To boundary value problems for a generalized hyperbolic equation with two fractional differentiation operators

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Abstract. Boundary value problems for a one-space-dimensional hyperbolic equation with two fractional differentiation operators and variable coefficients are considered. The method of finite differences is used for the approximate solution to the problems posed. Difference schemes for each of the problems are constructed, and by the method of energy inequalities for various relations between the orders of the fractional derivative (α , β) a priori estimates in differential and difference interpretations are obtained. The obtained estimates imply the uniqueness and stability of the solution with respect to the right-hand side and initial data, as well as the convergence of the solution of the difference problem to the solution of the original differential problem. Algorithms for the numerical solution of the problems posed are constructed.

1 Introduction

In recent years, interest has increased in the study of differential equations of fractional order, in which the unknown function under the sign of the fractional derivative is contained. This is due both to the development of the theory of fractional integro-differentiation itself and to the applications of such constructions in various fields of science [1]-[3]. Non-integer integrals and derivatives and fractional integro-differential equations have many applications in modern research in theoretical physics, mechanics, and applied mathematics. Fractional mathematical calculus is a powerful tool that can be used to obtain dynamic models in which integro-differential operators in time and coordinates describe power-law long-term memory and spatial non-locality of complex environments and processes [4].

In [5], mathematical models of the water regime in soils with a fractal structure were proposed and studied. These models are based on the differential equations considered in this paper with a time-fractional derivative. For the Cauchy problem, a unique representation of the solution is written out.

The equation with the Bessel operator considered in this paper arises in the transition from the three-dimensional equation

$$\partial_{0t}^{\alpha} u = Lu + f(x,t), (x,t) \in Q_T, \tag{0.1}$$

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where

$$Lu = \sum_{s=1}^{3} L_s u, x = (x_1, x_2, x_3),$$

$$L_s u = \frac{\partial}{\partial x_s}(k_s(x,t)\frac{\partial u}{\partial x_s}) + \partial_{0t}^{\beta}\frac{\partial}{\partial x_s}(\eta_s(x)\frac{\partial u}{\partial x_s}) + h_s(x,t)\frac{\partial u}{\partial x_s} - q_s(x,t)u.$$

to the cylindrical coordinate system (r, φ, z) in the case when the solution $u = u(r)$ does not depend on z orφ (there is axial symmetry) and equation (0.1) takes the form (we denote $x = r$) (see [6, p. 15], [13, p. 433]):

$$\partial_{0t}^{\alpha} u = \frac{1}{r}(rk(r,t)u_r)_r + \frac{1}{r}\partial_{0t}^{\alpha}(r\eta(r)u_r)_r + h(r,t)u_r - q(r,t)u + f(r,t), \quad (0.2)$$

and in the case of spherical symmetry, equation (0.1) takes the form:

$$\partial_{0t}^{\alpha} u = \frac{1}{r^2}(r^2 k(r,t)u_r)_r + \frac{1}{r^2}\partial_{0t}^{\alpha}(r^2\eta(r)u_r)_r + h(r,t)u_r - q(r,t)u + f(r,t), \quad (0.3)$$

where

$$k(r,t) = k_1(x,t) = k_2(x,t) = k_3(x,t), \eta(r) = \eta_1(x) = \eta_2(x) = \eta_3(x),$$
$$h(r,t) = h_1(x,t) = h_2(x,t) = h_3(x,t), q(r,t) = q_1(x,t) = q_2(x,t) = q_3(x,t)$$

are symmetry conditions on the coefficients due to the symmetry of r with respect to the variables x_1, x_2, x_3.

The works [7]-[9] are devoted to numerical methods for solving boundary value problems for various equations of fractional order. In these papers, results are obtained that allow, as in the classical case of $(\alpha = 1)$, to apply the method of energy inequalities to find a priori estimates of boundary value problems for a fractional order equation in differential and difference interpretations. In papers [10]-[12], boundary value problems are studied for various loaded differential equations of integer and fractional orders.

In this paper, we study boundary value problems for a one-space-dimensional hyperbolic equation with two fractional differentiation operators and variable coefficients. The method of finite differences is used for the approximate solution to the problems posed. Difference schemes are constructed for each of the problems, and by the method of energy inequalities for various relations between the orders of the fractional derivative $(\alpha,\ \beta)$, a priori estimates are obtained in differential and difference interpretations. The obtained estimates imply the uniqueness and stability of the solution with respect to the right-hand side and initial data, as well as the convergence of the solution of the difference problem to the solution of the original differential problem. An algorithm for the numerical solution to the problem is constructed.

2 Materials and Methods

2.1 Statement of the first boundary value problem

In a closed rectangle $\overline{Q}_T = \{(x,t): 0 \le x \le l,\ 0 \le t \le T\}$ we consider the first boundary value problem for a hyperbolic equation with two fractional differentiation operators and variable coefficients

$$\partial_{0t}^{\alpha} u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + \partial_{0t}^{\beta}\frac{\partial}{\partial x}\left(\eta(x)\frac{\partial u}{\partial x}\right) - q(x,t)u(x,t) + f(x,t),\ \ 0 < x < l,\ \ 0 < t \le T, \quad (1)$$

$$u(0,t) = u(l,t) = 0, 0 \le t \le T, \quad (2)$$

$$u(x,0) = u_0(x), 0 \le x \le l, \tag{3}$$

where

$$0 < c_0 \le k(x,t), \eta(x) \le c_1, |q(x,t)| \le c_2, \tag{4}$$

$\partial_{0t}^{\gamma} u = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\gamma} d\tau$ is a fractional derivative in the sense of Gerasimov–Caputo of order γ, $0 < \gamma < 1$.

We will assume that the solution of the problem under consideration exists and has the derivatives necessary in this presentation, the coefficients of the equation and boundary conditions satisfy the necessary smoothness conditions that provide the required order of approximation of the difference scheme.

Denote by $M_i, (i = 1,2,\dots,)$ positive constant numbers depending only on the input data of the original problem.

2.2 A priori estimate in differential form

To obtain an a priori estimate for the solution of problem (1)-(3) in differential form, we multiply equation (1) scalarly by u

$$(\partial_{0t}^{\alpha} u, u) = ((ku_x)_x, u) + \left(\partial_{0t}^{\beta}(\eta(x)u_x)_x, u\right) - (q(x,t)u, u) + (f(x,t), u) \tag{5}$$

where $(u,v) = \int_0^l uvdx, (u,u) = \| u \|_0^2$, where u, v are functions defined on $[0,l]$.

Using the Cauchy inequality with ε ([13], p. 100,) and Lemma 1 from [7], we transform the integrals in identity (5), then we obtain:

$$(\partial_{0t}^{\alpha} u, u) \ge \frac{1}{2}(1, \partial_{0t}^{\alpha} u^2) = \frac{1}{2}\partial_{0t}^{\alpha} \| u \|_0^2, \tag{6}$$

$$((ku_x)_x, u) = \int_0^l u(ku_x)_x dx = uku_x|_0^l - \int_0^l ku_x^2 dx, \tag{7}$$

$$\left(\partial_{0t}^{\beta}(\eta u_x)_x, u\right) = \int_0^l u\partial_{0t}^{\beta}(\eta u_x)_x dx = \eta u\partial_{0t}^{\beta} u_x|_0^l - \int_0^l \eta(x)u_x\partial_{0t}^{\beta} u_x dx \le \eta u\partial_{0t}^{\beta} u_x|_0^l - \frac{1}{2}\int_0^l \eta\partial_{0t}^{\beta}(u_x)^2 dx, \tag{8}$$

$$-(q(x,t)u, u) \le c_2 \| u \|_0^2, \tag{9}$$

$$(f, u) = \int_0^l fudx \le \frac{1}{2} \| u \|_0^2 + \frac{1}{2} \| f \|_0^2 \tag{10}$$

Taking into account the transformations (6)-(10), from (5), with regard to (2) we find

$$\partial_{0t}^{\alpha} \| u \|_0^2 + \partial_{0t}^{\beta} \int_0^l \eta(u_x)^2 dx + \| u_x \|_0^2 \le M_1 \| u \|_0^2 + M_2 \| f \|_0^2 \tag{11}$$

1) Let $\alpha > \beta$, then applying the fractional integration operator $D_{0t}^{-\alpha}$ to both sides of inequality (11), we obtain

$$\| u \|_0^2 + D_{0t}^{-(\alpha-\beta)} \| u_x \|_0^2 \le M_3 D_{0t}^{-\alpha} \| u \|_0^2 + M_4(D_{0t}^{-\alpha} \| f \|_0^2 + \| u_0(x) \|_0^2), \tag{12}$$

where $D_{0t}^{-\gamma} u = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{ud\tau}{(t-\tau)^{1-\gamma}}$ is the fractional Riemann-Liouville integral of order $\gamma, 0 < \gamma < 1$.

Based on Lemma 2 [7], from (12) we find an a priori estimate

$$\| u \|_1^2 \leq M_5 (D_{0t}^{-\alpha} \| f \|_0^2 + \| u_0(x) \|_0^2), \tag{13}$$

where $\| u \|_1^2 = \| u \|_0^2 + D_{0t}^{-(\alpha-\beta)} \| u_x \|_0^2$.

2) Let $\alpha = \beta$, then applying the fractional integration operator $D_{0t}^{-\alpha}$ to both sides of inequality (11), we obtain

$$\| u \|_0^2 + \| u_x \|_0^2 \leq M_6 D_{0t}^{-\alpha} \| u \|_0^2 + M_7 (D_{0t}^{-\alpha} \| f \|_0^2 + \| u_0(x) \|_0^2 + \| u'_0(x) \|_0^2) \tag{14}$$

Based on Lemma 2 [7], from (14) we find an a priori estimate

$$\| u \|_{W_2^1(0,l)}^2 \leq M_8 \left(D_{0t}^{-\alpha} \| f \|_0^2 + \| u_0(x) \|_{W_2^1(0,l)}^2 \right), \tag{15}$$

where $\| u \|_{W_2^1(0,l)}^2 = \| u \|_0^2 + \| u_x \|_0^2$.

3) Let $\alpha < \beta$, then applying the fractional integrationoperator $D_{0t}^{-\beta}$ to both sides of inequality (11), we obtain

$$\| u_x \|_0^2 + D_{0t}^{-(\beta-\alpha)} \| u \|_0^2 \leq M_9 D_{0t}^{-\beta} \| u \|_0^2 + M_{10} \left(D_{0t}^{-\beta} \| f \|_0^2 + \| u'_0(x) \|_0^2 \right) \tag{16}$$

Due to the condition $u(l,t) = 0$, the following identity holds [14, p.203]

$$u(x,t) = -\int_x^l u_x(x,t) dx,$$

then

$$u^2(x,t) = \left(-\int_x^l u_x(x,t) dx \right)^2 \leq (l-x) \int_x^l u_x^2(x,t) dx \leq l \int_0^l u_x^2(x,t) dx.$$

Integrating both sides over x from 0 to l, we obtain the inequality

$$\| u \|_0^2 \leq l^2 \| u_x \|_0^2 \tag{17}$$

Taking into account (17), from (16) we obtain

$$\| u_x \|_0^2 + D_{0t}^{-(\beta-\alpha)} \| u \|_0^2 \leq M_{11} D_{0t}^{-\beta} \| u_x \|_0^2 + M_{12} \left(D_{0t}^{-\beta} \| f \|_0^2 + \| u'_0(x) \|_0^2 \right).$$

Based on Lemma 2 [7], from the latter we find an a priori estimate

$$\| u \|_2^2 \leq M_{13} \left(D_{0t}^{-\beta} \| f \|_0^2 + \| u'_0(x) \|_0^2 \right), \tag{18}$$

where $\| u \|_2^2 = \| u_x \|_0^2 + D_{0t}^{-(\beta-\alpha)} \| u \|_0^2$.

2.3. Stability and convergence of the difference scheme

For an approximate solution of problem (1)-(3) we use the finite difference method. In the closed rectangle $\overline{Q}_T$ we introduce a uniform grid $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$, where $\overline{\omega}_h = \{x_i = ih, i = \overline{0,N}, h = l/N\}$, $\overline{\omega}_\tau = \{t_j = j\tau, j = 0,1,\ldots,j_0, \tau = T/j_0\}$.

On the uniform grid $\overline{\omega}_{h\tau}$, we associate the differential problem (1)-(3) with the difference scheme of the order of approximation $O(h^2 + \tau^2)$ for $\alpha = \beta$ and $O(h^2 + \tau^{2-\max\{\alpha,\beta\}})$ for $\alpha \neq \beta$:

$$\Delta_{0t_{j+\sigma}}^{\alpha} y = \left(a_i^j y_{\bar{x}}^{(\sigma)} \right)_{x,i} + \Delta_{0t_{j+\sigma}}^{\beta} (\gamma_i y_{\bar{x}})_{x,i} - d_i^j y_i^{(\sigma)} + \varphi_i^j, (x,t) \in \omega_{h,\tau}, \tag{19}$$

$$y_0^{(\sigma)} = y_N^{(\sigma)} = 0, \tag{20}$$

$$y(x,0) = u_0(x), \tag{21}$$

where $\Delta^{\gamma}_{0t_{j+\sigma}}y = \frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{s=0}^{j} c_{j-s}^{(\gamma,\sigma)} y_t^s$ is a discrete analogue of the fractional Gerasimova-Caputo derivative of order $\gamma, 0 < \gamma < 1$, which provides order of accuracy $O(\tau^{3-\gamma})$ if $\sigma = 1 - \frac{\gamma}{2}$, and $O(\tau^{2-\gamma})$ if $\sigma = 0.5$ [8].

$$a_0^{(\gamma,\sigma)} = \sigma^{1-\gamma}, a_l^{(\gamma,\sigma)} = (l+\sigma)^{1-\gamma} - (l-1+\sigma)^{1-\gamma}, l \geq 1,$$

$$b_l^{(\gamma,\sigma)} = \frac{1}{2-\gamma}[(l+\sigma)^{2-\gamma} - (l-1+\sigma)^{2-\gamma}] - \frac{1}{2}[(l+\sigma)^{1-\alpha} + (l-1+\sigma)^{1-\gamma}], l \geq 1,$$

$$\text{if } j = 0, c_0^{(\gamma,\sigma)} = a_0^{(\gamma,\sigma)};$$

$$\text{if } j > 0 c_s^{(\gamma,\sigma)} = \begin{cases} a_0^{(\gamma,\sigma)} + b_1^{(\gamma,\sigma)}, & s = 0; \\ a_s^{(\gamma,\sigma)} + b_{s+1}^{(\gamma,\sigma)} - b_s^{(\gamma,\sigma)}, & 1 \leq s \leq j-1; \\ a_j^{(\gamma,\sigma)} - b_j^{(\gamma,\sigma)}, & s = j, \end{cases}$$

$$\sigma = 1 - \frac{\gamma}{2}, \text{ if } \alpha = \beta \text{ and } \sigma = 0.5, \text{ if } \alpha \neq \beta,$$

$$c_s^{(\gamma,\sigma)} > \frac{1-\gamma}{2}(s+\sigma)^{-\gamma} > 0, \quad y^{(\sigma)} = \sigma y^{j+1} + (1-\sigma)y^j, a^{(+1)} = a_{i+1},$$

$$a_i^j = k(x_{i-0.5}, t_{j+\sigma}), \gamma_i = \eta(x_{i-0.5}), \varphi_i^j = f(x_i, t_{j+\sigma}), d_i^j = q(x_i, t_{j+\sigma}),$$

An a priori estimate of the solution of problem (19)-(21) will be found by the method of energy inequalities; then we multiply (19) scalarly by $y^{(\sigma)}$:

$$\left(\Delta^{\alpha}_{0t_{j+\sigma}}y, y^{(\sigma)}\right) = \left(\left(a y_{\bar{x}}^{(\sigma)}\right)_x, y^{(\sigma)}\right) + \left(\Delta^{\beta}_{0t_{j+\sigma}}(\gamma y_{\bar{x}})_x, y^{(\sigma)}\right) - \left(d y^{(\sigma)}, y^{(\sigma)}\right) + \left(\varphi, y^{(\sigma)}\right) \tag{22}$$

where scalar products and the norm have the form:

$$(u,v) = \sum_{i=1}^{N-1} u_i v_i h, (u,v] = \sum_{i=1}^{N} u_i v_i h, (u,u) = (1,u^2) = \| u \|_0^2.$$

Taking into account Lemma 1 [8], we transform the sums included in identity (22), then we get:

$$\left(\Delta^{\alpha}_{0t_{j+\sigma}}y, y^{(\sigma)}\right) \geq \frac{1}{2}\Delta^{\alpha}_{0t_{j+\sigma}} \| y \|_0^2 \tag{23}$$

$$\left((a y_{\bar{x}}^{(\sigma)})_x, y^{(\sigma)}\right) = a y_{\bar{x}}^{(\sigma)} y^{(\sigma)}|_0^N - \left(a, (y_{\bar{x}}^{(\sigma)})^2\right] \tag{24}$$

$$\left(\Delta^{\beta}_{0t_{j+\sigma}}(\gamma y_{\bar{x}})_x, y^{(\sigma)}\right) = \gamma y^{(\sigma)} \Delta^{\beta}_{0t_{j+\sigma}} y_{\bar{x}}|_0^N - \left(\gamma, y_{\bar{x}}^{(\sigma)} \Delta^{\beta}_{0t_{j+\sigma}}(y_{\bar{x}})\right] \leq -\left(\frac{\gamma}{2}, \Delta^{\beta}_{0t_{j+\sigma}}(y_{\bar{x}})^2\right] \leq -\frac{c_0}{2}\Delta^{\beta}_{0t_{j+\sigma}} \| y_{\bar{x}}]|_0^2 \tag{25}$$

$$-\left(d y^{(\sigma)}, y^{(\sigma)}\right) \leq c_2 \| y^{(\sigma)} \|_0^2 \tag{26}$$

$$\left(\varphi, y^{(\sigma)}\right) \leq \frac{1}{2} \| y^{(\sigma)} \|_0^2 + \frac{1}{2} \| \varphi \|_0^2 \tag{27}$$

From (22), taking into account (23)-(27), we obtain

$$\Delta^{\alpha}_{0t_{j+\sigma}} \| y \|_0^2 + \Delta^{\beta}_{0t_{j+\sigma}} \| y_{\bar{x}}]|_0^2 + \| y_{\bar{x}}^{(\sigma)}]|_0^2 \leq M_1 \| y^{(\sigma)} \|_0^2 + M_2 \| \varphi \|_0^2.$$

We rewrite the latter in a different form

$$\Delta^{\alpha}_{0t_{j+\sigma}} \| y \|_0^2 + \Delta^{\beta}_{0t_{j+\sigma}} \| y_{\bar{x}}]|_0^2 \leq M_3^{\sigma} \| y^{j+1} \|_0^2 + M_4^{\sigma} \| y^j \|_0^2 + M_2 \| \varphi \|_0^2 \tag{28}$$

1) Let $\alpha > \beta$, then on the basis of Lemma 7 [9], from (28) we obtain

$$\| y^{j+1} \|_0^2 \leq M_5 \left(\| y^0 \|_0^2 + \max_{0 \leq j' \leq j} \| \varphi^{j'} \|_0^2 \right) \tag{29}$$

2) Let $\alpha = \beta$, then by virtue of (20) and Lemma 7 [9], from (28) we obtain:

$$\| y^{j+1} \|_{W_2^1(0,l)}^2 \leq M_6 \left(\| y^0 \|_{W_2^1(0,l)}^2 + \max_{0 \leq j' \leq j} \| \varphi^{j'} \|_0^2 \right), \tag{30}$$

where $\| y^0 \|_{W_2^1(0,l)}^2 = \| y^0 \|_0^2 + \| y_{\bar{x}}^0]|_0^2$.

3) Let $\alpha < \beta$, then by virtue of (20), the inequality $\| y \|_0^2 \leq l^2 \| y_{\bar{x}}]|_0^2$ [13], and the lemma 7 [9], from (28) we get:

$$\| y_{\bar{x}}^{j+1}]|_0^2 \leq M_7 (\| y_{\bar{x}}^0]|_0^2 + \max_{0 \leq j' \leq j} \| \varphi^{j'} \|_0^2) \tag{31}$$

We investigate the question of the solvability of problem (19)-(21). For this consider a homogeneous problem ($\varphi = 0,\ u_0(x) = 0$) whose solution certainly exists

$$\Delta_{0t_{j+\sigma}}^{\alpha} y = \left(a_i^j y_{\bar{x}}^{(\sigma)} \right)_{x,i} + \Delta_{0t_{j+\sigma}}^{\beta} (\gamma_i y_{\bar{x}})_{x,i} - d_i^j y_i^{(\sigma)}, \tag{32}$$

$$y_0^{(\sigma)} = y_N^{(\sigma)} = 0, y(x,0) = 0 \tag{33}$$

Let $y(x,t)$ be one of the solutions of the homogeneous problem (32)-(33). From inequality (29) it follows that $\| y^{j+1} \|_0^2 \leq 0$, from (30) $-\| y^{j+1} \|_{W_2^1(0,l)}^2 \leq 0$, and from (31) $-\| y_{\bar{x}}^{j+1} \|_0^2 \leq 0$, but $\| y^{j+1} \|_0^2 = 0$ and $\| y_{\bar{x}}^{j+1} \|_0^2 = 0$ only when $y = 0$, $x \in \overline{w}_h$. Therefore, from a priori estimates (29) for $\alpha > \beta$, (30) for $\alpha = \beta$, (31) for $\alpha < \beta$ it follows that the only solution of the homogeneous problem (32)-(33) is $y = 0$. Thus, a solution of problem (19)-(21) exists and is unique for any $\varphi, u_0(x)$.

Let $u(x,t)$ be a solution of problem (1)-(3), $y(x_i, t_j) = y_i^j$ be a solution of the difference problem (19)-(21). To evaluate the accuracy of difference scheme (19)-(21), consider the difference $z_i^j = y_i^j - u_i^j$, where $u_i^j = u(x_i, t_j)$. Then, substituting $y = z + u$ into relations (19)-(21), we obtain the problem for the function z:

$$\Delta_{0t_{j+\sigma}}^{\alpha} z = \left(a_i^j z_{\bar{x}}^{(\sigma)} \right)_{x,i} + \Delta_{0t_{j+\sigma}}^{\beta} (\gamma_i z_{\bar{x}})_{x,i} - d_i^j z_i^{(\sigma)} + \Psi_i^j, (x,t) \in \omega_{h,\tau}, \tag{34}$$

$$z_0^{(\sigma)} = z_N^{(\sigma)} = 0, \tag{35}$$

$$z(x,0) = 0, x \in \overline{\omega}_h, \tag{36}$$

where $\Psi = O(h^2 + \tau^2)$ if $\alpha = \beta$ and $\Psi = O(h^2 + \tau^{2-\max\{\alpha,\beta\}})$ if $\alpha \neq \beta$ are approximation errors of differential problem (1)-(3) by difference scheme (19)-(21) in the class of the solution $u = u(x,t)$ of problem (1)-(3).

Due to the fact that problem (34)-(36) is linear, we have:

1) Let $\alpha > \beta$, then applying a priori estimate (29) to the solution of problem (34)-(36), we obtain the inequality

$$\| z^{j+1} \|_0^2 \leq M \max_{0 \leq j' \leq j} \| \Psi^{j'} \|_0^2 \tag{37}$$

2) Let $\alpha = \beta$, then applying a priori estimate (30) to the solution of problem (34)-(36), we obtain the inequality

$$\| z^{j+1} \|^2_{W_2^1(0,l)} \leq M \max_{0\leq j' \leq j} \| \Psi^{j'} \|^2_0 \tag{38}$$

3) Let $\alpha < \beta$, then applying a priori estimate (31) to the solution of problem (34)-(36), we obtain the inequality

$$\| z_{\bar{x}}^{j+1}] |^2_0 \leq M \max_{0\leq j' \leq j} \| \Psi^{j'} \|^2_0 \tag{39}$$

2.4. Formulation of a boundary value problem for an equation with a Bessel operator and a priori estimate in differential form

In a closed rectangle $\overline{Q}_T = \{(x,t): 0 \leq x \leq l,\ 0 \leq t \leq T\}$ consider the problem for a hyperbolic equation of fractional order with the Bessel operator

$$\partial_{0t}^{\alpha} u = \frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m k(x,t)\frac{\partial u}{\partial x}\right) + \frac{1}{x^m}\partial_{0t}^{\beta}\frac{\partial}{\partial x}\left(x^m \eta(x)\frac{\partial u}{\partial x}\right) + r(x,t)\frac{\partial u}{\partial x} - q(x,t)u(x,t) + f(x,t),\ \ 0 < x < l, 0 < t \leq T, \tag{40}$$

$$\lim_{x\to 0} x^m \Pi(x,t) = 0,\ \ 0 \leq t \leq T, \tag{41}$$

$$u(l,t) = 0,\ \ 0 \leq t \leq T, \tag{42}$$

$$u(x,0) = u_0(x),\ \ 0 \leq x \leq l, \tag{43}$$

where

$$0 < c_0 \leq k(x,t), \eta(x) \leq c_1,\ \ q(x,t) \geq 0,$$

$$|r(x,t)|, |r_x(x,t)|, |k_x(x,t)| \leq c_2,\ \ m \geq 0, \tag{44}$$

For $x = 0$, the boundedness condition for the solution $|u(0,t)| < \infty$ is set, which is equivalent to condition (41), which in turn is equivalent to the identity $\Pi(0,t) = 0$ [13, p. 173] if the functions $r(0,t), k(0), q(0,t), f(0,t)$ and the integral are finite, $\Pi(x,t) = k(x,t)u_x(x,t) + \eta(x)\partial_{0t}^{\beta} u_x(x,t)$.

To obtain an a priori estimate for the solution of problem (40)-(43) in differential form, we multiply equation (40) scalarly by $x^m u$:

$$(\partial_{0t}^{\alpha} u, x^m u) = ((x^m k u_x)_x, u) + (\partial_{0t}^{\beta}(x^m \eta u_x)_x, u) + (r u_x, x^m u) - (q u(x,t), x^m u) + (f, x^m u) \tag{45}$$

where $(u,v) = \int_0^l uv dx, (u,u) = \| u \|^2_0$, where u, v are functions defined on $[0,l]$.

Let us transform the integrals in identity (45) using Cauchy inequality with ε ([13, p. 100]) and Lemma 1; then from [7]we find:

$$(\partial_{0t}^{\alpha} u, x^m u) \geq \frac{1}{2}\left(x^m, \partial_{0t}^{\alpha}(u^2)\right) = \frac{1}{2}\partial_{0t}^{\alpha} \| x^{\frac{m}{2}} u \|^2_0, \tag{46}$$

$$((x^m k u_x)_x, u) = \int_0^l u(x^m k u_x)_x dx = x^m u k u_x|_0^l - \int_0^l x^m k u_x^2 dx \leq x^m u k u_x|_0^l - c_0 \| x^{\frac{m}{2}} u_x \|^2_0 \tag{47}$$

$$\left(\partial_{0t}^{\beta}(x^m \eta u_x)_x, u\right) = \int_0^l u \partial_{0t}^{\beta}(x^m \eta u_x)_x dx = x^m \eta u \partial_{0t}^{\beta} u_x|_0^l - \int_0^l x^m \eta(x) u_x \partial_{0t}^{\beta} u_x dx \leq x^m \eta u \partial_{0t}^{\beta} u_x|_0^l - \frac{1}{2}\int_0^l \eta \partial_{0t}^{\beta}(x^{\frac{m}{2}} u_x)^2 dx \tag{48}$$

$$(ru_x, x^m u) = \int_0^l rx^m uu_x dx \le \frac{c_2^2}{4\varepsilon}\int_0^l x^m u^2 dx + \varepsilon \int_0^l x^m u_x^2 dx \le \varepsilon \parallel x^{\frac{m}{2}} u_x \parallel_0^2 + M_1(\varepsilon) \parallel x^{\frac{m}{2}} u \parallel_0^2, \tag{49}$$

$$-(q(x,t)u, x^m u) = -\left(q(x,t), (x^{\frac{m}{2}} u)^2\right), \tag{50}$$

$$(f, x^m u) = \int_0^l x^m fu dx \le \frac{1}{2} \parallel x^{\frac{m}{2}} u \parallel_0^2 + \frac{1}{2} \parallel x^{\frac{m}{2}} f \parallel_0^2 \tag{51}$$

Taking into account conditions (41), (42) and transformations (46)-(51), from (45) we find

$$\partial_{0t}^{\alpha} \parallel x^{\frac{m}{2}} u \parallel_0^2 + \partial_{0t}^{\beta} \int_0^l \eta (x^{\frac{m}{2}} u_x)^2 dx + \parallel x^{\frac{m}{2}} u_x \parallel_0^2 \le \varepsilon M_2 \parallel x^{\frac{m}{2}} u_x \parallel_0^2 + M_3(\varepsilon) \parallel x^{\frac{m}{2}} u \parallel_0^2 + M_4 \parallel x^{\frac{m}{2}} f \parallel_0^2 \tag{52}$$

1) Consider the case when $\alpha > \beta$, then applying the fractional integrationoperator $D_{0t}^{-\alpha}$ to both sides of inequality (52), for $\varepsilon = \frac{1}{2M_2}$ we get

$$\parallel x^{\frac{m}{2}} u \parallel_0^2 + D_{0t}^{-(\alpha-\beta)} \parallel x^{\frac{m}{2}} u_x \parallel_0^2 \le M_5 D_{0t}^{-\alpha} \parallel x^{\frac{m}{2}} u \parallel_0^2 + M_6 \left(D_{0t}^{-\alpha} \parallel x^{\frac{m}{2}} f \parallel_0^2 + \parallel x^{\frac{m}{2}} u_0(x) \parallel_0^2\right) \tag{53}$$

Based on Lemma 2 [7], from (53) we find the a priori estimate

$$\parallel x^{\frac{m}{2}} u \parallel_0^2 + D_{0t}^{-(\alpha-\beta)} \parallel x^{\frac{m}{2}} u_x \parallel_0^2 \le M_7 \left(D_{0t}^{-\alpha} \parallel x^{\frac{m}{2}} f \parallel_0^2 + \parallel x^{\frac{m}{2}} u_0(x) \parallel_0^2\right) \tag{54}$$

2) Consider the case when $\alpha = \beta$, then applying the fractional integration operator $D_{0t}^{-\alpha}$ to both sides of inequality (52), we obtain

$$\parallel x^{\frac{m}{2}} u \parallel_0^2 + \parallel x^{\frac{m}{2}} u_x \parallel_0^2 + D_{0t}^{-\alpha} \parallel x^{\frac{m}{2}} u_x \parallel_0^2 \le M_8 D_{0t}^{-\alpha} \left(\parallel x^{\frac{m}{2}} u \parallel_0^2 + \parallel x^{\frac{m}{2}} u_x \parallel_0^2\right) + M_9 \left(D_{0t}^{-\alpha} \parallel x^{\frac{m}{2}} f \parallel_0^2 + \parallel x^{\frac{m}{2}} u_0(x) \parallel_0^2 + \parallel x^{\frac{m}{2}} u'_0(x) \parallel_0^2\right) \tag{55}$$

Based on Lemma 2 [7], from (55) we find the a priori estimate

$$\parallel x^{\frac{m}{2}} u \parallel_0^2 + \parallel x^{\frac{m}{2}} u_x \parallel_0^2 + D_{0t}^{-\alpha} \parallel x^{\frac{m}{2}} u_x \parallel_0^2 \le M_{10} \left(D_{0t}^{-\alpha} \parallel x^{\frac{m}{2}} f \parallel_0^2 + \parallel x^{\frac{m}{2}} u_0(x) \parallel_0^2 + \parallel x^{\frac{m}{2}} u'_0(x) \parallel_0^2\right) \tag{56}$$

3) Let us consider the case when $\alpha < \beta$, then applying the fractional integrationoperator $D_{0t}^{-\beta}$ to both parts of inequality (52), for $\varepsilon = \frac{1}{2M_2}$ we get

$$\parallel x^{\frac{m}{2}} u_x \parallel_0^2 + D_{0t}^{-(\beta-\alpha)} \parallel x^{\frac{m}{2}} u \parallel_0^2 \le M_{11} D_{0t}^{-\beta} \parallel x^{\frac{m}{2}} u \parallel_0^2 + M_{12} \left(D_{0t}^{-\beta} \parallel x^{\frac{m}{2}} f \parallel_0^2 + \parallel x^{\frac{m}{2}} u'_0(x) \parallel_0^2\right) \tag{57}$$

Then from (57), in view of condition (17) we obtain

$$\parallel x^{\frac{m}{2}} u_x \parallel_0^2 + D_{0t}^{-(\beta-\alpha)} \parallel x^{\frac{m}{2}} u \parallel_0^2 \le$$
$$\le M_{13} D_{0t}^{-\beta} \parallel x^{\frac{m}{2}} u_x \parallel_0^2 + M_{14} \left(D_{0t}^{-\beta} \parallel x^{\frac{m}{2}} f \parallel_0^2 + \parallel x^{\frac{m}{2}} u'_0(x) \parallel_0^2\right).$$

Based on Lemma 2 [7], from the latter we find an a priori estimate

$$\| x^{\frac{m}{2}} u_x \|_0^2 + D_{0t}^{-(\beta-\alpha)} \| x^{\frac{m}{2}} u \|_0^2 \leq M_{15} \left(D_{0t}^{-\beta} \| x^{\frac{m}{2}} f \|_0^2 + \| x^{\frac{m}{2}} u'_0(x) \|_0^2 \right) \quad (58)$$

2.5. Stability and convergence of the difference scheme

On the uniform grid $\overline{\omega}_{h\tau}$, we associate the differential problem (40)-(43) with the difference scheme of the order of approximation $O(h^2+\tau^2)$ for $\alpha=\beta$ and $O(h^2+\tau^{2-\max\{\alpha,\beta\}})$ for $\alpha \neq \beta$:

$$\overline{\chi}\Delta_{0t_{j+\sigma}}^{\alpha} y = \frac{\chi}{x_i^m}(x_{i-0.5}^m a_i^j y_{\bar{x}}^{(\sigma)})_x + \frac{1}{x_i^m}\Delta_{0t_{j+\sigma}}^{\beta}(x_{i-0.5}^m \gamma_i y_{\bar{x},i})_x + \frac{b^{-j}}{x_i^m}\left(x_{i-0.5}^m a_i^j y_{\bar{x},i}^{(\sigma)}\right) + \frac{b^{+j}}{x_i^m}\left(x_{i+0.5}^m a_{i+1}^j y_{x,i}^{(\sigma)}\right) - d_i^j y_i^{(\sigma)} + \varphi_i^j, (x,t) \in \omega_{h,\tau}, \quad (59)$$

$$\chi_0 a_1 y_{(x,0)}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\beta}\left(\gamma_1 y_{x,0}\right) = \frac{0.5h}{m+1}\left(\Delta_{0t_{j+\sigma}}^{\alpha} y_0 + d_0^j y_0^{(\sigma)}\right) - \mu, t \in \overline{\omega}_\tau, x = 0, \quad (60)$$

$$y_N^{(\sigma)} = 0, t \in \overline{\omega}_\tau, x = l, \quad (61)$$

$$y(x,0) = u_0(x), x \in \overline{\omega}_h, t = 0, \quad (62)$$

where

$$a_i^j = k\left(x_{i-0.5}, t_{j+\sigma}\right), \ \ \gamma_i = \eta(x_{i-0.5}), \ \ b_i^{\pm j} = \frac{\overline{\chi}_i r^{\pm j}(x, t_{j+\sigma})}{k(x_i, t_{j+\sigma})},$$

$$d_i^j = \begin{cases} \overline{\chi}_i d_i^{j+\sigma}, i \neq 0, N, \\ d_i^{j+\sigma}, i = 0, N. \end{cases} \quad \varphi_i^j = \begin{cases} \overline{\chi}_i f_i^{j+\sigma}, i \neq 0, N, \\ f_i^{j+\sigma}, i = 0, N. \end{cases} \quad \hbar = \begin{cases} 0.5h, i = 0, \\ h, i \neq 0, N, \end{cases}$$

$$r\left(x, t_{j+\sigma}\right) = r^+\left(x, t_{j+\sigma}\right) + r^-\left(x, t_{j+\sigma}\right), \ \ |r(x, t_{j+\sigma})| = r^+(x, t_{j+\sigma}) - r^-(x, t_{j+\sigma}),$$

$$\overline{\chi}_i = 1 + \frac{m(m-1)h^2}{24x_i^2}, i = \overline{1, N-1}, \ \ r^+(x, t_{j+\sigma}) = 0.5(r(x, t_{j+\sigma}) + |r(x, t_{j+\sigma})|) \geq 0,$$

$$r^-\left(x, t_{j+\sigma}\right) = 0.5\left(r\left(x, t_{j+\sigma}\right) - \left|r\left(x, t_{j+\sigma}\right)\right|\right) \leq 0, \ \ \chi_0 = \frac{1}{1 + \frac{0.5h|r_0|}{(m+1)a_1}}, \ \ \overline{x}^m = x_{i-0.5}^m,$$

$$\mu = \frac{0.5h}{m+1}\varphi_0, \ \ r_N = r(l,t) = r_N^{j+\sigma} \geq 0, \ \ r_0 = r(0,t) = r_0^{j+\sigma} \leq 0,$$

$$y^{(\sigma)} = \sigma y^{j+1} + (1-\sigma) y^j, a^{(+1)} = a_{i+1}, \ \ \chi(x_i, t_{j+\sigma}) = \frac{1}{1 + R(x_i, t_{j+\sigma})},$$

$R(x_i, t_{j+\sigma}) = \frac{0.5h|r(x_i, t_{j+\sigma})|\overline{\chi}_i}{k(x_{i-0.5}, t_{j+\sigma})}$ is the difference Reynolds number.

$$Y = \hat{y} + y, \ \ \hat{y} = y^{j+1}, \ \ y_t = \frac{\hat{y} - y}{\tau}, \ \ y = y_i^j = y(x_i, t_j), \ \ t^* = t_{j+\sigma}.$$

An a priori estimate of the solution of problem (59)-(62) will be found by the method of energy inequalities; for this we multiply (59) scalarly by $x^m y^{(\sigma)}$:

$$(\overline{\chi}\Delta_{0t_{j+\sigma}}^{\alpha} y, x^m y^{(\sigma)}) = (\chi(x_{i-0.5}^m a_i^j y_{\bar{x}}^{(\sigma)})_x, y^{(\sigma)}) + (\Delta_{0t_{j+\sigma}}^{\beta}(x_{i-0.5}^m \gamma_i y_{\bar{x},i})_x, y^{(\sigma)}) + (b^{-j} x_{i-0.5}^m a_i^j y_{\bar{x},i}^{(\sigma)}, y^{(\sigma)}) + (b^{+j} x_{i+0.5}^m a_{i+1}^j y_{x,i}^{(\sigma)}, y^{(\sigma)}) - (d_i^j y_i^{(\sigma)}, y^{(\sigma)}) + \left(\varphi, x^m y^{(\sigma)}\right) \quad (63)$$

We estimate the sums in (63), taking into account Lemma 1 [8]:

$$(\overline{\chi}\Delta_{0t_{j+\sigma}}^{\alpha} y, x^m y^{(\sigma)}) \geq M_1(\tfrac{1}{2}, \Delta_{0t_{j+\sigma}}^{\alpha}(x^{\frac{m}{2}} y)^2) \geq \tfrac{1}{4}(1, \Delta_{0t_{j+\sigma}}^{\alpha}(x^{\frac{m}{2}} y)^2) \geq \tfrac{1}{4}\Delta_{0t_{j+\sigma}}^{\alpha} \| x^{\frac{m}{2}} y \|_0^2 \quad (64)$$

where

$$M_1 = \begin{cases} 1, if m = 0, m \geq 1 \\ \frac{1}{2}, if m \in (0,1), h \leq h_0 = \sqrt{\frac{12x^2}{m(1-m)}}. \end{cases}$$

$$(\chi(x_{i-0.5}^m a_i^j y_{\bar{x}}^{(\sigma)})_x, y^{(\sigma)}) = \chi x_{i-0.5}^m a_i^j y_{\bar{x}}^{(\sigma)} y^{(\sigma)}|_0^N - (x_{i-0.5}^m a_i^j y_{\bar{x}}^{(\sigma)}, (\chi y^{(\sigma)})_{\bar{x}}] = \chi x_{i-0.5}^m a_i^j y^{(\sigma)} y_{\bar{x}}^{(\sigma)}|_0^N - (\bar{x}^m a \chi_{\bar{x}}, y_{\bar{x}}^{(\sigma)} y^{(\sigma)}] - (x_{i-0.5}^m a \chi^{(-1)}, (y_{\bar{x}}^{(\sigma)})^2] \leq \chi x_{i-0.5}^m a_i y^{(\sigma)} y_{\bar{x}}^{(\sigma)}|_0^N - (\bar{x}^m a \chi_{\bar{x}}, y_{\bar{x}}^{(\sigma)} y^{(\sigma)} - \frac{1}{1+hM_2}(x_{i-0.5}^m a \chi, (y_{\bar{x}}^{(\sigma)})^2] \leq x_{i-0.5}^m y^{(\sigma)} \chi a_i^j y_{\bar{x}}^{(\sigma)}|_0^N - M_3 \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + \frac{\varepsilon}{2} \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + M_4^{\varepsilon} \parallel x^{\frac{m}{2}} y^{(\sigma)} \parallel_0^2 \quad (65)$$

$$\left(\Delta_{0t_{j+\sigma}}^{\beta}(\bar{x}^m \gamma_i y_{\bar{x},i})_x, y^{(\sigma)}\right) = \bar{x}^m y^{(\sigma)} \Delta_{0t_{j+\sigma}}^{\beta}(\gamma_i y_{\bar{x}})|_0^N - \left(\bar{x}^m y_{\bar{x}}^{(\sigma)}, \Delta_{0t_{j+\sigma}}^{\beta}(\gamma_i y_{\bar{x},i})\right] \leq y^{(\sigma)} \bar{x}^m \Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_i y_{\bar{x}})|_0^N - \frac{c_0}{2} \Delta_{0t_{j+\sigma}}^{\beta} \parallel \bar{x}^m y_{\bar{x}} \Big]|_0^2 \quad (66)$$

$$(b^{-j} x_{i-0.5}^m a_i^j y_{\bar{x},i}^{(\sigma)}, y^{(\sigma)}) + (b^{+j} x_{i+0.5}^m a_{i+1}^j y_{x,i}^{(\sigma)}, y^{(\sigma)}) - (d_i^j y^{(\sigma)}, y^{(\sigma)}) \leq \frac{\varepsilon}{2} \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + M_5^{\varepsilon} \parallel x^{\frac{m}{2}} y^{(\sigma)} \parallel_0^2 \quad (67)$$

$$\left(\varphi, x^m y^{(\sigma)}\right) \leq \frac{1}{2} \parallel x^{\frac{m}{2}} y^{(\sigma)} \parallel_0^2 + \frac{1}{2} \parallel x^{\frac{m}{2}} \varphi \parallel_0^2 \quad (68)$$

Taking into account transformations (64)-(68), from (63) we find

$$\frac{1}{4} \Delta_{0t_{j+\sigma}}^{\alpha} \parallel x^{\frac{m}{2}} y \parallel_0^2 + M_3 \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + \frac{c_0}{2} \Delta_{0t_{j+\sigma}}^{\beta} \parallel \bar{x}^m y_{\bar{x}}]|_0^2 \leq \bar{x}^m y^{(\sigma)} (\chi a y_{\bar{x}}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_i y_{\bar{x}}))|_0^N + \varepsilon \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + M_6^{\varepsilon} \parallel x^{\frac{m}{2}} y^{(\sigma)} \parallel_0^2 + \frac{1}{2} \parallel x^{\frac{m}{2}} \varphi \parallel_0^2 \quad (69)$$

We transform the first expression on the right-hand side of (69), then we get

$$\bar{x}^m y^{(\sigma)} [\chi a y_{\bar{x}}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\beta}(\gamma_i y_{\bar{x}})]|_0^N = -x_{0.5}^m y_0^{(\sigma)} (\chi_0 a_1 y_{x,0}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\beta} \gamma_1 y_{x,0}) = -x_{0.5}^m y_0^{(\sigma)} [\frac{0.5h}{m+1} (\Delta_{0t_{j+\sigma}}^{\alpha} y_0 + d_0^j y_0^{(\sigma)}) - \mu] = x_{0.5}^m y_0^{(\sigma)} \mu - \frac{0.5h}{m+1} x_{0.5}^m y_0^{(\sigma)} \Delta_{0t_{j+\sigma}}^{\alpha} y_0 - \frac{0.5h}{m+1} x_{0.5}^m d_0^j (y_0^{(\sigma)})^2 \leq M_7^{\varepsilon_1} \left(x_{0.5}^{\frac{m}{2}} \varphi_0\right)^2 h - \frac{0.5h}{m+1} x_{0.5}^m d_0^j (y_0^{(\sigma)})^2 + M_8^{\varepsilon_1} \parallel x^{\frac{m}{2}} y \parallel_0^2 + \varepsilon_1 (x_{0.5}^{\frac{m}{2}} y_0^{(\sigma)})^2 h - \frac{0.5h}{2(m+1)} \Delta_{0t_{j+\sigma}}^{\alpha} (x_{0.5}^{\frac{m}{2}} y_0)^2 \quad (70)$$

Taking into account (70), from (69) with $\varepsilon = \frac{M_3}{2}$ we obtain

$$\Delta_{0t_{j+\sigma}}^{\alpha} \parallel x^{\frac{m}{2}} y \parallel_1^2 + \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + \Delta_{0t_{j+\sigma}}^{\beta} \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}]|_0^2 \leq M_9 \parallel x^{\frac{m}{2}} y^{(\sigma)} \parallel_1^2 + M_{10} \left(\parallel x^{\frac{m}{2}} \varphi \parallel_0^2 + \left(x_{0.5}^{\frac{m}{2}} \varphi_0\right)^2 h \right) \quad (71)$$

where $\parallel x^{\frac{m}{2}} y \parallel_1^2 = \parallel x^{\frac{m}{2}} y \parallel_0^2 + (x_{0.5}^{\frac{m}{2}} y_0)^2$.

We rewrite (71) in another form

$$\Delta_{0t_{j+\sigma}}^{\alpha} \parallel x^{\frac{m}{2}} y \parallel_1^2 + \Delta_{0t_{j+\sigma}}^{\beta} \parallel \bar{x}^{\frac{m}{2}} y_{\bar{x}}]|_0^2 \leq M_{11}^{\sigma} \parallel x^{\frac{m}{2}} y^{j+1} \parallel_1^2 + M_{12}^{\sigma} \parallel x^{\frac{m}{2}} y^j \parallel_1^2 +$$

$$M_{13}\left(\| x^{\frac{m}{2}}\varphi \|_0^2 + \left(x_{0.5}^{\frac{m}{2}}\varphi_0\right)^2 h\right) \tag{72}$$

1) Consider the case when $\alpha > \beta$; then, based on Lemma 7 [9], from (72) we obtain an a priori estimate

$$\| x^{\frac{m}{2}} y^{j+1} \|_1^2 \leq M_{14}\left(\| x^{\frac{m}{2}} y^0 \|_1^2 + \max_{0\leq j'\leq j}\left(\| x^{\frac{m}{2}}\varphi \|_0^2 + \left(x_{0.5}^{\frac{m}{2}}\varphi_0\right)^2 h\right)\right) \tag{73}$$

2)Consider the case when $\alpha = \beta$; then, based on Lemma 7 [9], from (72) we obtain an a priori estimate

$$\| x^{\frac{m}{2}} y^{j+1} \|_1^2 + \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}]|_0^2 \leq M_{15}(\| x^{\frac{m}{2}} y^0 \|_1^2 + \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}^0]|_0^2 + \max_{0\leq j'\leq j}(\| x^{\frac{m}{2}}\varphi \|_0^2 + \left(x_{0.5}^{\frac{m}{2}}\varphi_0\right)^2 h)) \tag{74}$$

3) Consider the case when $\alpha < \beta$. By virtue of the condition $y(l,t) = 0$, the following equality holds:

$$y_i = y_N - \sum_{s=i}^{N} y_{\bar{x},s} h = -\sum_{s=i}^{N} y_{\bar{x},s} h.$$

We square both sides, then, based on the Cauchy-Schwarz inequality, we get

$$y_i^2 = \left(-\sum_{s=i}^{N} y_{\bar{x},s} h\right)^2 \leq \sum_{s=i}^{N} y_{\bar{x},s}^2 h \sum_{s=i}^{N} h = (x_N - x_i)\sum_{s=i}^{N} y_{\bar{x},s}^2 h \leq l\sum_{s=i}^{N} y_{\bar{x},s}^2 h.$$

We now multiply both sides by h and sum over i from 1 to N

$$\sum_{i=1}^{N} y_i^2 h \leq \sum_{i=1}^{N}\left(l\sum_{s=i}^{N} y_{\bar{x},s}^2 h\right) h = \sum_{s=1}^{N} y_{\bar{x},s}^2 h \sum_{s=1}^{N} lh \leq l^2 \sum_{s=1}^{N} y_{\bar{x},s}^2 h \leq l^2 \| y_{\bar{x}}]|_0^2.$$

Therefore,

$$\| y]|_0^2 \leq l^2 \| y_{\bar{x}}]|_0^2 \tag{75}$$

Then, by virtue of (75), Lemma 7 [9], from (72) we obtain

$$\Delta_{0t_{j+\sigma}}^{\alpha}\left(\| x^{\frac{m}{2}} y \|_0^2 + (x_{0.5}^{\frac{m}{2}} y_0)^2\right) + \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + \Delta_{0t_{j+\sigma}}^{\beta} \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}]|_0^2 \leq M_{16}^{\varepsilon_1} \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + \varepsilon_1 M_{17}(x_{0.5}^{\frac{m}{2}} y_{\bar{x},1}^{(\sigma)})^2 h + M_{18}^{\varepsilon_1}(\| x^{\frac{m}{2}}\varphi \|_0^2 + \left(x_{0.5}^{\frac{m}{2}}\varphi_0\right)^2 h) \tag{76}$$

Choosing $\varepsilon_1 = \frac{1}{2M_{17}}$ in (76), we obtain

$$\Delta_{0t_{j+\sigma}}^{\alpha} \| x^{\frac{m}{2}} y \|_0^2 + \Delta_{0t_{j+\sigma}}^{\beta} \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}]|_0^2 \leq M_{19} \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]|_0^2 + M_{20}\left(\| x^{\frac{m}{2}}\varphi \|_0^2 + \left(x_{0.5}^{\frac{m}{2}}\varphi_0\right)^2 h\right), \tag{77}$$

We rewrite (77) in another form

$$\Delta_{0t_{j+\sigma}}^{\alpha} \| x^{\frac{m}{2}} y \|_0^2 + \Delta_{0t_{j+\sigma}}^{\beta} \| x^{\frac{m}{2}} y_{\bar{x}}]|_0^2 \leq$$
$$M_{21}^{\sigma} \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{j+1}]|_0^2 + M_{22}^{\sigma} \| \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{j}]|_0^2 + M_{23} F^j,$$

where $F^j = \| x^{\frac{m}{2}}\varphi \|_0^2 + \left(x_{0.5}^{\frac{m}{2}}\varphi_0\right)^2 h.$

Then, based on Lemma 7 [9], from the latter we obtain an a priori estimate

$$\| x^{\frac{m}{2}} y_{\bar{x}}^{j+1}]|_0^2 \le M_{24}(\| \bar{x}^{\frac{m}{2}} y_{\bar{x}}^{0}]|_0^2 + \max_{0\le j'\le j}(\| x^{\frac{m}{2}}\varphi \|_0^2 + \left(x_{0.5}^{\frac{m}{2}}\varphi_0\right)^2 h)) \tag{78}$$

where $M_{14}, M_{15}, M_{24} = const > 0$, independents of h and τ.

2.6. Algorithm for the numerical solution to differential problem (1)-(3)

For the numerical solution to problem (1)-(3), we bring the difference scheme (19)-(21) to the calculated form. Then equation (1) is reduced to the following form

$$A_i y_{i-1}^{j+1} - C_i y_i^{j+1} + B_i y_{i+1}^{j+1} = -F_i^j, i = \overline{1, N-1},$$

where

$$A_i = \tau\sigma a_i^j + \gamma_i \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)}, B_i = \tau\sigma a_{i+1}^j + \gamma_{i+1} \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)},$$

$$C_i = A_i + B_i + h^2 \frac{\tau^{1-\alpha} c_0^{(\alpha,\sigma)}}{\Gamma(2-\alpha)} + h^2\tau\sigma d_i,$$

$$F_i^j = AA_i y_{i-1}^j - CC_i y_i^j + BB_i y_{i+1}^j + h^2\tau\varphi_i^j - h^2 \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j-1} c_{j-s}^{(\alpha,\sigma)} (y_i^{s+1} - y_i^s) +$$

$$+ \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} ((\gamma_{i+1} y_{i+1})^{s+1} - (\gamma_{i+1} y_{i+1})^s) -$$

$$- \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} (((\gamma_i + \gamma_{i+1}) y_i)^{s+1} - ((\gamma_i + \gamma_{i+1}) y_i)^s) +$$

$$+ \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} ((\gamma_i y_{i-1})^{s+1} - (\gamma_i y_{i-1})^s),$$

$$AA_i = \tau(1-\sigma) a_i^j - \gamma_i \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)}, BB_i = \tau(1-\sigma) a_{i+1}^j - \gamma_{i+1} \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)},$$

$$CC_i = AA_i + BB_i - h^2 \frac{\tau^{1-\alpha} c_0^{(\alpha,\sigma)}}{\Gamma(2-\alpha)} + h^2\tau(1-\sigma) d_i.$$

2.7 Algorithm for the numerical solution to differential problem (40)-(43)

For the numerical solution to problem (40)-(43), we bring the difference scheme (59)-(62) to the calculated form. Then equation (40) is reduced to the following form

$$A_i y_{i-1}^{j+1} - C_i y_i^{j+1} + B_i y_{i+1}^{j+1} = -F_i^j, \ \ i = \overline{1, N-1},$$

where

$$A_i = (\tau\sigma\kappa_i^j a_i^j + \gamma_i \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)} - \tau h\sigma b_i^{-j} a_i^j) x_{i-0.5}^m,$$

$$B_i = (\tau\sigma\kappa_i^j a_{i+1}^j + \gamma_{i+1} \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)} + \tau h\sigma b_i^{+j} a_{i+1}^j) x_{i+0.5}^m,$$

$$C_i = A_i + B_i + x_i^m \overline{\kappa}_i h^2 \frac{\tau^{1-\alpha} c_0^{(\alpha,\sigma)}}{\Gamma(2-\alpha)} + x_i^m \sigma \tau h^2 d_i^j,$$

$$F_i^j = AA_i y_{i-1}^j - CC_i y_i^j + BB_i y_{i+1}^j +$$

$$+ x_i^m h^2 \tau \varphi_i^j - x_i^m \overline{\kappa}_i h^2 \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j-1} c_{j-s}^{(\alpha,\sigma)} (y_i^{s+1} - y_i^s) +$$

$$+ \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} ((\gamma_{i+1} y_{i+1})^{s+1} - (\gamma_{i+1} y_{i+1})^s) -$$

$$- \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} (((\gamma_i + \gamma_{i+1}) y_i)^{s+1} - ((\gamma_i + \gamma_{i+1}) y_i)^s) +$$

$$+ \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} ((\gamma_i y_{i-1})^{s+1} - (\gamma_i y_{i-1})^s),$$

$$AA_i = (\tau(1-\sigma)\kappa_i^j a_i^j - \gamma_i \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)} - \tau h (1-\sigma) b_i^{-j} a_i^j) x_{i-0.5}^m,$$

$$BB_i = (\tau(1-\sigma)\kappa_i^j a_{i+1}^j - \gamma_{i+1} \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)} + \tau h (1-\sigma) b_i^{+j} a_{i+1}^j) x_{i+0.5}^m,$$

$$CC_i = AA_i + BB_i - x_i^m \overline{\kappa}_i h^2 \frac{\tau^{1-\alpha} c_0^{(\alpha,\sigma)}}{\Gamma(2-\alpha)} + x_i^m (1-\sigma) \tau h^2 d_i^j.$$

Boundary condition (41) takes the form

$$y_0^{j+1} = \kappa_1 y_1^{j+1} + \tilde{\mu}_1,$$

where

$$\kappa_1 = \frac{\tau\sigma\kappa_0 a_1 + \gamma_1 \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)}}{\tau\sigma\kappa_0 a_1^j + \gamma_1 \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)} + 0.5 h^2 \frac{\tau^{1-\alpha} c_0^{(\alpha,\sigma)}}{\Gamma(2-\alpha)(m+1)} + \frac{0.5\sigma\tau h^2 d_0^j}{m+1}},$$

$$\tilde{\mu}_1 = [\mu_1 h\tau + \tau(1-\sigma)\kappa_0 a_1 (y_1^j - y_0^j) - \gamma_1 \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)} (y_1^j - y_0^j) +$$

$$+ \frac{0.5h^2}{(m+1)} \frac{\tau^{1-\alpha} c_0^{(\alpha,\sigma)}}{\Gamma(2-\alpha)} y_0 - \frac{0.5h^2}{m+1} \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j-1} c_{j-s}^{(\alpha,\sigma)} (y_0^{s+1} - y_0^s) +$$

$$+ \frac{\tau^{1-\beta}}{\Gamma(2-\beta)(m+1)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} ((\gamma_1 y_1)^{s+1} - (\gamma_1 y_1)^s) -$$

$$- \frac{0.5(1-\sigma)\tau h^2 d_0^j y_0^j}{m+1} - \frac{\tau^{1-\beta}}{\Gamma(2-\beta)(m+1)} \sum_{s=0}^{j-1} c_{j-s}^{(\beta,\sigma)} ((\gamma_1 y_0)^{s+1} - (\gamma_1 y_0)^s)] /$$

$$/ [\tau\sigma\kappa_0 a_1^j + \gamma_1 \frac{\tau^{1-\beta} c_0^{(\beta,\sigma)}}{\Gamma(2-\beta)} + 0.5 h^2 \frac{\tau^{1-\alpha} c_0^{(\alpha,\sigma)}}{\Gamma(2-\alpha)(m+1)} + \frac{0.5\sigma\tau h^2 d_0^j}{m+1}].$$

Boundary condition (42) takes the form

$$y_N^{j+1} = 0.$$

Conditions for the stability of the sweep method

$$A_i \neq 0, \ B_i \neq 0, \ C_i \geq |A_i| + |B_i|, \ i = 1,2,\ldots,N-1,$$
$$\kappa_{11} \leq 1, \ \kappa_{21} \leq 1, \ |\kappa_{11}| + |\kappa_{21}| < 2$$

with $\sigma \neq 0$ are fulfilled.

Thus, difference schemes (19)-(21), (40)-(43) are reduced to tridiagonal systems of linear algebraic equations, which are solved by the sweep method [13].

3 Results

Theorem 1. If $k(x,t) \in C^{1,0}(Q_T)$, $\eta(x) \in C^1[0,l]$, $r(x,t), q(x,t), f(x,t) \in C(Q_T)$, $u(x,t) \in C^{2,0}(Q_T) \cap C^{1,0}(\overline{Q}_T)$, $\partial_{0t}^{\alpha} u(x,t) \in C(Q_T)$, $\partial_{0t}^{\alpha} u_{xx}(x,t) \in C(\overline{Q}_T)$ and conditions (4) are satisfied, then estimates: (13) in the case when $\alpha > \beta$; (15) in the case when $\alpha = \beta$; (18) in the case when $\alpha < \beta$ are valid for the solution $u(x,t)$ of problem (1)-(3)

The a priori estimates (13), (15), (18) imply the uniqueness and stability of the solution with respect to the right-hand side and the initial data

Theorem 2. Let conditions (4) be satisfied, then exist $\tau_0 = \tau_0(c_0, c_1, c_2, \alpha, \beta, \sigma)$, such that if $\tau \leq \tau_0$, then estimates: (29) in the case when $\alpha > \beta$; (30) in the case when $\alpha = \beta$; (31) in the case when $\alpha < \beta$ are valid for the solution of the difference problem (19)-(21).

The a priori estimates (29)-(31) imply the uniqueness and stability of the solution **of** the difference scheme (19)-(21) with respect to the initial data and the right-hand side, as well as the convergence of the solution of the difference problem (19)-(21) to the solution of the differential problem (1)-(3) so that if there exist such τ_0 then for $\tau \leq \tau_0$ the a priori estimates are valid:

1) in case when $\alpha > \beta$: $\| y^{j+1} - u^{j+1} \|_0^2 \leq M\left(h^2 + \tau^{2-\max\{\alpha,\beta\}}\right)$;

2) in case when $\alpha = \beta$: $\| y^{j+1} - u^{j+1} \|_{W_2^1(0,l)}^2 \leq M(h^2 + \tau^2)$;

3) in case when $\alpha < \beta$: $\| y_{\bar{x}}^{j+1} - u_{\bar{x}}^{j+1}] |_0^2 \leq M\left(h^2 + \tau^{2-\max\{\alpha,\beta\}}\right)$,

where $M -$ const>0, independent of h and τ.

Theorem 3. If $k(x,t) \in C^{1,0}(\overline{Q}_T)$, $\eta(x) \in C^1[0,l]$, $q(x,t), f(x,t) \in C(\overline{Q}_T)$, $u(x,t) \in C^{2,0}(Q_T) \cap C^{1,0}(\overline{Q}_T)$, $\partial_{0t}^{\alpha} u(x,t) \in C(\overline{Q}_T$, $\partial_{0t}^{\alpha} u_{xx}(x,t) \in C(\overline{Q}_T)$ and conditions (4), (44) are satisfied; then estimates: (54) in the case when $\alpha > \beta$; (56) in the case when $\alpha = \beta$; (58) in the case when $\alpha < \beta$ are valid for the solution $u(x,t)$ of problem (40)-(43)

The a priori estimates (54), (56), (58) imply the uniqueness and stability of the solution with respect to the right-hand side and the initial data

Theorem 4. Let conditions (4), (44) be satisfied, then exists $\tau_0 = \tau_0(c_0, c_1, c_2, \alpha, \beta, \sigma)$, such that if $\tau \leq \tau_0$, then estimates: (73), in the case when $\alpha > \beta$; (74) in the case when $\alpha = \beta$; (78) in the case when $\alpha < \beta$ are valid for the solution of the difference problem (59)-(62).

The a priori estimates (73), (74), (78) imply the uniqueness and stability of the solution of the difference scheme (59)-(62) with respect to the initial data and the right-hand side, as well as the convergence of the solution of the difference problem (59)-(62) to the solution of the differential problem (40)-(43) so that if there exist such τ_0 then for $\tau \leq \tau_0$ the a priori estimates are valid:

1) in case when $\alpha > \beta$: $\| x^{\frac{m}{2}}(y^{j+1} - u^{j+1}) \|_0^2 \leq M\left(h^2 + \tau^{2-\max\{\alpha,\beta\}}\right)$;

2)in case when $\alpha = \beta$: $\| x^{\frac{m}{2}}(y^{j+1} - u^{j+1}) \|_0^2 + \| \bar{x}^{\frac{m}{2}}(y_{\bar{x}}^{j+1} - u_{\bar{x}}^{j+1})] |_0^2 \leq M(h^2 + \tau^2)$;

3) in case when $\alpha < \beta$: $\| \bar{x}^{\frac{m}{2}}(y_{\bar{x}}^{j+1} - u_{\bar{x}}^{j+1})] |_0^2 \leq M\left(h^2 + \tau^{2-\max\{\alpha,\beta\}}\right)$,

where $M - const > 0$, independent of h and τ.

4 Conclusion

Boundary value problems are considered for a one-space-dimensional hyperbolic equation

with two fractional differentiation operators and variable coefficients. The method of finite differences is used for the approximate solution to the problems posed. Difference schemes are constructed for each of the problems, and by the method of energy inequalities for various relations between the orders of the fractional derivative $(\alpha,\ \beta)$, a priori estimates are obtained in differential and difference interpretations. The resulting estimates imply the uniqueness and stability of the solution with respect to the right-hand side and the initial data, as well as the convergence of the solution of the difference problem to the solution of the corresponding original differential problem at a rate of $O(h^2+\tau^2)$ for $\alpha=\beta$ and $O(h^2+\tau^{2-\max\{\alpha,\beta\}})$ for $\alpha\neq\beta$. Algorithms for the numerical solution of the problems posed are constructed.

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