

# Optimal recovery of the derivate from the confined analytic function

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**Abstract.** In the article the author finds out the best straight-line method to recover the derivate in zero-point from the confined analytic function determined in the unit circle according to values of the function and values of its derivate in given points creating the regular polygon with zero as the center. The paper consists of three parts. The first part one contains notions and results necessary for the solution of the objective. The second part determines the measure of inaccuracy of the best approximation method and draws out the corresponding extremum function. It is established that the extremum function is the only one to an approximation of the constant multiplier modulo equal to one. The third part calculates coefficients of the best straight-line method of recovery. It is established that the best straight-linear method of recovery is the only one.

## 1 Introduction

The works of A.I. Matasov [1], P.A. Akimov, A.A. Golovan (see [2]-[6]) were considered estimation problems which have a broad applicability. The authors of these article make references to the problems of optimal recovery of some classes of analytic functions given in the unit circle. Current work considers one more problem of this kind.

Aims of the optimal recovery of some classes of analytic functions were set for the first time in the work of K.Yu. Osipenko [7]. Then this problematic was developed in the works [8]-[11], works by R.R. Akopian [12]-[15], G.G. Magaril-Iliyaev [16]-[18], author of the present article [19]-[21] and other mathematicians. The optimal recovery issues in articles by these authors were studied in Hardy and Sobolev clesses. In this paper we study the problem of optimal recovery of derivatives from confined analytic functions defined in the unit circle at zero with values of functions and their derivatives at points forming a regular polygon. This allows to obtain coefficients in the simple concrete form.

Let  $W$  be a certain set laying in the linear complex space  $X$  and  $L, l_1, \dots, l_n$  — linear complex functionals defined on  $X$ . If  $S(t_1, \dots, t_n)$  is any complex function then the measure of inaccuracy of the approximation with using the method  $S$  is the value

$$r_n(S) = \sup_{x \in W} |L(x) - S(l_1(x), \dots, l_n(x))|.$$

Method  $S_0(t_1, \dots, t_n)$  is the best method of approximation if

$$r_n(S_0) = \inf_S r_n(S).$$

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In the work [7] the existence of the best straight-linear method of approximation is proved (under some conditions on the set  $W$ ),  $S_0 = \sum_{k=1}^n c_k l_k(x)$ , herewith  $c_k$  – are certain constant complex numbers. Meanwhile, the measure of inaccuracy of the best method of approximation (or recovery) can be calculated using the formula

$$r_n(S_0) = \sup_{\substack{x \in W \\ l_1(x) = \dots = l_n(x) = 0}} |L(x)|. \tag{1}$$

Let  $K = \{z: |z| < 1\}$  be a unit circle and  $\Gamma = \{z: |z| = 1\}$  — a unit circuit. We denote via  $B^1(K) = \{f(z): |f(z)| \leq 1, z \in K\}$  – the set of analytic functions defined in  $K$ . Let us determine points in the circle  $K$  which looks like

$$z_1 = R, z_2 = Re^{\frac{2\pi i}{n}}, \dots, z_n = Re^{\frac{(n-1)2\pi i}{n}}, \tag{2}$$

Where  $R$  is the constant number ( $0 < R < 1$ ).

Let  $L(f) = f'(0), l_1(f) = f(z_1), l_2(f) = f'(z_1), \dots, l_{2n-1}(f) = f(z_n), l_{2n} = f'(z_n)$  where  $f(z) \in B^1(K)$ .

According to the work [22], if  $\omega(\zeta)$  can be summarized with  $\Gamma$  ( $\zeta \in \Gamma$ ) function then the followind duality correlation is carried out

$$\sup_{f \in B^1(K)} \left| \int_{\Gamma} f(\zeta) \omega(\zeta) d\zeta \right| = \min_{\varphi \in H_1} \int_{\Gamma} |\omega(\zeta) - \varphi(\zeta)| |d\zeta|, \tag{3}$$

Where  $H_1$  – Hardy class (Hardy classes see [23]). The extreme function  $f^*(z)$  exist on the left side of the correlation (3). Moreover, it ( $f^*(z) \in B^1(K)$ ) is one and only with the accuracy to the multiplier  $e^{i\delta}, \delta \in R$ . The extremum function  $\varphi^*(z)$  on the right side of the correlation (3) ( $\varphi^*(z) \in H_1$ ), generally speaking, is not the only one. If  $\omega(\zeta)$  is an end value on the limit  $\Gamma$  of meromorphic in  $\bar{K}$  function  $\omega(z)$  with poles  $\beta_1, \dots, \beta_m$  (each pole is repeated as many times as its order), then the function

$$R(z) = f^*(z)[\omega(z) - \varphi^*(z)] \tag{4}$$

is analytic function (except poles) till the limit  $\Gamma$  and has in  $\bar{K}$

$$\nu = m - 1 \tag{5}$$

zeros  $\alpha_1, \dots, \alpha_{m-1}$  ( $|\alpha_k| \leq 1, k = 1, \dots, m - 1$ ). Moreover, it is proved (see [22])

$$R(z) = C \frac{\prod_{k=1}^{m-1} (z - \alpha_k)(1 - \bar{\alpha}_k z)}{\prod_{k=1}^m (z - \beta_k)(1 - \bar{\beta}_k z)}, \tag{6}$$

where  $C$  is a certain complex constant. Denote it via

$$B(z) = \prod_{k=1}^n \frac{z - z_k}{1 - \bar{z}_k z} \tag{7}$$

end product by Blyashke (here point  $z_1, \dots, z_n$  look like (2)).

In the work [24] was stated that in this case the end product by Blyashke looks like

$$B(z) = \frac{z^n - R^n}{1 - R^n z^n}. \tag{8}$$

It should be noted that the following correlation is still carried out

$$z_k B'(z_k) = \frac{nR^n}{1 - R^{2n}}. \tag{9}$$

The statement is true. Because (see (8))

$$B'(z_k) = \frac{1}{z_k} \frac{nR^n}{1 - R^{2n}}$$

The correlation (9) is a result.

## 2 Determination of the Measure of Inaccuracy of the Best Recovery Method

**Lemma 1.** The following correlation takes place

$$\sup_{g(z) \in B^1(K)} |g'(0)| = 1, \tag{10}$$

and the extremum function  $g^*(z)$  of the problem (10) is the only with accuracy to the constant multiplier modulo equal to one and looking like

$$g^* = e^{i\delta} z, \quad \delta \in R. \tag{11}$$

**Confirmation.** Let us denote

$$d = \sup_{g \in B^1(K)} \frac{1}{2\pi} \left| \int_{\Gamma} \frac{g(\zeta)}{\zeta^2} d\zeta \right|. \tag{12}$$

If  $g(z) \in B^1(K)$ , then

$$\frac{1}{2\pi} \left| \int_{\Gamma} \frac{g(\zeta)}{\zeta^2} d\zeta \right| \leq \frac{1}{2\pi} \int_{\Gamma} |d\zeta| = 1.$$

It means that  $d \leq 1$ . After it we will observe the function  $g^*(z) = z$ . It is clear that  $g^*(z) \in B^1(K)$ .  $d \geq 1$  because  $g^*(z)' = 1$ . Hence,  $d = 1$ . As the extremum function of the problem (12) is the only one with the accuracy to the multiplier  $e^{i\delta}$ ,  $\delta \in R$  (see introduction), the extremum function of the problem (10) looks like (11). Lemma is confirmed. Let us note that (see (8))

$$B(0) = -R^n. \tag{13}$$

Let it be

$$Q(z) = B^2(z). \tag{14}$$

Because

$$Q'(z) = 2B(z)B'(z), \quad Q''(z) = 2(B'(z))^2 + 2B(z)B''(z),$$

Then (see (7), (14), (9))

$$Q(z_k) = Q'(z_k) = 0, \quad Q''(z_k) = 2(B'(z_k))^2 \neq 0, \quad k = 1, \dots, n. \tag{15}$$

Denote it via (see (1))

$$r_1(0, z_1, \dots, z_n) = \sup_{\substack{f \in B^1(K) \\ f(z_1)=f'(z_1)=\dots=f(z_n)=f'(z_n)=0}} |f'(0)| - \tag{16}$$

The measure of inaccuracy of the best method of approximation of the value  $f'(0)$  according to values  $f(z_1), f'(z_1), \dots, f(z_n), f'(z_n)$ , where  $f(z) \in B^1(K)$ .

**Theorem 1.** The extremum function  $F(z)$  of the problem (16) is the only one with accuracy to the multiplier  $e^{i\delta}$  ( $\delta \in R$ ) and it looks like

$$F(z) = e^{i\delta} z \left( \frac{z^n - R^n}{1 - R^n z^n} \right)^2, \tag{17}$$

and the measure of inaccuracy of the best recovery method is calculated using the formula

$$r_1(0, z_1, \dots, z_n) = R^{2n}. \tag{18}$$

**Confirmation.** Let us denote it via

$$A = \{f(z): f(z) \in B^1(K), f(z_1) = f'(z_1) = \dots = f(z_n) = f'(z_n) = 0\}$$

Is a class of analytic functions defined in the circle  $K$ . Let us propose that  $f(z) \in A$ . Let us observe the function

$$g(z) = \frac{f(z)}{B^2(z)}.$$

If  $|z| = 1$ , then

$$|g(z)| = \left| \frac{f(z)}{B^2(z)} \right| = |f(z)| \leq 1.$$

It means that  $g(z) \in B^1(K)$ . Vice versa, if  $f(z) = B^2(z)g(z)$  and the function  $g(z) \in B^1(K)$ , then  $f(z) \in A$ . The function  $F(z)$  (see (17)) belongs to the class A. From that (see (13), (10))

$$r_1(0, z_1, \dots, z_n) = \sup_{f \in A} |f'(0)| = \sup_{g \in B^1(K)} |B^2(0)g'(0)| = R^{2n} \sup_{g \in B^1(K)} |g'(0)| = R^{2n}.$$

It is clear that the extremum function of the problem (16) looks like (17) (see (11)).

### 3 Calculation of Coefficients of the Best Straight-Line Method

Let

$$\sum_{k=1}^n (c_k f(z_k) + \alpha_k f'(z_k)) -$$

the best straight-linear method of recovery ( $c_k, \alpha_k$  – certain constant numbers). Then

$$\sup_{f \in B^1(K)} \left| f'(0) - \sum_{k=1}^n (c_k f(z_k) + \alpha_k f'(z_k)) \right| = r_1(0, z_1, \dots, z_n). \tag{19}$$

Let us rewrite the last equation into another form (see (18))

$$\sup_{f \in B^1(K)} \left| \int_{\Gamma} \left( \frac{1}{\zeta^2} - \sum_{k=1}^n \frac{c_k}{\zeta - z_k} - \sum_{k=1}^n \frac{\alpha_k}{(\zeta - z_k)^2} \right) f(\zeta) d\zeta \right| = 2\pi R^{2n}. \tag{20}$$

Denote via

$$\omega(\zeta) = \frac{1}{\zeta^2} - \sum_{k=1}^n \frac{c_k}{\zeta - z_k} - \sum_{k=1}^n \frac{\alpha_k}{(\zeta - z_k)^2}. \tag{21}$$

It should be noted that the function  $F(z)$  is the extremum function of the problem (19) (see (17), (8)). It means that the same is for the problem (20). As the extremum function of the problem (20) is the only one with accuracy to the multiplier  $e^{i\delta}, \delta \in R$ , the extremum function  $f^*(z)$  on the left side of the equation (20) looks like

$$f^*(z) = e^{i\delta} F(z), \tag{22}$$

and the function  $F(z)$  of the type (17).

**Theorem 2.** Coefficients of the best straight-linear method of approximation is the only one and can be found using the formula

$$c_k = \frac{1 - R^{2n}}{n^2 z_k} ((n - 1)R^{2n} + n + 1), \tag{23}$$

$$\alpha_k = -\frac{(1 - R^{2n})^2}{n^2}, k = 1, \dots, n. \tag{24}$$

**Confirmation.** The function (see (4), (22))

$$R(z) = e^{i\delta} z B^2(z) \left[ \frac{1}{z^2} - \sum_{k=1}^n \frac{c_k}{z - z_k} - \sum_{k=1}^n \frac{\alpha_k}{(z - z_k)^2} - 2\pi i \varphi^*(z) \right]$$

has the one and only simple pole in the point  $z = 0$  and is not equal to zero in the closed circle  $\bar{K}$  (see (5), (7));  $\varphi^*(z)$  is the extremum function on the right side of the equation (3) at the corresponding function  $\omega(\zeta)$  looking like (21). Hence (see (6))

$$zB^2(z) \left[ \frac{1}{z^2} - \sum_{k=1}^n \frac{c_k}{z - z_k} - \sum_{k=1}^n \frac{\alpha_k}{(z - z_k)^2} - 2\pi i \varphi^*(z) \right] = \frac{C}{z},$$

where C is a certain constant number. Because

$$C = B^2(z) \left[ 1 - z^2 \sum_{k=1}^n \frac{c_k}{z - z_k} - z^2 \sum_{k=1}^n \frac{\alpha_k}{(z - z_k)^2} - 2\pi i z^2 \varphi^*(z) \right],$$

(see (13))

$$C = \lim_{z \rightarrow 0} B^2(z) \left[ 1 - z^2 \sum_{k=1}^n \frac{c_k}{z - z_k} - z^2 \sum_{k=1}^n \frac{\alpha_k}{(z - z_k)^2} - 2\pi i z^2 \varphi^*(z) \right] = B^2(0) = R^{2n}.$$

Hence

$$\frac{1}{z^2} - \sum_{k=1}^n \frac{c_k}{z - z_k} - \sum_{k=1}^n \frac{\alpha_k}{(z - z_k)^2} - 2\pi i \varphi^*(z) = \frac{R^{2n}}{z^2 B^2(z)}. \quad (25)$$

As each zero  $z_k$  of the function  $B^2(z)$  is dual (see (9)), it is easy to assure that

$$B^2(z) = (B'(z_k))^2 (z - z_k)^2 P(z); \quad (26)$$

notably  $P(z_k) = 1$  and  $P(z)$  is the analytic function defined in the certain area of the point  $z_k$ . Actually, we have (see (14), (15))

$$B^2(z) = \frac{2(B'(z_k))^2}{2!} (z - z_k)^2 + \dots$$

The equations (26) follows from that. Thus (see (25), (9))

$$\begin{aligned} -\alpha_k &= R^{2n} \lim_{z \rightarrow z_k} \frac{(z - z_k)^2}{z^2 B^2(z)} = R^{2n} \lim_{z \rightarrow z_k} \frac{1}{z^2 (B'(z_k))^2 P(z)} = \frac{R^{2n}}{\left(\frac{nR^n}{1 - R^{2n}}\right)^2} = \\ &= \frac{(1 - R^{2n})^2}{n^2}. \end{aligned}$$

Hence the validity of formulas (24) follows. After this we observe the following functions

$$\psi(z) = \frac{z^n - R^n}{z - z_k} = z^{n-1} + z^{n-2} z_k + \dots + z z_k^{n-2} + z_k^{n-1}. \quad (27)$$

Let us calculate  $\psi(z_k)$  и  $\psi'(z_k)$ . Since  $(z - z_k)\psi(z) = z^n - R^n$ , then  $\psi(z) + (z - z_k)\psi'(z) = n z^{n-1}$ ,  $2\psi'(z) + (z - z_k)\psi''(z) = n(n - 1)z^{n-2}$ .

From this

$$\psi(z_k) = n z_k^{n-1} = \frac{n z_k^n}{z_k} = \frac{n R^n}{z_k} \quad (28)$$

(however, the formula (28) can be obtained right from the formula (27));

$$\psi'(z_k) = \frac{n(n - 1)R^n}{2z_k^2}. \quad (29)$$

Finally, let us calculate coefficients  $c_k$ . Denote as

$$q(z) = z\psi(z).$$

Then

$$q(z_k) = nR^n, \quad (30)$$

$$q'(z_k) = \frac{n(n + 1)R^n}{2z_k}. \quad (31)$$

In fact, the formula (30) follows directly from (28).

As  $q'(z) = \psi(z) + z\psi'(z)$ , we obtain the formula (31) from the formulas (28), (29). We have (see (25), (30), (31))

$$\begin{aligned} -c_k &= R^{2n} \lim_{z \rightarrow z_k} \left( \frac{(z - z_k)^2}{z^2 B^2(z)} \right)' = R^{2n} \lim_{z \rightarrow z_k} \left( \frac{(z - z_k)^2 (1 - R^n z^n)^2}{z^2 (z^n - R^n)^2} \right)' = \\ &= R^{2n} \lim_{z \rightarrow z_k} \left( \frac{(z - z_k)^2 (1 - R^n z^n)^2}{z^2 (z - z_k)^2 \psi^2(z)} \right)' = R^{2n} \lim_{z \rightarrow z_k} \left( \frac{(1 - R^n z^n)^2}{q^2(z)} \right)' = \\ &= R^{2n} \lim_{z \rightarrow z_k} \left( \left( \frac{1 - R^n z^n}{q(z)} \right)^2 \right)' = \\ &= R^{2n} \lim_{z \rightarrow z_k} 2 \frac{1 - R^n z^n}{q(z)} \frac{-nR^n z^{n-1} q(z) - (1 - R^n z^n) q'(z)}{q^2(z)} = \\ &= 2R^{2n} \frac{1 - R^{2n}}{nR^n} \left( \frac{\frac{-nR^{2n}}{z_k} - (1 - R^{2n}) \frac{n(n+1)}{2z_k}}{\frac{nR^{2n}}{z_k} + (1 - R^{2n}) \frac{(n+1)}{2z_k}} \right) = \\ &= -\frac{1 - R^{2n}}{n^2} \frac{(n-1)R^{2n} + n + 1}{z_k}. \end{aligned}$$

From this we confirm the validity of the formula (23). It is clear that coefficients  $c_k$  and  $\alpha_k$  are the only one. Let us note that coefficients  $\alpha_k$  are negative and equal with all  $k = 1, \dots, n$ .

## 4 Results

Thus, the work has the solution of the stated problem. The measure of inaccuracy of the best method of approximation of the values  $f'(0)$  of the values  $f(z_1), f'(z_1), \dots, f(z_n), f'(z_n)$  where  $f(z) \in B^1(K)$  and the corresponding extremum function is written out. The coefficients of the best straight-linear method of recovery are calculated in this article.

The formulas have a convenient look for their calculation.

## 5 Conclusion

We decided to represent the stated problem in frames of the confined analytic functions. It would be interesting to solve an analogous problem in the Hardy space and if the derivatives have a higher order. It means that it could be possible to observe the problem of the optimal recovery of values  $f'(0)$  of the values  $f(z_1), f'(z_1), \dots, f(z_n), f'(z_n)$  where  $f(z) \in H^p$ ,  $1 \leq p < \infty$  and numbers  $z_1, z_2, \dots, z_n$  look like (2).

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