(k)}\\ \boldsymbol{v}_{2i+1}^{(k)}\end{bmatrix}$$
$$\boldsymbol{Y}_{i}^{(k)}=\bar{\boldsymbol{H}}\boldsymbol{A}_{b}^{(k)}\boldsymbol{D}_{i}+\boldsymbol{V}_{i}^{(k)} \tag{15}$$

where $\bar{\boldsymbol{H}}=diag([\boldsymbol{HF}^{\boldsymbol{H}} \quad \boldsymbol{HF}^{\boldsymbol{H}}]^{T})$ and $\boldsymbol{H}$ is the convolution matrix of the channel mentioned in Eq. 5, $\boldsymbol{F}$ is the matrix of the discrete Fourier transform, $\boldsymbol{V}_{i}^{(k)}=\left[\boldsymbol{v}_{2i}^{(k)} \quad \boldsymbol{v}_{2i+1}^{(k)}\right]^{T}$ is an additive noise vector. We assume that the noise is complex Gaussian, zero mean, with variance of $s_{v^{(k)}}^{2}$. As this approach is dependent on the cross-correlation of each pair of neighboring symbols, the resulting symbol can be written as:

$$\boldsymbol{y}_{2i}^{(k)}=\boldsymbol{HF}^{\boldsymbol{H}}\left(\boldsymbol{A}_{11}^{(k)}\boldsymbol{d}_{2i}+\boldsymbol{A}_{12}^{(k)}\boldsymbol{d}_{2i+1}\right)+\boldsymbol{v}_{2i}^{(k)} \tag{16}$$

$$\boldsymbol{y}_{2i+1}^{(k)} = \boldsymbol{HF}^{\boldsymbol{H}}\left(\boldsymbol{A}_{21}^{(k)}\boldsymbol{d}_{2i} + \boldsymbol{A}_{22}^{(k)}\boldsymbol{d}_{2i+1}\right) + \boldsymbol{v}_{2i+1}^{(k)} \tag{17}$$

As a result of the design of precoding system, the cross-correlation of two successive symbols develops a unique formation as follows:

$$\boldsymbol{R}^{(k)} = \boldsymbol{E}\left[\boldsymbol{y}_{2i}^{(k)}\boldsymbol{y}_{2i+1}^{(k)\boldsymbol{H}}\right]$$

$$\boldsymbol{R}^{(k)} = \boldsymbol{E}\left[\left(\boldsymbol{HF}^{\boldsymbol{H}}\left(\boldsymbol{A}_{11}^{(k)}\boldsymbol{d}_{2i} + \boldsymbol{A}_{12}^{(k)}\boldsymbol{d}_{2i+1}\right) + \boldsymbol{v}_{2i}^{(k)}\right)\left(\left(\boldsymbol{d}_{2i}^{\boldsymbol{H}}\boldsymbol{A}_{21}^{(k)\boldsymbol{H}} + \boldsymbol{d}_{2i+1}^{\boldsymbol{H}}\boldsymbol{A}_{22}^{(k)\boldsymbol{H}}\right)\boldsymbol{FH}^{\boldsymbol{H}} + \boldsymbol{v}_{2i+1}^{(k)\boldsymbol{H}}\right)\right] \tag{18}$$

Since we have already assumed that the signal and noise vector are uncorrelated, then we can write $\boldsymbol{R}^{(k)}$ into the following:

$$\boldsymbol{R}^{(k)} = \boldsymbol{E}\left[\left(\boldsymbol{HF}^{\boldsymbol{H}}\left(\boldsymbol{A}_{11}^{(k)}\boldsymbol{d}_{2i} + \boldsymbol{A}_{12}^{(k)}\boldsymbol{d}_{2i+1}\right)\left(\boldsymbol{d}_{2i}^{\boldsymbol{H}}\boldsymbol{A}_{21}^{(k)\boldsymbol{H}} + \boldsymbol{d}_{2i+1}^{\boldsymbol{H}}\boldsymbol{A}_{22}^{(k)\boldsymbol{H}}\right)\boldsymbol{FH}^{\boldsymbol{H}}\right)\right] + \boldsymbol{E}\left[\left(\boldsymbol{v}_{2i}^{(k)}\boldsymbol{v}_{2i+1}^{(k)\boldsymbol{H}}\right)\right] \tag{19}$$

$$\boldsymbol{R}^{(k)} = \boldsymbol{HF}^{\boldsymbol{H}}\left(\boldsymbol{A}_{11}^{(k)}\boldsymbol{A}_{21}^{(k)\boldsymbol{H}} + \boldsymbol{A}_{12}^{(k)}\boldsymbol{A}_{22}^{(k)\boldsymbol{H}}\right)\boldsymbol{FH}^{\boldsymbol{H}}$$

$$\boldsymbol{R}^{(k)} = \boldsymbol{HP}^{(k)}\boldsymbol{H}^{\boldsymbol{H}} \tag{20}$$

Where,

$$\boldsymbol{P}^{(k)} = \boldsymbol{F}^{\boldsymbol{H}}\left(\boldsymbol{A}_{11}^{(k)}\boldsymbol{A}_{21}^{(k)\boldsymbol{H}} + \boldsymbol{A}_{12}^{(k)}\boldsymbol{A}_{22}^{(k)\boldsymbol{H}}\right)\boldsymbol{F} \tag{21}$$

According to linear algebra, multiplication and summation of a circulant matrix produce also circulant matrix, since all of the proposed precoding matrices are circulant (or have some circulant property) then the matrix $\boldsymbol{P}^{(k)}$ is a diagonal matrix with $(P_0^{(k)}, P_1^{(k)}, ..., P_{N-1}^{(k)})$ to be the elements of main diagonal. This is due to diagonalization by the FFT matrix.

The same conditions and structures mentioned in Eq. (**Ошибка! Источник ссылки не найден.**-(12) except for noise component $s_{v^{(k)}}^{2}$, also applied to this method to estimate the normalized channel vector and a_k .

These OFDM systems with a block precoding matrix have been investigated in several studies [8–10]. The primary benefit of this kind is that it requires half as many operations to obtain a correlation matrix as the first model did.

A zero forcing equalizer is implemented for data detection:

$$\breve{\boldsymbol{D}}_{\boldsymbol{i}}^{(k)} = \left(\boldsymbol{A}_{\boldsymbol{b}}^{(k)}\right)^{-1}\bar{\boldsymbol{H}}_{\boldsymbol{ep}}^{(k)}\boldsymbol{Y}_{\boldsymbol{i}}^{(k)} \tag{22}$$

Where, $\breve{\boldsymbol{D}}_{\boldsymbol{i}}^{(k)}$ is the estimated symbols block vector, $\bar{\boldsymbol{H}}_{\boldsymbol{ep}}^{(k)}$ is the pseudo−inverse matrix of $\bar{\boldsymbol{H}}_{\boldsymbol{e}}^{(k)} = diag([\breve{\boldsymbol{H}}_{\boldsymbol{e}}^{(k)}\boldsymbol{F}^{\boldsymbol{H}} \quad \breve{\boldsymbol{H}}_{\boldsymbol{e}}^{(k)}\boldsymbol{F}^{\boldsymbol{H}}]^T)$.

The precoder must satisfy some important conditions, such as:

C1- The precoding matrix product $\boldsymbol{A}^{(k)}\boldsymbol{A}^{(k)\boldsymbol{H}}$ should be circulant. Since the estimate equation was already derived using this assumption. So the matrix $\boldsymbol{P}^{(k)}$ be diagonal as in Eq. 8 and Eq. 21.

C2- The precoding matrix must have a full rank, which can be achieved by using a special design of the precoding matrix.

C3- The average symbol power must be preserved by the precoder, one the other hand:

$$\boldsymbol{E}\left[\left\| \boldsymbol{A}^{(k)}\boldsymbol{d}_i \right\|^2\right] = tr\left(\boldsymbol{A}^{(k)}\boldsymbol{d}_i\boldsymbol{d}_i^{\boldsymbol{H}}\boldsymbol{A}^{(k)\boldsymbol{H}}\right) \leq \boldsymbol{E}\left[\left\| \boldsymbol{d}_i \right\|^2\right] \tag{23}$$

For the practical case, the design of the precoding matrix allows auto-correlation or cross-correlation of blocks of the precoding matrix to be at least a single diagonal.

Many papers [2,11] have used this form of precoding matrix to estimate the OFDM channel as in Eq. 24:

$$\boldsymbol{A}^{(1)} = \begin{bmatrix} \sqrt{r} & \sqrt{\frac{1-r}{N-1}} & \cdots & \sqrt{\frac{1-r}{N-1}} \\ \sqrt{\frac{1-r}{N-1}} & \sqrt{r} & \cdots & \sqrt{\frac{1-r}{N-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{1-r}{N-1}} & \sqrt{\frac{1-r}{N-1}} & \cdots & \sqrt{r} \end{bmatrix} \tag{24}$$

where $0 < r \leq 1$ .This precoding matrix satisfies the requirements in C2 (proof in Appendix B), and depending on the value of r , it may also meet the requirements in C3.

Eq. 24 represents the precoding matrix used in the first system. In the second system, the precoding matrix will have the following structure:

$$\boldsymbol{A}_{\boldsymbol{b}}^{(1)} = \begin{bmatrix} \boldsymbol{A}_{11}^{(1)} & \boldsymbol{A}_{12}^{(1)} \\ \boldsymbol{A}_{21}^{(1)} & \boldsymbol{A}_{22}^{(1)} \end{bmatrix} \tag{25}$$

We choose the structure of the blocked matrix according to [2,11] as follows:

$$\boldsymbol{A}_{11}^{(1)} = \boldsymbol{A}_{22}^{(1)} = \frac{2}{3}\boldsymbol{A}^{(1)} \tag{26}$$

$$\boldsymbol{A}_{12}^{(1)} = \boldsymbol{A}_{21}^{(1)} = \frac{1}{3}\boldsymbol{A}^{(1)} \tag{27}$$

This matrix is implemented in the second system for the first case of precoding of the matrix.

We propose two precoding matrices $\boldsymbol{A}^{(2)}$ and $\boldsymbol{A}^{(3)}$ as follows:

1- Generate two row vectors $\boldsymbol{r}_{\boldsymbol{N}}$, $\boldsymbol{r}_{\boldsymbol{N}/4}$ following the equation in Eq. 28.

$$r_W(i) = \begin{cases} 2r\,e^{\left(-2r\,\frac{i-1}{N}\right)} & ,r > 0 \quad for\ \frac{i-1}{W} \in \mathbf{Z} \\ \frac{1}{\sqrt{N}} & \quad otherwise \end{cases} \tag{28}$$

where $i = 1, 2, ..., N$.

2- Constructing the corresponding Toeplitz matrix $\boldsymbol{A}_{\boldsymbol{T}}^{(l)}$, as in Eq. 29.

$$\boldsymbol{A}_{\boldsymbol{T}}^{(l)} = Toeplitz\left(\left[r_{W_l}(1) \quad flip\left(r_{W_l}(2:N)\right)\right], \boldsymbol{r}_{\boldsymbol{W}_l}^{T}\right) \tag{29}$$

where $l = 2,3$, $\boldsymbol{W}_2 = N$, $\boldsymbol{W}_3 = N/4$.

3- The last step to get the final matrix is to use SVD on $\boldsymbol{A}_{\boldsymbol{T}}^{(l)}$, then we can construct the matrix as follows:

$$\left[\boldsymbol{U}^{(l)}, \boldsymbol{S}^{(l)}, \boldsymbol{V}^{(l)}\right] = \boldsymbol{SVD}\left(\boldsymbol{A}_{\boldsymbol{T}}^{(l)}\right) \tag{30}$$

$$\boldsymbol{A}^{(l)} = \boldsymbol{U}^{(l)} \boldsymbol{S}^{(l)} \tag{31}$$

where $\boldsymbol{U}^{(l)}$ is a matrix of orthonormal vectors, and $\boldsymbol{S}^{(l)}$ is a diagonal matrix of singular values .

The construction of both $\boldsymbol{A}_{\boldsymbol{b}}^{(2)}$, $\boldsymbol{A}_{\boldsymbol{b}}^{(3)}$ is according to Eq. 25, Eq. 26 and Eq. 27 by considering $\boldsymbol{A}^{(2)}$ and $\boldsymbol{A}^{(3)}$ obtained by the previous steps. Since the matrix $\boldsymbol{A}^{(2)}$ has the same general structure as $\boldsymbol{A}^{(1)}$ and follows the conditions in Appendix B, then it is a full rank matrix. In Appendix C the proof of full rank property of $\boldsymbol{A}^{(3)}$ at $W = N/4$.

3 Results and discussion

We examine the performance of the proposed estimator under various scenarios. The channel model with the exponential power delay profile [12,13] is used:

$$E\left[\left\|h_l\right\|^2\right] = \exp\left(-\frac{l}{t_{rms}/T_s}\right), l = 0, ..., L_c \tag{32}$$

The phase of each channel ray is uniformly distributed over [0, 2π). This channel is implemented using Naftali model [14] as follows:

$$h_l(k) = N\left(0, \frac{1}{2}s_l^2\right) + jN\left(0, \frac{1}{2}s_l^2\right), \quad k = 1, \dots, n \, , \; l = 0, \dots, L_c \tag{33}$$

Where $N\left(0, 0.5s_l^2\right)$ is a Gaussian random variable with zero mean and variance $0.5s_l^2$ where $s_l^2 = E\left[\left\|h_l\right\|^2\right]$, n is the number of realization of the channel impulse response. The parameter for channel model are $t_{rms}/T_s = 10$, $n = 40$.

OFDM parameters selected according to the 802.11a protocol [15] which call for modulation type of 16QAM with $N = 64$, length of zeros padding $L = 16$ and the channel in 3-tapped channel with $L_c = 2$. The simulation carried out with different numbers of OFDM symbols and with Monte-Carlo run $n_w = 160$.

The normalized estimation mean square errors (NMSE) is defined as:

$$NMSE(k) = \frac{1}{n \times n_w} \sum_{i}^{n_w} \sum_{j}^{n} \frac{var\left(\boldsymbol{h}_{(j)} - (\tilde{\boldsymbol{h}}_e^{(k)})_{(i,j)}\right)}{var\left(\boldsymbol{h}_{(j)}\right)} \tag{34}$$

where j is the index of different channel realizations and i is the index of Monte-Carlo runs. With MATLAB environment, we first generate the channel realization then for each data realization (Monte-Carlo run) each channel realization is implemented. We investigated several examples in this paper as follows:

Example 1: Fig. Fig**.** shows that, when compared to other results of the same scenario, using the matrix $\boldsymbol{A}^{(2)}$ at $r = 0.1$ shows the best performance, especially for values of SNR >15 dB .

At the same value of r , the performance of the first two matrices is close at $r = 0.3$ and 0.6 with the advantage to the matrix $\boldsymbol{A}^{(2)}$.

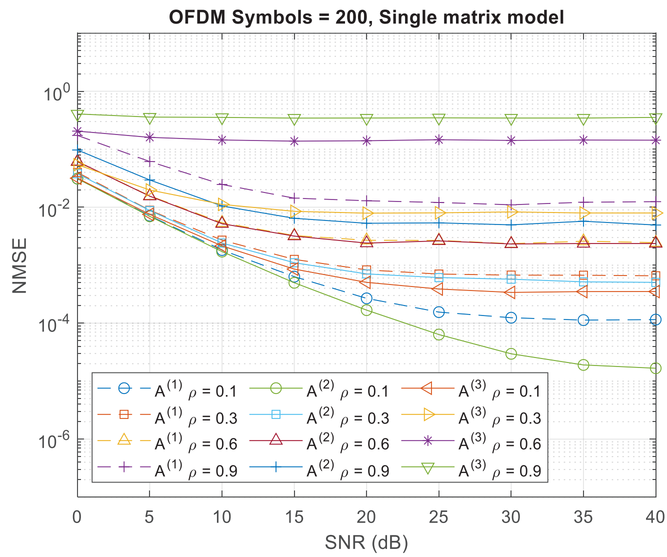

Fig. 2. NMSE for the first system with lower OFDM symbols.

In Fig. Fig**.** , an overview of the results of the bit error rate show several conclusions depending on the value of SNR and r .

 https://doi.org/10.1051/e3sconf/202447402025

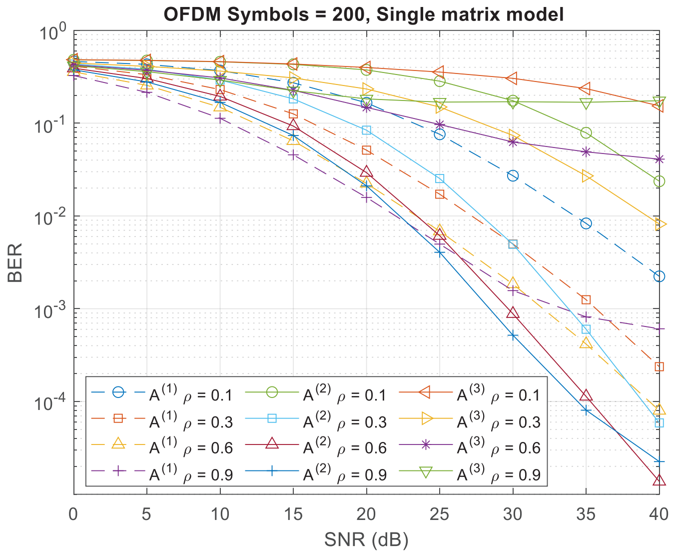

Fig. 3. BER for the first system with lower OFDM symbols.

The performance of the matrices at values $r = 0.3$, 0.6 and 0.9 depends on the value of the SNR. For lower SNR, with matrix $\boldsymbol{A}^{(1)}$ the system performs better than $\boldsymbol{A}^{(2)}$ (SNR < 22.5 dB at $r = 0.9$ and SNR < 24 dB at $r = 0.6$ and SNR < 30 dB at $r = 0.3$), whereas the converse is true at higher SNR. For a comparison of the same model at different r values we observe that the best performance of BER is dependent on SNR up to limit after which the curves enter a zone where dependence of the values on SNR is decreasing, as an example at $r = 0.9$, with $\boldsymbol{A}^{(1)}$ at SNR > 30 dB , with $\boldsymbol{A}^{(2)}$ at SNR > 40 dB .

Example 2: In Fig. Fig**.** and Fig. Fig**.** , we examined the proposed matrices in the second system. The NMSE results in Fig. Fig**.** shows the same behavior as the comparison between the proposed matrices and the circulant matrix in the previous scenario, which proves that the overall results do not depend on the system type.

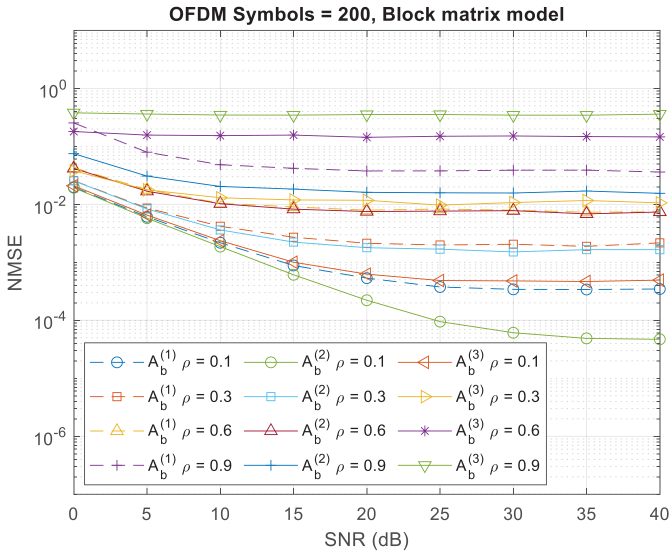

Fig. 4. NMSE for the second system with lower OFDM symbols.

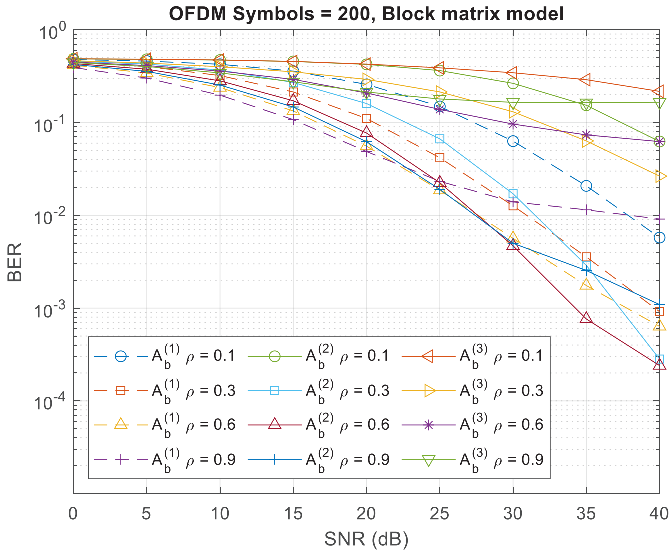

Fig. 5. BER for the second system with lower OFDM symbols.

In Fig. Fig. , at the same value of r , the matrix $A_b^{(3)}$ shows worse performance compared to other matrices. As in the previous scenario, the matrix $A_b^{(1)}$ at the same r value has better performance than $A_b^{(2)}$ with a lower SNR (SNR < 23 dB at $r = 0.9$, SNR < 28 dB at $r = 0.6$, SNR < 33 dB at $r = 0.3$) and vice versa with a higher SNR.

Example 3: In this example, we compare the results of both systems. In Fig. Fig. , we consider a comparison of curves with the best NMSE results of both systems for each matrix, in which we can prove the advantage of using precoding matrix $A^{(2)}$ over the other matrices.

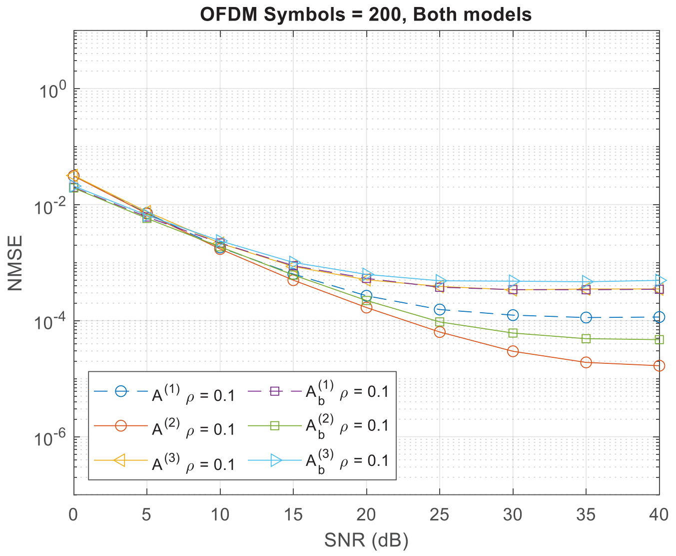

Fig. 6. NMSE comparison of both systems using curves with the best performance of each precoding matrix.

Since the number of OFDM symbols affects the estimation results, in Fig. Fig**.** we investigate the variations of NMSE curves with the best performance at different numbers of OFDM symbols at lower and higher SNR values.

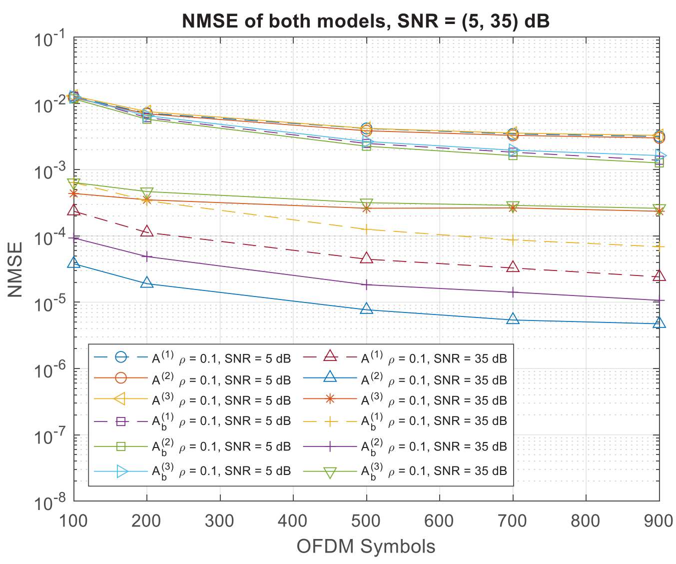

Fig. 7. Comparison of NMSE of both systems with the number of OFDM symbols.

At higher SNR, the best performance is achieved by implementing the proposed matrices $\boldsymbol{A}^{(2)}$, $\boldsymbol{A}_b^{(2)}$ with the advantage of the first model over all the results. Since the number of multiplications required to calculate $\boldsymbol{R}^{(k)}$ in the second model is half that of the first model, the results of NMSE using $\boldsymbol{A}^{(2)}$ as a comparison with the rest matrices shows an impact even with the remainder matrices of the first model. This makes it a good option for practical systems.

Given that the behavior of the probability of a bit error has varied for various precoding matrices, it is worth studying the performance of the probability of a bit error depending on the number of OFDM symbols. In Fig. Fig**.** , Fig. Fig**.** and Fig. Fig**.** we consider two different cases of SNR. With a lower SNR, the behavior of all systems is close even with different numbers of OFDM symbols. With a higher SNR, we can notice that the matrix $\boldsymbol{A}^{(2)}$ shows better performance of NMSE overall results and at different values of OFDM symbols.

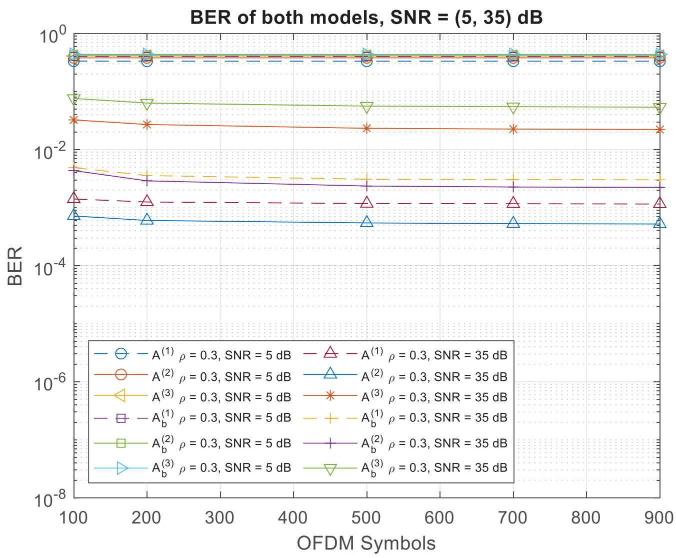

Fig. 8. Comparison of BER in both systems with the number of OFDM symbols at $r = 0.3$.

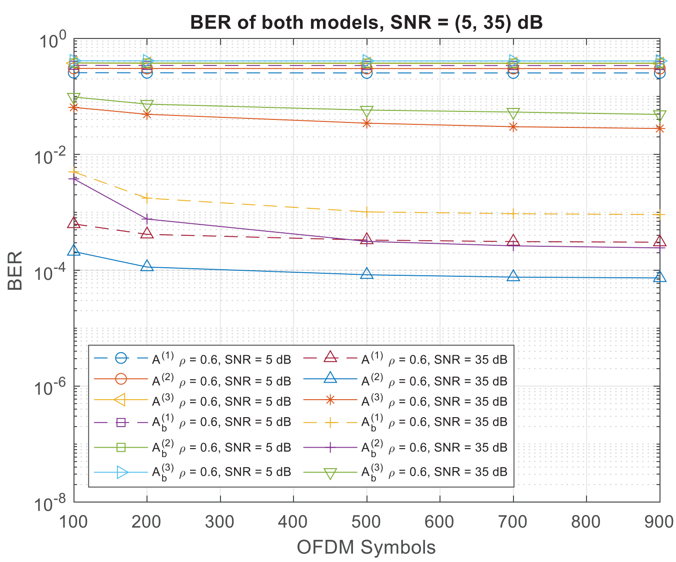

Fig. 9. Comparison of BER in both systems with the number of OFDM symbols at $r = 0.6$.

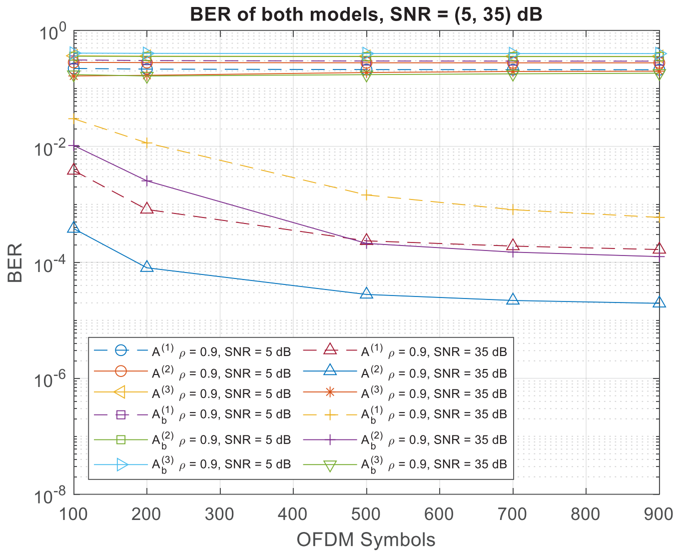

Fig. 10. Comparison of BER in both systems with the number of OFDM symbols at $r = 0.9$.

4 Conclusion

This article demonstrates a precoding blind channel estimation method for OFDM systems with zero padding. A new design of estimator is proposed with a new design of the precoding matrix that been investigated and compared with a conventional circulant matrix. NMSE results and bit error probabilities were studied for different cases of SNR, OFDM symbols and r values for the same channel conditions. A proof of full rank property shows that a different matrices can be obtained which may lead to the optimum matrix design which can be implemented to this system for the best performance of both BER and NMSE. Under some circumstances, the novel method can be used to estimate channel order, making it a potential contender for further research.

The proof of Eq. 11 is as follows:

We define a vector of length t with complex exponents as follows [16]:

$$\boldsymbol{j}_{m,t} = \begin{bmatrix} 1 & e^{-jw_m} & e^{-j2w_m} & \cdots & e^{-j(t-1)w_m} \end{bmatrix} \quad (35)$$

where $\omega_m = \dfrac{2\pi m}{N}$, N length of OFDM symbols. Multiplying $\boldsymbol{j}_{m,N+L}$ by $\boldsymbol{y}_i^{(k)}$:

$$\boldsymbol{j}_{m,N+L}\boldsymbol{y}_i^{(k)} = \boldsymbol{j}_{m,N+L}\boldsymbol{HF}^H\boldsymbol{s}_i^{(k)} + \boldsymbol{j}_{m,N+L}\boldsymbol{v}_i^{(k)} \quad (36)$$

Observing the first term in Eq. 36 , then $\boldsymbol{j}_{m,N+L}\boldsymbol{H}$ can be viewed as:

$$\boldsymbol{j}_{m,N+L}\boldsymbol{H} = \begin{bmatrix} 1 & e^{-jw_m} & e^{-j2w_m} & \cdots & e^{-j(N+L-1)w_m} \end{bmatrix} \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \ddots & \vdots \\ \vdots & h_1 & \ddots & \ddots \\ h_L & \vdots & \ddots & h_0 \\ 0 & h_L & \ddots & h_1 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & h_L \end{bmatrix} \quad (37$$

$$\boldsymbol{j}_{m,N+L}\boldsymbol{H} = \begin{bmatrix} \sum_{n=0}^{L} h_n e^{-jn\omega_m} & \sum_{n=0}^{L} h_n e^{-j(n+1)\omega_m} & \cdots & \sum_{n=0}^{L} h_n e^{-j(n+N-1)\omega_m} \end{bmatrix} \quad (40)$$

$$\boldsymbol{j}_{m,N+L}\boldsymbol{H} = \left(\sum_{n=0}^{L} h_n e^{-jn\omega_m} \right) \begin{bmatrix} 1 & e^{-jw_m} & e^{-j2w_m} & \cdots & e^{-j(N-1)w_m} \end{bmatrix} \quad (38)$$

Since $W_N^m = e^{-j\frac{2\pi m}{N}} = e^{-jw_m}$. The common element in Eq. 38 can be expressed as vector multiplication of $\boldsymbol{h}$ by $\boldsymbol{j}_{m,L}$ and substituting in Eq. 36:

$$\boldsymbol{j}_{m,N+L}\boldsymbol{y}_{\boldsymbol{i}}^{(k)} = \left(\boldsymbol{j}_{m,L}\boldsymbol{h}\right)\begin{bmatrix} 1 & W_N^m & \cdots & W_N^{m(N-1)} \end{bmatrix}\boldsymbol{F}^{\boldsymbol{H}}\boldsymbol{s}_{\boldsymbol{i}}^{(k)} + \boldsymbol{j}_{m,N+L}\boldsymbol{v}_{\boldsymbol{i}}^{(k)} \quad (39)$$

Since $\boldsymbol{F}^{\boldsymbol{H}}$ contains orthonormal vectors, then:

$$\begin{bmatrix} 1 & W_N^m & \cdots & W_N^{m(N-1)} \end{bmatrix}\boldsymbol{F}^{\boldsymbol{H}}\boldsymbol{s}_{\boldsymbol{i}}^{(k)} = \sqrt{N}\, s_{i,\mathrm{m}}^{(k)} \quad (40)$$

The results of $\boldsymbol{j}_{m,N+L}\boldsymbol{v}_{\boldsymbol{i}}^{(k)}$ is also a Gaussian random variable that is still uncorrelated with $\boldsymbol{s}_{\boldsymbol{i}}^{(k)}$, then Eq. 39 can be rewritten as:

$$\boldsymbol{j}_{m,N+L}\boldsymbol{y}_{\boldsymbol{i}}^{(k)} = \sqrt{N}\left(\boldsymbol{j}_{m,L}\boldsymbol{h}\right) s_{i,\mathrm{m}}^{(k)} + \bar{v}_{i,\mathrm{m}}^{(k)} \quad (41)$$

The channel vector $\boldsymbol{h}$ can be represented by its normalized vector with a scalar value a_k . Using the fact that noise and signal are not correlated, then a series of equations is given to find a_k as follows:

$$\boldsymbol{j}_{m,N+L}\boldsymbol{y}_{\boldsymbol{i}}^{(k)} = a_k\sqrt{N}\left(\boldsymbol{j}_{m,L}\boldsymbol{h}_n^{(k)}\right) s_{i,\mathrm{m}}^{(k)} + \bar{v}_{i,\mathrm{m}}^{(k)} \quad (42)$$

$$s_{i,\mathrm{m}}^{(k)*}\left(\boldsymbol{j}_{m,N+L}\boldsymbol{y}_{\boldsymbol{i}}^{(k)}\right) = a_k\sqrt{N}\left(\boldsymbol{j}_{m,L}\boldsymbol{h}_n^{(k)}\right)\left|s_{i,\mathrm{m}}^{(k)}\right|^2 + s_{i,\mathrm{m}}^{(k)*}\bar{v}_{i,\mathrm{m}}^{(k)} \quad (43)$$

$$E\left(s_{i,\mathrm{m}}^{(k)*}\left(\boldsymbol{j}_{m,N+L}\boldsymbol{y}_{i}^{(k)}\right)\right)=a_{k}\sqrt{N}\left(\boldsymbol{j}_{m,L}\boldsymbol{h}_{n}^{(k)}\right)E\left(\left|s_{i,\mathrm{m}}^{(k)}\right|^{2}\right)+E\left(s_{i,\mathrm{m}}^{(k)*}\overline{v}_{i,\mathrm{m}}^{(k)}\right) \tag{44}$$

Then a_k can be calculated using the equation:

$$a_{k}=\frac{E\left(s_{i,\mathrm{m}}^{(k)*}\left(\boldsymbol{j}_{m,N+L}\boldsymbol{y}_{i}^{(k)}\right)\right)}{\sqrt{N}\left(\boldsymbol{j}_{m,L}\boldsymbol{h}_{\boldsymbol{n}}^{(k)}\right)E\left(\left|s_{i,\mathrm{m}}^{(k)}\right|^{2}\right)} \tag{45}$$

The proof of the full rank property of a circulant matrix $\boldsymbol{A}^{(1)}$ is as follows: -

According to linear algebra, if the matrix $\boldsymbol{A}^{(1)}$ is a full rank which implies that the matrix $\boldsymbol{A}_{\boldsymbol{T}}^{(1)}$ (Eq. 46) is also a full rank (since they share the same structure) then the vectors representing its columns are linearly independent, which means that if the vector $\boldsymbol{\beta}\in\mathbf{R}^{N}$, $\boldsymbol{\beta}=[b_1,b_2,...,b_N]^T$, then:

If $\boldsymbol{A}_{\boldsymbol{T}}^{(1)}\boldsymbol{\beta}=\boldsymbol{0}$ and $\boldsymbol{A}_{\boldsymbol{T}}^{(1)}$ is full rank, then the vector $\boldsymbol{\beta}$ must equal to $\boldsymbol{0}$, in other words for $i=1,2,\ldots,N \quad b_i\in\boldsymbol{\beta}$ then $b_i=0$.

We formulate the problem as follows: -

$$\boldsymbol{A}_{\boldsymbol{T}}^{(1)}=\begin{bmatrix} x & y & \cdots & y \\ y & x & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ y & \cdots & \cdots & x \end{bmatrix} \tag{46}$$

Where $x=\sqrt{r}$, $y=\sqrt{\dfrac{1-r}{N-1}}$. Assuming that the vector $\boldsymbol{\beta}$ is not a null vector, then $\boldsymbol{A}_{\boldsymbol{T}}^{(1)}\boldsymbol{\beta}$ can be viewed as follows:

$$\boldsymbol{A}_{\boldsymbol{T}}^{(1)}\boldsymbol{\beta}=\begin{bmatrix} x & y & \cdots & y \\ y & x & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ y & \cdots & \cdots & x \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{47}$$

Constructing N equations in the following ways:

Row (1)

$$b_1x+\sum_{i=2}^{N}b_iy=0 \tag{48}$$

Row (2)

$$b_2x+\sum_{i=1,i\neq 2}^{N}b_iy=0 \tag{49}$$

The general form for any row j can be written as follows:

Row (j) $$b_j x + \sum_{i=1, i \neq j}^{N} b_i y = 0 \quad (50)$$

Sum the *N* row equations, then we get:

$$x \sum_{j=1}^{N} b_j + y \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} b_i = 0 \quad (51)$$

By observing the second term in Eq. 51, we find the following:

$$\sum_{j=1}^{N} \left(\sum_{i=1, i \neq j}^{N} b_i \right) = \left(b_2 + b_3 + \ldots + b_N\right) + \left(b_1 + b_3 + \ldots + b_N\right) + \ldots \left(b_1 + b_2 + \ldots + b_{N-1}\right) \quad (52)$$

Taking the common elements in Eq. 52, then the equation can be written as:

$$\sum_{j=1}^{N} \left(\sum_{i=1, i \neq j}^{N} b_i \right) = \left(N-1\right) b_2 + \left(N-1\right) b_3 + \ldots + \left(N-1\right) b_N = \left(N-1\right) \sum_{j=1}^{N} b_j \quad (53)$$

Substituting Eq. 53 in Eq. 51:

$$x \sum_{j=1}^{N} b_j + y \left(N-1\right) \sum_{j=1}^{N} b_j = \left(x + y \left(N-1\right)\right) \left(\sum_{j=1}^{N} b_j \right) = 0 \quad (54)$$

Which lead as to the following conclusions:

Either $x = -y \left(N-1\right)$ which is not true in our case.

Or $$\sum_{j=1}^{N} b_j = 0 \quad (55)$$

From Eq. 55 we can formulate the following equation for any $j = 1, 2, \ldots, N$.

$$b_j = \sum_{i=1, i \neq j}^{N} -b_i \quad (56)$$

Substituting Eq. 56 in the general form equation Eq. 50 :

Row (j) $$-x \sum_{i=1, i \neq j}^{N} b_i + y \sum_{i=1, i \neq j}^{N} b_i = 0 \quad (57)$$

Factoring the common element in Eq. 57 we get the following conclusions:

Either $x = y$ which is not true in our case

Or $$\sum_{i=1, i \neq j}^{N} b_i = 0 = b_j \quad (58)$$

Which means that the vectors $\boldsymbol{\beta} = \boldsymbol{0}$ and $\boldsymbol{A}^{(1)}$ is a full rank. The general conditions for matrices of the form in Eq. 46 to be of full rank are as follows:

$$x \neq -y\left(N-1\right) \tag{59}$$

$$x \neq y \tag{60}$$

Since the matrix $\boldsymbol{A}^{(2)}$ is constructed by $\boldsymbol{A}_{\boldsymbol{T}}^{(2)}$ that has the same structure as the matrix $\boldsymbol{A}^{(1)}$ and correspond to the general conditions in Eq. 59 and Eq. 60, then the matrix $\boldsymbol{A}_{\boldsymbol{T}}^{(2)}$ is of full rank, which implies that $\boldsymbol{A}^{(2)}$ is also a full rank matrix.

Proof of the full rank property of matrix $\boldsymbol{A}^{(3)}$. From Appendix B, $\boldsymbol{A}^{(3)}$ is a complete rank if and only if $\boldsymbol{A}_{\boldsymbol{T}}^{(3)}$ is a complete rank , that is : if and only if $\boldsymbol{\beta} \in \mathbf{R}^{N}$, $\boldsymbol{\beta} = [b_1, b_2, ..., b_N]^T$, and $\boldsymbol{A}_{\boldsymbol{T}}^{(3)}\boldsymbol{\beta} = \boldsymbol{0}$ then this implies that the vector $\boldsymbol{\beta}$ must be equal to the zero vector $\boldsymbol{0}$, which means that $b_i = 0$ for $i = 1, 2, \ldots, N$, $b_i \in \boldsymbol{\beta}$.

Defining a set of indexes $\boldsymbol{z}^{(k)} = \left\{z_1^{(k)}, z_2^{(k)}, z_3^{(k)}, z_4^{(k)}\right\}$, $k = \left(\left(\left(row\ \ index\right) - 1\right)\ mod\ \dfrac{N}{4}\right) + 1$, mod is modulo operation which represent the index of the rows, and from Eq. 28 we define $z_i^{(k)}$ at $w = N/4$ as follows:

$$z_i^{(k)} = \frac{\left(i-1\right)N}{4} + k \tag{61}$$

Also we define another equation for the simplicity of the rest of the proof as follows:

$$x\left(z_i^{(1)}\right) = 2r\, e^{\left(-2r\,\frac{z_i^{(1)}-1}{N}\right)} = \bar{x}_i \tag{62}$$

Recalling Eq. 28, the vector $\boldsymbol{r}_{N/4}$ have the following structure:

$$\boldsymbol{r}_{N/4} = \left[\bar{x}_1, y, ..., \bar{x}_2, y, ..., \bar{x}_3, y, ... \bar{x}_4, y, ... y\right] \tag{63}$$

Then matrix $\boldsymbol{A}_{\boldsymbol{T}}^{(3)}$ corresponding to vector $\boldsymbol{r}_{N/4}$ and with Eq. 29 can be viewed as:

$$
\boldsymbol{A}_{\boldsymbol{T}}^{(3)} = \begin{bmatrix}
\bar{x}_1 & y & \cdots & \bar{x}_2 & y & \cdots & \bar{x}_3 & y & \cdots & \bar{x}_4 & y & \cdots \\
y & \ddots & \ddots & y & \ddots & \ddots & y & \ddots & \ddots & y & \ddots & \ddots \\
\vdots & \ddots & \bar{x}_1 & \ddots & \ddots & \bar{x}_2 & \ddots & \ddots & \bar{x}_3 & \ddots & \ddots & \bar{x}_4 \\
\bar{x}_4 & y & \ddots & \bar{x}_1 & y & \ddots & \bar{x}_2 & y & \ddots & \bar{x}_3 & y & \ddots \\
y & \ddots & \ddots & y & \ddots & \ddots & y & \ddots & \ddots & y & \ddots & \ddots \\
\vdots & \ddots & \bar{x}_4 & \ddots & \ddots & \bar{x}_1 & \ddots & \ddots & \bar{x}_2 & \ddots & \ddots & \bar{x}_3 \\
\bar{x}_3 & y & \ddots & \bar{x}_4 & y & \ddots & \bar{x}_1 & y & \ddots & \bar{x}_2 & y & \ddots \\
y & \ddots & \ddots & y & \ddots & \ddots & \ddots & \ddots & \ddots & y & \ddots & \ddots \\
\vdots & \ddots & \bar{x}_3 & \ddots & \ddots & \bar{x}_4 & \ddots & \ddots & \bar{x}_1 & \ddots & \ddots & \bar{x}_2 \\
\bar{x}_2 & y & \ddots & \bar{x}_3 & y & \ddots & \bar{x}_4 & \ddots & \ddots & \bar{x}_1 & y & \ddots \\
y & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \cdots & \bar{x}_2 & y & \cdots & \bar{x}_3 & \cdots & \cdots & \bar{x}_4 & \cdots & \cdots & \bar{x}_1
\end{bmatrix}
\begin{array}{l}
\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \frac{N}{4} \\
\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \frac{N}{4} \\
\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \frac{N}{4} \\
\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \frac{N}{4}
\end{array}
\qquad (64)
$$

From Eq. 64 and $\boldsymbol{A}_{\boldsymbol{T}}^{(3)}\boldsymbol{\beta} = \boldsymbol{0}$, we can construct $\frac{N}{4}$ groups of equations as follows:

G1
$$b_{z_1^{(k)}}\bar{x}_1 + b_{z_2^{(k)}}\bar{x}_2 + b_{z_3^{(k)}}\bar{x}_3 + b_{z_4^{(k)}}\bar{x}_4 + \sum_{i=1, i\notin \boldsymbol{z}^{(k)}}^{N} b_i y = 0 \qquad (65)$$

G2
$$b_{z_1^{(k)}}\bar{x}_4 + b_{z_2^{(k)}}\bar{x}_1 + b_{z_3^{(k)}}\bar{x}_2 + b_{z_4^{(k)}}\bar{x}_3 + \sum_{i=1, i\notin \boldsymbol{z}^{(k)}}^{N} b_i y = 0 \qquad (66)$$

G3
$$b_{z_1^{(k)}}\bar{x}_3 + b_{z_2^{(k)}}\bar{x}_4 + b_{z_3^{(k)}}\bar{x}_1 + b_{z_4^{(k)}}\bar{x}_2 + \sum_{i=1, i\notin \boldsymbol{z}^{(k)}}^{N} b_i y = 0 \qquad (67)$$

G4
$$b_{z_1^{(k)}}\bar{x}_2 + b_{z_2^{(k)}}\bar{x}_3 + b_{z_3^{(k)}}\bar{x}_4 + b_{z_4^{(k)}}\bar{x}_1 + \sum_{i=1, i\notin \boldsymbol{z}^{(k)}}^{N} b_i y = 0 \qquad (68)$$

Summation over each group, we get the following equations:

G1
$$\sum_{k=1}^{N/4}\left(b_{z_1^{(k)}}\bar{x}_1 + b_{z_2^{(k)}}\bar{x}_2 + b_{z_3^{(k)}}\bar{x}_3 + b_{z_4^{(k)}}\bar{x}_4\right) + \sum_{k=1}^{N/4}\left(\sum_{i=1, i\notin \boldsymbol{z}^{(k)}}^{N} b_i y\right) = 0 \qquad (69)$$

G2
$$\sum_{k=1}^{N/4}\left(b_{z_1^{(k)}}\bar{x}_4 + b_{z_2^{(k)}}\bar{x}_1 + b_{z_3^{(k)}}\bar{x}_2 + b_{z_4^{(k)}}\bar{x}_3\right) + \sum_{k=1}^{N/4}\left(\sum_{i=1, i\notin \boldsymbol{z}^{(k)}}^{N} b_i y\right) = 0 \qquad (70)$$

G3 $$\sum_{k=1}^{N/4}\left(b_{z_1^{(k)}}\bar{x}_3+b_{z_2^{(k)}}\bar{x}_4+b_{z_3^{(k)}}\bar{x}_1+b_{z_4^{(k)}}\bar{x}_2\right)+\sum_{k=1}^{N/4}\left(\sum_{i=1,i\notin z^{(k)}}^{N} b_i y\right)=0 \tag{71}$$

G4 $$\sum_{k=1}^{N/4}\left(b_{z_1^{(k)}}\bar{x}_2+b_{z_2^{(k)}}\bar{x}_3+b_{z_3^{(k)}}\bar{x}_4+b_{z_4^{(k)}}\bar{x}_1\right)+\sum_{k=1}^{N/4}\left(\sum_{i=1,i\notin z^{(k)}}^{N} b_i y\right)=0 \tag{72}$$

We sum Eq. 69, Eq. 70, Eq. 71 and Eq. 72 then we get:

$$\left(\sum_{j=1}^{4}\bar{x}_j\right)\sum_{k=1}^{N/4}\left(b_{z_1^{(k)}}+b_{z_2^{(k)}}+b_{z_3^{(k)}}+b_{z_4^{(k)}}\right)+\sum_{k=1}^{N/4}\sum_{j=1}^{4}\left(\sum_{i=1,i\notin z^{(k)}}^{N} b_i y\right)=0 \tag{73}$$

We observe that the second summation in the first term above is equal to the sum of all elements of $\boldsymbol{\beta}$. Simplifying the second term of Eq. 73 as follows:

$$\sum_{k=1}^{N/4}\sum_{j=1}^{4}\left(\sum_{i=1,i\notin z^{(k)}}^{N} b_i y\right)=4y\sum_{k=1}^{N/4}\left(\sum_{i=1,i\notin z^{(k)}}^{N} b_i\right)=4y\left(\frac{N}{4}-1\right)\sum_{i=1}^{N} b_i \tag{74}$$

Then Eq. 73 can be rewritten as:

$$\left(\sum_{j=1}^{4}\bar{x}_j\right)\left(\sum_{i=1}^{N} b_i\right)+4y\left(\frac{N}{4}-1\right)\sum_{i=1}^{N} b_i \tag{75}$$

Then we can conclude the following:

Either $$\left(\sum_{j=1}^{4}\bar{x}_j\right)=-4y\left(\frac{N}{4}-1\right)$$ which is not true

Or $$\sum_{i=1}^{N} b_i=0 \tag{76}$$

For any value of k and from Eq. 76 we can write:

$$\sum_{i=1,i\notin z^{(k)}}^{N} b_i=\sum_{i=1,i\in z^{(k)}}^{N} -b_i \tag{77}$$

Substituting Eq. 77 in Eq. 65, Eq. 66, Eq. 67 and Eq. 68, we get the new set of equations:

G1 $$b_{z_1^{(k)}}\left(\bar{x}_1-y\right)+b_{z_2^{(k)}}\left(\bar{x}_2-y\right)+b_{z_3^{(k)}}\left(\bar{x}_3-y\right)+b_{z_4^{(k)}}\left(\bar{x}_4-y\right)=0 \tag{78}$$

G2 $$b_{z_1^{(k)}}\left(\bar{x}_4-y\right)+b_{z_2^{(k)}}\left(\bar{x}_1-y\right)+b_{z_3^{(k)}}\left(\bar{x}_2-y\right)+b_{z_4^{(k)}}\left(\bar{x}_3-y\right)=0 \tag{79}$$

G3 $$b_{z_1^{(k)}}\left(\bar{x}_3-y\right)+b_{z_2^{(k)}}\left(\bar{x}_4-y\right)+b_{z_3^{(k)}}\left(\bar{x}_1-y\right)+b_{z_4^{(k)}}\left(\bar{x}_2-y\right)=0 \tag{80}$$

G4 $$b_{z_1^{(k)}}\left(\bar{x}_2-y\right)+b_{z_2^{(k)}}\left(\bar{x}_3-y\right)+b_{z_3^{(k)}}\left(\bar{x}_4-y\right)+b_{z_4^{(k)}}\left(\bar{x}_1-y\right)=0 \tag{81}$$

The last four equations at any value of k can be written as a matrix multiplication by a vector:

$$\begin{bmatrix} \left(\bar{x}_1-y\right) & \left(\bar{x}_2-y\right) & \left(\bar{x}_3-y\right) & \left(\bar{x}_4-y\right) \\ \left(\bar{x}_4-y\right) & \left(\bar{x}_1-y\right) & \left(\bar{x}_2-y\right) & \left(\bar{x}_3-y\right) \\ \left(\bar{x}_3-y\right) & \left(\bar{x}_4-y\right) & \left(\bar{x}_1-y\right) & \left(\bar{x}_2-y\right) \\ \left(\bar{x}_2-y\right) & \left(\bar{x}_3-y\right) & \left(\bar{x}_4-y\right) & \left(\bar{x}_1-y\right) \end{bmatrix} \begin{bmatrix} b_{z_1^{(k)}} \\ b_{z_2^{(k)}} \\ b_{z_3^{(k)}} \\ b_{z_4^{(k)}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{82}$$

Since the 4×4 matrix in Eq. 82 is a full rank, which can be proven by calculating its determinant, then the vector $\begin{bmatrix} b_{z_1^{(k)}} & b_{z_2^{(k)}} & b_{z_3^{(k)}} & b_{z_4^{(k)}} \end{bmatrix}^T$, must be the null vector, and since $k=1,2,\ldots,\frac{N}{4}$ that prove the elements of vector $\boldsymbol{\beta}$ must be zero, and so , $\boldsymbol{A}^{(3)}$ is a full rank matrix. The general conditions for matrix $\boldsymbol{A}^{(3)}$ to be of full rank are as follows:

$$\left(\sum_{j=1}^{4}\bar{x}_j\right) \neq -4y\left(\frac{N}{4}-1\right) \tag{83}$$

$$\begin{bmatrix} \left(\bar{x}_1-y\right) & \left(\bar{x}_2-y\right) & \left(\bar{x}_3-y\right) & \left(\bar{x}_4-y\right) \\ \left(\bar{x}_4-y\right) & \left(\bar{x}_1-y\right) & \left(\bar{x}_2-y\right) & \left(\bar{x}_3-y\right) \\ \left(\bar{x}_3-y\right) & \left(\bar{x}_4-y\right) & \left(\bar{x}_1-y\right) & \left(\bar{x}_2-y\right) \\ \left(\bar{x}_2-y\right) & \left(\bar{x}_3-y\right) & \left(\bar{x}_4-y\right) & \left(\bar{x}_1-y\right) \end{bmatrix} \text{must be full rank} \tag{84}$$

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