# Solution of the Heat and Mass Transfer Problem for Soil Radiant Heating Conditions Using the Error Function

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**Abstract.** Achieving high yields of agricultural crops requires the ability to predict soil temperature and moisture regimes, taking into account soil heating technology. The object of study is soil heated by a ceiling infrared emitter. The subject of study is one-dimensional non-stationary fields of soil moisture content and temperature. The objective of the study is to predict soil temperature and moisture regimes under radiant heating conditions. Research methods: analytical methods for solving differential equations of heat and mass transfer using the error function. Research results: the top 5 mm layer of milled peat with an initial moisture content of 3.7 kg/kg will reach a final moisture content of 1.0 kg/kg in about 6 hours during infrared drying. As a result of radiant heating, the soil will heat up from an initial temperature of 5 °C to a final temperature of 20 °C in approximately 3 hours. The analytical solution of the mass transfer differential equation can be used for theoretical studies of drying of capillary-porous materials, for example, to determine the drying period or the thickness of the material layer that will dry to a given final moisture content. The analytical solution of the heat transfer differential equation can be used to control the operating mode of the infrared radiation source, for example, to determine the periods of its operation and switching off in case the soil surface temperature reaches the maximum (critical) value. The mathematical solutions considered in the article do not take into account the cross processes of heat and mass transfer, which is a promising direction for further scientific research.

**Keywords:** Soil Moisture Content, Soil Temperature, Heat and Mass Transfer Differential Equations; One-Dimensional Non-stationary Field, Soil, Radiant Heating, Error Function.

## 1 Introduction

For the favorable growth, development, and fruiting of plants, and for growing seedlings in cultivation structures, such as industrial greenhouses or orangeries, it is necessary to maintain not only the optimal temperature and humidity of the air in the room, but also the required temperature and humidity regime of the soil [1-7]. Traditional heating systems based on convective heating of the internal air have a number of inherent drawbacks [8-12]: irrational energy consumption per unit of agricultural products produced; significant initial financial costs associated with the purchase of the necessary engineering equipment and materials; thermal inertia of the heat consumption system, etc. Such a technology as soil heating using underground pipelines in which a heat carrier (hot water or steam) circulates has also not proven itself due to its high cost and critical consequences in the event of an emergency. Therefore, the use of ceiling infrared emitters as heat sources for heating cultivation structures, along with other alternative heating methods, such as solar or geothermal, is becoming increasingly relevant [13-15]. Due to its principle of operation, environmental friendliness and energy efficiency, radiant soil heating using infrared radiation sources is currently one of the most advanced and promising types of heating in the world [16-19].

Despite the development of new methods for heating cultivation structures, the issues of finding, predicting, and regulating the temperature and humidity regime of the soil remain open and relevant for heat and mass transfer theory [20, 21]. The accepted methods of formulating, solving, and analyzing these multifaceted problems are completely insufficient for their implementation. The initial and boundary conditions, and, consequently, the final formulas should adequately reflect the physics of the processes taking place [22]. The correct solution of the considered problems requires additional knowledge about the thermophysical properties of the soil, characterizing the material's ability to diffuse energy and matter, as well as about the heat and mass transfer processes occurring at the interface "soil - environment" [23]. The application of classical heat and mass transfer solutions to the object of study - soil, taking into account the features of radiant heating, will make it possible to prevent possible deviations of environmental parameters from optimal (or permissible) values and prevent the death of agricultural products.

Predicting the temperature and humidity regime of the soil under radiant heating conditions can be associated with an analytical description of the diffusion of heat and mass in a colloidal capillary-porous body. The mathematical "reflection" of real processes occurring in the soil layer, taking into account external and internal factors, consists in the compilation and solution of differential heat and mass transfer equations in partial derivatives [24]. Solving such equations presents certain mathematical difficulties. Only in rare cases is it possible to obtain exact analytical solutions [25].

Thanks to the modern achievements in the field of creating electronic computers that work with large volumes of memory at high information processing speeds, the results of analytical solutions are issued in a short time without any particular technical difficulties. In addition, analytical methods are very compact, and the issue of computational complexity fades into the background in connection with the development of mathematical editors that allow differentiation and integration operations with rational, irrational, transcendental and complex functions to be performed in a matter of seconds according to given equations.

Object of study: Soil heated by a ceiling infrared emitter.

Subject of study: One-dimensional non-stationary fields of soil moisture content and temperature.

Objective of the study: To predict the temperature and humidity regime of the soil under radiant heating conditions.

Research tasks:

1. Formulation of the problem. Formulation of conditions of uniqueness corresponding to radiant heating of the soil surface.
2. Obtaining a solution to the differential equations of heat and mass transfer using the error function [26-30].
3. Consideration of an analytical solution of the differential equations of heat and mass transfer on a specific example.
4. Description of the obtained results of the solution.

## 2 Methodology

The problem statement of mass transfer is as follows: Given a semi-infinite body, namely soil (Figure 1), with an initial moisture content $W_0$, kg/kg. Over time $t$, s, moisture evaporates from the soil surface into the surrounding environment under the influence of radiant heating, with a constant time-dependent intensity $j$, $\mathrm{kg/(m^2 \cdot s)}$. moisture evaporates from the soil surface into the surrounding environment under the influence of radiant heating, with a constant time-dependent intensity $z$ -axis direction only. The task is to find the distribution of moisture content$W(z, t)$.

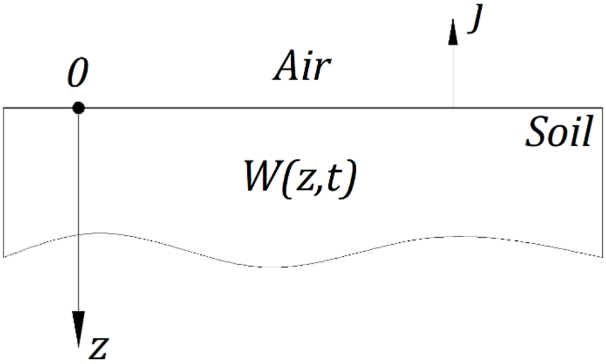

**Fig. 1.** Statement of the Mass Transfer Problem.

Under radiant heating, the intensity of moisture evaporation from the soil surface can be determined using Dalton's formula:

$$j = \alpha_m \left(p_{surf} - p_\infty\right), \mathrm{kg/(m^2 \cdot s)}, \tag{1}$$

where $\alpha_m$ is the coefficient of external moisture exchange (mass transfer coefficient), measured in $\mathrm{kg/(m^2 \cdot s \cdot Pa)}$, which is calculated through similarity criteria equations of mass exchange processes or approximately computed using Lewis's formula; $p_{surf}$ and $p_\infty$ are respectively the partial pressure of water vapor at the soil surface and far from it (in the surrounding environment), Pa.

The differential equation of mass transfer and the conditions of uniqueness for its solution have the following form (Figure 1):

$$\frac{\partial W(z,t)}{\partial t} = D \frac{\partial W^2(z,t)}{\partial z^2} (0 < z < \infty, t > 0), \tag{2}$$

$$W(z, 0) = W_0, \tag{3}$$

$$D\rho \frac{\partial W(0,t)}{\partial z} + j = 0, \tag{4}$$

$$W(\infty,t) = W_0, \frac{\partial W(\infty,t)}{\partial z} = 0, \tag{5}$$

where $W$ is the moisture content, measured in $\mathrm{kg/kg}$; $z$ is the coordinate, m; $t$ is time, s; $D$ is the soil diffusion coefficient, $\mathrm{m^2/s}$; ρ is the soil density, $\mathrm{kg/m^3}$; $j$ is the intensity of moisture evaporation from the soil surface, $\mathrm{kg/(m^2 \cdot s)}$.

The solution to the differential equation of mass transfer (2) under the initial (3) and boundary (4), (5) conditions is known and is expressed as follows:

$$W(z,t) = W_0 - \frac{2j}{\rho}\sqrt{\frac{t}{D}}\,\mathrm{ierfc}(u_1), \tag{6}$$

$$\mathrm{ierfc}(u_1) = \frac{1}{\sqrt{\pi}}e^{-u_1^2} - u_1\,\mathrm{erfc}(u_1), \tag{7}$$

$$u_1 = \frac{z}{2\sqrt{Dt}}. \tag{8}$$

The heat transfer problem is formulated as follows: given a semi-infinite body, namely soil (Figure 2), with an initial temperature $T_0$, °C. Under the influence of radiant heating with a constant time-dependent heat flux density $q$, $\mathrm{W/m^2}$, the soil surface is heated over time $t$, s. Temperature changes occur along one direction only, along the $z$-axis. The task is to find the temperature distribution $T(z,t)$.

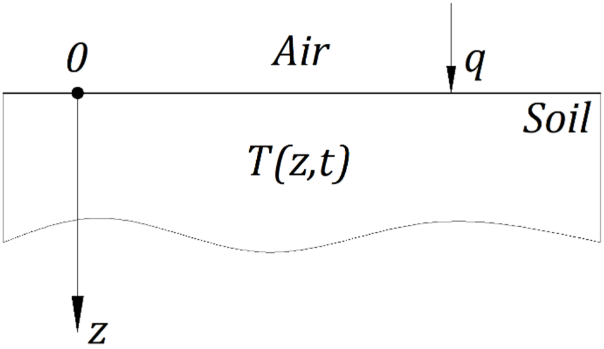

**Fig. 2.** Statement of the Heat Transfer Problem.

The heat flux density of the soil surface can be determined through the heat balance equation:

$$q = \left(1 - \frac{R}{100}\right)q_{inf} - q_{vp} - q_{conv}, \mathrm{W/m^2}, \tag{9}$$

where $R$ is the surface soil reflectance, %; $q_{inf}$ is the heat flux density emitted from the infrared radiation source, $\mathrm{W/m^2}$; $q_{vp}$ is the heat flux density consumed by the evaporation of moisture from the soil surface (directly proportional to the moisture evaporation intensity $j$), $\mathrm{W/m^2}$; $q_{conv}$ is the heat flux density from convective heat exchange between the soil surface and the surrounding medium (air), $\mathrm{W/m^2}$, determined by Newton's equation.

The differential equation of heat transfer and the conditions for its uniqueness have the following form (Figure 2):

$$\frac{\partial T(z,t)}{\partial t} = \alpha\frac{\partial T^2(z,t)}{\partial z^2}\,(0 < z < \infty, t > 0), \tag{10}$$

$$T(z,0) = T_0, \tag{11}$$

$$k\frac{\partial T(0,t)}{\partial z} + q = 0, \tag{12}$$

$$T(\infty,t) = T_0, \frac{\partial T(\infty,t)}{\partial z} = 0, \tag{13}$$

where $T$ is the temperature, °C; $z$ is the coordinate, m; $t$ is time, s; α is the soil thermal diffusivity coefficient, $\mathrm{m^2/s}$; $k$ is the soil thermal conductivity coefficient, $\mathrm{W/(m \cdot K)}$; $q$ is the heat flux density on the soil surface, $\mathrm{W/m^2}$.

The solution to the differential equation of heat transfer (10) under the initial (11) and boundary (12), (13) conditions is known and is expressed as follows:

$$T(z,t) = T_0 + \frac{2q}{k}\sqrt{\alpha t}\,\mathrm{ierfc}(u_2), \tag{14}$$

$$\mathrm{ierfc}(u_2) = \frac{1}{\sqrt{\pi}}e^{-u_2^2} - u_2\,\mathrm{erfc}(u_2), \tag{15}$$

$$u_2 = \frac{z}{2\sqrt{\alpha t}}. \tag{16}$$

## 3 Results and Discussion

Let's consider the solution to the differential equation of mass transfer (2) using the example of milled peat (Figure 3) with the following initial data: $\rho = 74\,\mathrm{kg/m^3}$; $D = 2.0 \cdot 10^{-8}\,\mathrm{m^2/s}$; $W_0 = 3.7\,\mathrm{kg/kg}$; $j = 170 \cdot 10^{-6}\,\mathrm{kg/(m^2 \cdot s)}$.

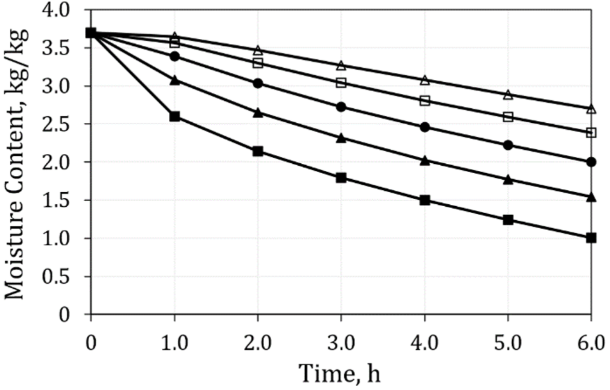

**Fig. 3.** Solution to the differential equation of mass transfer: ■ – 0 mm; ▲ – 5 mm; ● – 10 mm; □ – 15 mm; △ – 20 mm.

Let's illustrate the solution to the differential equation of heat transfer (10) using the example of milled peat (Figure 4) with the following initial data: $k = 0.302\,\mathrm{W/(m \cdot K)}$; $\alpha = 14.8 \cdot 10^{-8}\,\mathrm{m^2/s}$; $T_0 = 5$ °C; $q = 100\,\mathrm{W/m^2}$.

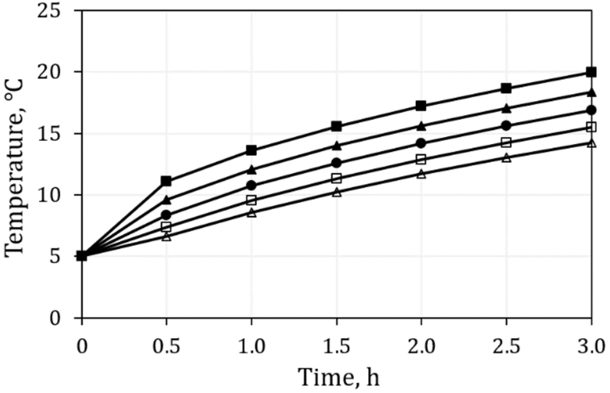

**Fig. 4.** Solution to the differential equation of heat transfer: ■ – 0 mm; ▲ – 5 mm; ● – 10 mm; □ – 15 mm; △ – 20 mm.

Based on the mathematical solutions (6)-(8) and (14)-(16) corresponding to the differential equations of heat and mass transfer (2) and (10) respectively, a program has been developed using the mathematical editor Mathcad to determine the moisture content and temperature of the semi-infinite body.

## 4 Conclusions

According to the results of computational simulations, the upper layer of milled peat, 5 mm thick, with an initial moisture content of $3.7\,\mathrm{kg/kg}$ will reach a final moisture content of $1.0\,\mathrm{kg/kg}$ after approximately 6 h of infrared drying (Figure 3). As a result of radiant heating, the soil will reach a final temperature of 20 °C from an initial value of 5 °C in approximately 3 h (Figure 4).

Analytical solutions to the differential equations of heat and mass transfer (2) and (10), represented respectively by algebraic functions (6)-(8) and (14)-(16), enable the investigation of the influence of initial parameters (such as moisture evaporation intensity from the soil surface or the magnitude of heat flux density) on the temperature-moisture regime of the soil. This is a determining factor for creating a favorable microclimate in cultivation structures for seedling growth or for the cultivation of greens, vegetables, and flowers in enclosed soil. Additionally, equations (6)-(8) can be used for theoretical studies of drying capillary-porous bodies (such as chunks, ground, or milled peat), while equations (14)-(16) can be applied for managing the operation mode of the infrared radiation source.

At this stage of scientific research, the differential equations of heat and mass transfer (2) and (10) do not include terms accounting for cross processes of heat and mass exchange: thermodiffusion (Soret effect) and vapor diffusion processes (heat energy transfer along with water vapor movement through soil capillaries). Thus, it is of scientific interest to obtain solutions to the differential equations of heat and mass transfer using the method of Fourier integral transformation, which accounts for vapor diffusion processes in soil, as well as to consider the method of combined application of Laplace integral transformation and the Bubnov-Galerkin variational method (in this case, both vapor diffusion processes and the phenomenon of thermo-moisture conductivity are taken into account).

## 5 Funding

This research was funded by the Ministry of Science and Higher Education of the Russian Federation within the framework of the state assignment No. 075-03-2022-010 dated 14 January 2022 and No. 075--01568-23-04 dated 28 March 2023(Additional agreement 075-03-2022-010/10 dated 09 November 2022, Additional agreement 075-03-2023-004/4 dated 22 May 2023), FSEG-2022-0010.

## References

1. Hsu, Hsin and Dirmeyer, Paul. *Soil moisture-evaporation coupling shifts into new gears under increasing $CO_2$*. Nature Communications. **14**. (2023). https://doi.org/10.1038/s41467-023-36794-5
2. Mustafaev, G. and Magomedova, A. and Samir, Mursalov and Sapukova, A. and Khalikov, M. *The influence of irrigation methods on the water regime and productivity of greenhouse tomatoes*. Vegetable crops of Russia. 71-75. (2021). https://doi.org/10.18619/2072-9146-2021-2-71-75
3. Sabziparvar, Ali-Akbar and Jahromi, Fatemeh. *Evaluating the most effective climatic parameters affecting the monthly mean soil temperature estimates using the PLS method*. Arabian Journal of Geosciences. **15**. (2022). https://doi.org/10.1007/s12517-022-10297-x
4. Singh, R., Hussain, A., Fezaa, L.H.A., Gupta, G., Singh, A.P., Dogra, A. *Greenhouse Automation System Using ESP32 and ThingSpeak for Temperature, Humidity, and Light*

*Control*. In: Tiwari, S., Trivedi, M.C., Kolhe, M.L., Singh, B.K. (eds) Advances in Data and Information Sciences. ICDIS 2023. Lecture Notes in Networks and Systems. Vol. **796**. Springer. Singapore. (2024). https://doi.org/10.1007/978-981-99-6906-7_43

5. Si, Changqing and Qi, Fei and Ding, Xiaoming and He, Fen and Gao, Zhenjun and Feng, Qian and Zheng, Liang. *CFD Analysis of Solar Greenhouse Thermal and Humidity Environment Considering Soil–Crop–Back Wall Interactions*. Energies. **16**. 2305. (2023). https://doi.org/10.3390/en16052305
6. Wu, Mingge. *Research on decoupling greenhouse temperature and humidity based on feedback linearization*. Journal of Electrical Systems. **20**. 394-399. (2024). https://doi.org/10.52783/jes.1192
7. Polishchyk, Dmytro and Lysenko, Vitaliy and Osadchiy, Serhii and Zaiets, Nataliia. *Intellectual Scenario-synergetic Control of the Humidity and Temperature Regime of the Greenhouse Facilities*. International Journal of Computing. 311-317. (2022). https://doi.org/10.47839/ijc.21.3.2686
8. Meng, Haiyan and Tian, Zhe and Wu, Xia and Yakai, Lu and Haoran, Mai. *Method for Determining Design Heating Load of Rural Residential Buildings Considering Indoor Temperature Uncertainty*. Applied Sciences. **14**. 1835. (2024). https://doi.org/10.3390/app14051835
9. Mikielewicz, Jarosław and Ochrymiuk, Tomasz and Cenian, Adam. *Comparison of traditional with low temperature district heating systems based on organic Rankine cycle*. Energy. **245**. 123280. (2022). https://doi.org/10.1016/j.energy.2022.123280
10. Sangoi, Juliana and Scolaro, Taylana and Ghisi, Enedir. *Water Heating Systems Performance in Multi-Family Residential Buildings in Brazil*. Water. **15**. 3145. (2023). https://doi.org/10.3390/w15173145
11. Krajčík, Michal and Arıcı, Müslüm and Ma, Zhenjun. *Trends in research of heating, ventilation and air conditioning and hot water systems in building retrofits: Integration of review studies*. Journal of Building Engineering. **76**. 107426. (2023). https://doi.org/10.1016/j.jobe.2023.107426
12. Ignatiev, K. and Giniyatullin, E. and Ziganshin, M. *Research of the energy efficiency of a combined air and water heating of a public building*. Safety and Reliability of Power Industry. **14**. 124-131. (2021). https://doi.org/10.24223/1999-5555-2021-14-2-124-131
13. Ptitsyna, Elena and Kuvaldin, A. and Ptitsyn, D. *The effective modes of infrared radiators and heating systems*. E3S Web of Conferences. **140**. 04005. (2019). https://doi.org/10.1051/e3sconf/201914004005
14. Puttige, Anjan Rao and Feng, Kailun and Lu, Weizhuo and Olofsson, Thomas. *Are radiators ready for the challenges of the future: A review of advancements in radiators*. E3S Web of Conferences. **356**. 03024. (2022). https://doi.org/10.1051/e3sconf/202235603024
15. Liu, Jian and Lin, Zhang. *Energy and Exergy Performances of Floor, Ceiling, Wall Radiator and Stratum Ventilation Heating Systems for Residential Buildings*. Energy and Buildings. **220**. 110046. (2020). https://doi.org/10.1016/j.enbuild.2020.110046
16. Wang, Tianhui and Liu, Yanfeng and Wang, Dengjia and Gao, Weijun. *Experimental evaluation on asymmetrical thermal sensation in modular radiant heating system*. Building and Environment. **222**. 109433. (2024). https://doi.org/10.1016/j.buildenv.2022.109433
17. Joe, Goosang and Park, Sanghoon. (2023). *A Study on Establishing Thermal Output Conditions of Radiant Ceiling Heating Panels for Improving Thermal Comfort of Perimeter Zone in Buildings*. Applied Sciences. **13**. 6744. https://doi.org/10.3390/app13116744
18. Noro, Marco and Mancin, Simone and Busato, Filippo and Cerboni, Francesco. *Innovative Hybrid Condensing Radiant System for Industrial Heating: An Energy and*

*Economic Analysis*. Sustainability. **15**. 3037. (2023). https://doi.org/10.3390/su15043037

19. Behroozeh, Samira and Hayati, Dariush and Karami, Ezatollah. *Factors influencing energy consumption efficiency in greenhouse cropping systems*. Environment, Development and Sustainability. 1-36. (2024). https://doi.org/10.1007/s10668-024-04851-8
20. Nagarajah, Arravinthen and Azizan, Fathin and Zalani, Fatnin. *Assessing Changes in Soil Moisture Distribution for Before and After Irrigation in a Harumanis Greenhouse*. Advanced and Sustainable Technologies (ASET). **3**. 36-44. (2024). https://doi.org/10.58915/aset.v3i.580
21. Nam, Sang-Woon. *The Effect of Soil Warming on the Greenhouse Heating Load*. Journal of The Korean Society of Agricultural Engineers. **48**. (2006). https://doi.org/10.5389/KSAE.2006.48.5.051
22. Zhu, Xi and Gao, Zongjun and Chen, Tao and Wang, Wanli and Lu, Chuan and Zhang, Qinglian. *Study on the Thermophysical Properties and Influencing Factors of Regional Surface Shallow Rock and Soil in China*. Frontiers in Earth Science. **10**. (2022). https://doi.org/10.3389/feart.2022.864548
23. Pikus, Grigoriy. *Thermophysical properties of the soil massif*. Magazine of Civil Engineering. **92**. (2020). https://doi.org/10.18720/MCE.92.2
24. Ngwarati, Tendai and Kevin, Mtondo and Francis, Hove and Sibanda, Simelokuhle. Optimization of temperature, humidity and airflow in African Indigenous Vegetable (AIV) greenhouse. i-manager's Journal on Future Engineering and Technology. **18**. 6. (2023). https://doi.org/10.26634/jfet.18.4.19760
25. Huang, Shuanggen and Liu, Quanyao and Wu, Yan and Chen, Minmin and Yin, Hua and Zhao, Jinhui. *Edible Mushroom Greenhouse Environment Prediction Model Based on Attention CNN-LSTM*. Agronomy. **14**. 473. (2024). https://doi.org/10.3390/agronomy14030473
26. Nas, Merve and Yalçin, Sibel and Bayram, Hasan. *A Class of Analytic Functions Defined by a Second Order Differential Inequality and Error Function*. International Journal of Applied and Computational Mathematics. **10**. (2024). https://doi.org/10.1007/s40819-024-01681-0
27. Mohammed, Nada and Kuffi, Emad. *The complex integral transform complex Sadik transform of error function The Complex Integral Transform Complex Sadik transform of Error Function*. Journal of Interdisciplinary Mathematics. **26**. (2023). https://doi.org/10.47974/JIM-1613
28. A. Mehdi, Sadiq and Kuffi, Emad and Ajel Mansour, Iman. *Applying SEE integral transform on error and complementary error functions*. **2834**. 080008. (2023). https://doi.org/10.1063/5.0161592
29. Irmak, Hüseyin and Salem, Fatma. *Comprehensive Results for the Error Functions in the Complex Plane and Some of Their Consequences*. Equations. **3**. 50-58. (2023). https://doi.org/10.37394/232021.2023.3.7
30. Ferreira, Chelo and López, José and Sinusía, Ester. *Uniform Convergent Expansions of the Error Function in Terms of Elementary Functions*. Mediterranean Journal of Mathematics. **20**. (2023). https://doi.org/10.1007/s00009-023-02297-2