Application of branching processes to solve boundary value problems for the semi-linear Helmholtz equation

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Abstract. This work devoted to construct a probabilistic model for solving boundary value problems for the semi-linear Helmholtz equation. Probabilistic representations of problem solving in the form of the mathematical expectation of some random variable are obtained. In accordance with probabilistic representation, branching processes are constructed and modeling formulas are specified for each of the branching processes. It has been proven that the constructed branching processes degenerate with probability one and the average number of particles of each n-th generation is less than or equal to one. For the considered problems, unbiased estimates were constructed on the trajectories of the corresponding random processes. Unbiased estimates which constructed on the trajectories of a branching process with a limited average number of branches, easily modeled in computer and has limited variance. Based on the proposed estimates, computational experiments were carried out. The results of the computational experiment show that using the constructed algorithm it is possible to solve semi-linear problems that occur in practice.

1 Introduction

In this work we will study a probabilistic representation of the solution of the Helmholtz boundary problem for the non-linear problem

$$-\Delta u(x) + cu(x) = f(u),\ x \in D,\ u|_{\Gamma} = \psi \tag{1}$$

where, $f(u)$ in our case could be a function $\exp(\mathrm{u}), \sin(\mathrm{u})$ or $\cos(\mathrm{u})$. About some review and existents of the solution semi-linear elliptic boundary value problem was written in the paper [1]. Under the assumption of the existence of a solution, an unbiased estimator is constructed on the trajectories of the proposed branching process "walk on spheres". To do this, using Green's formula, a special integral equation is written that connects the value of the function with its integrals over a ball and a sphere of maximum radius centered at a point and entirely contained in the region under consideration. It is proved that under certain conditions there exists a fixed point for the nonlinear integral operator corresponding to the integral equation. In this case, the iteration process method converges and classical Monte Carlo methods could be used. A probabilistic representation of the solution of the problem in

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the form of the mathematical expectation of some random variable is obtained. In accordance with the probabilistic representation, a branching process of walk on spheres is constructed and an unbiased estimator of the solution of the problem with finite variance is constructed on its trajectories. Recently appeared several papers devoted to above mentioned problems [2-4,15-17].

2 Formulation of the problem

First of all, let us present some well-known results from [5], for the solution of the Dirichlet problem in the three-dimensional case for the linear Helmholtz equation $\Delta u - cu = -f$, where $c > 0$. The function $f(x)$ satisfies the Golder condition in the domain D bounded by the regular boundary Γ, on which the boundary condition $u|_\Gamma = \varphi(x)$ is given:

$$\Delta u(x) - cu(x) = -f,\ x \in D,\ u|_\Gamma = \varphi \tag{2}$$

Let's introduce the notation: $\bar{D}$ - closure of the region D; function $d(x) = \min_{y\in\Gamma}|x-y|$ is the distance from the point x to the boundary Γ; $\Gamma_\varepsilon = \{x \in \bar{D}: d(x) < \varepsilon\}, \varepsilon$ - neighborhood of Γ; the sphere $S(x)$ is the largest of the spheres centered at the point x that lie entirely in $\bar{D}$, i.e. $S(x) = \{y \in \bar{D}: |y-x| = d(x)\}$. Let it be necessary to estimate the solution $u(x_0)$ of the stated problem at a given point $x_0 \in D$. It is known from the theory of fundamental solutions [2] that $u(x_0)$ can be represented as

$$u(x_0) = \frac{d_0\sqrt{c}}{4\pi d_0^2 \operatorname{sh}\left(d_0\sqrt{c}\right)} \int_{S(x_0)} u(s)ds + \int_{|x-x_0|<d_0} \frac{\operatorname{sh}\left[(d_0 - |x-x_0|)\sqrt{c}\right]}{4\pi|x-x_0|\operatorname{sh}\left(d_0\sqrt{c}\right)} f(x)dx \tag{3}$$

where $d_0 = d(x_0)$. The relation (2) must be supplemented with the following equality, which takes into account the boundary condition of the problem:

$$u(x_0) = \varphi(x_0), \qquad x_0 \in \Gamma \tag{4}$$

For the function $u(x)$, we can write the following integral equation:

$$u(x) = \int_D k(x,y)u(y)dy + \phi(y), \tag{5}$$

where

$$k(x,y) = \begin{cases} \dfrac{d\sqrt{c}}{\operatorname{sh}\left(d\sqrt{c}\right)}\delta_r(y), & x \notin \Gamma_\varepsilon, \\ 0, & x \in \Gamma_\varepsilon, \end{cases}$$

$$\phi(x) = \begin{cases} \dfrac{1}{4\pi}\displaystyle\int_{|x-y|<d} \frac{sh\left[(d-|x-y|)\sqrt{c}\right]}{|x-y|\,sh\left(d\sqrt{c}\right)} f(y)d(y), & x \notin \Gamma_\varepsilon \\ u(x), & x \in \Gamma_\varepsilon \end{cases}$$

Here $d = d(x), \delta_x(y)$ is the generalized density corresponding to the uniform probability distribution on the sphere $S(x)$. The probabilistic approach to solving the problem is associated with the Markov chain, which is called a walk-in sphere or a spherical process.

We will consider a probabilistic approach to solving the problem (2,4) in the special case $f(x) = f(u)$ which is mentioned in (1). Recently, new works have appeared devoted to solving this kind of boundary value problems using the Monte Carlo method [6,7].

3 Probabilistic representation construction of an unbiased estimators of the solutions

Applying the equation (3) to the equation $-\Delta u(x) + cu(x) = f(u)$ in (1) we get the following integral representation:

$$u(x) = \frac{R\sqrt{c}}{4\pi R^2 \operatorname{sh}(R\sqrt{c})} \int_{S_R} u(y)dy + \int_{K_R} \frac{\operatorname{sh}[(R-|x-y|)\sqrt{c}]}{4\pi|x-y|\operatorname{sh}(R\sqrt{c})} f(u(y))dy \qquad (6)$$

Here $R(x) = \min_{y\in\Gamma}|x-y|$ is the distance from the point x to the boundary, K_R is the ball of radius R centered at the point x, S_R - corresponding sphere. The first integral in (6) is the integral over the surface of the sphere S_R. Since the surface area of the sphere S_R is equal to $4\pi R^2$, after introducing this expression under the sign of the first integral, it will represent a uniform distribution on the sphere:

$$u(x) = \frac{R\sqrt{c}}{\operatorname{sh}(R\sqrt{c})} \int_{S_R} u(x+R\omega)d\omega +$$
$$+\frac{\operatorname{sh}(R\sqrt{c}) - R\sqrt{c}}{\operatorname{sh}(R\sqrt{c})} \int_{K_R} \frac{\operatorname{sh}[(R-|x-y|)\sqrt{c}]}{4\pi|x-y|(\operatorname{sh}(R\sqrt{c}) - R\sqrt{c})} \cdot \frac{f(u(y))}{c} dy$$

Let's rewrite this expression as follows

$$u(x) = q\int_{S_R} u(y)d\omega + (1-q)\int_{K_R} p(x,y)\frac{f(u(y))}{c}dy \qquad (7)$$

the following notation was used here: $q = \frac{R\sqrt{c}}{\operatorname{sh}(R\sqrt{c})}$, ω is the uniform distribution on S_R,

$$p(x,y) = \frac{\operatorname{sh}[(R-|x-y|)\sqrt{c}]}{4\pi|x-y|(\operatorname{sh}(R\sqrt{c}) - R\sqrt{c})}$$

transition density from x to $y(x, y \in K_R)$. Let D is bounded domain in R^3 with regular border Γ, $f(x,u) = \sum_{n=0}^{\infty} a_n(x)u^n(x)$, where coefficients satisfy the conditions $\lim_{n\to\infty} \sqrt[n]{\bar{a}_n} = 0$, where $\sup_{x\in D}|a_n(x)| \le \bar{a}_n$ and $\sum_{n=0}^{\infty} \bar{a}_n n < \infty$. Suppose $\varphi(x) \in C(\bar{\Gamma})$, $a_n(x) \in C(\bar{D})$ $(n = 0,1,2,\dots)$ and $c > 0$. In this case there are exist [8] the only continuous solutions of semi-linear problem (1). Using this assertion, the equation (7) will be in the form

$$u(x) = q\int_{S_R(x)} u(y)d\omega + (1-q)\int_{K_R(x)} p(x,y)\sum_{n=0}^{\infty}\frac{a_n(y)}{c}u^n(y)dy. \qquad (8)$$

In accordance with (8), it is natural to construct unbiased estimates for solving problem (1) on the trajectories of the process of walk on spheres with branching, in which particles of the same type.

When using Monte Carlo methods, the convergence of the iteration method for the problems under consideration is assumed. Let us define the conditions under which the nonlinear integral operator corresponding to the probabilistic representation (8) and defined

below is a compression operator and thus has a fixed point (for fixed points for nonlinear operators, see, for example, [8-10]. Let us define the operators $\Re_1$and $\Re_2$acting on the functions from $C(\bar{D})$ by the formulas:

$$\Re_1 u(x) = \int_{\bar{D}} \frac{u(y)}{4\pi R^2} I_{S_R(x)}(y) d\mu(y), \qquad (x) \in \bar{D};$$

$$\Re_2 u(x) = \int_{\bar{D}} p\,(x,y) u(y)\; I_{K_R(x)}(y) d\mu(y), \qquad (x) \in \bar{D};$$

Here the measure μ - is defined on σ - the algebra of Borel subsets $\bar{D}$ by the equality $\mu(A) = \lambda(A) + S(A \cap \Gamma)$, λ - the Lebesgue measure in R^3 , *S*- the surface area. Note that for fixed *x* the following integrals are equal to one:

$$\int_{\bar{D}} \frac{1}{4\pi R^2} I_{S_R(x)}(y) d\mu(y) = 1, \qquad \int_{\bar{D}} p\,(x,y)\; I_{K_R(x)}(y) d\mu(y) = 1.$$

Let's consider a nonlinear integral operator $\boldsymbol{F}$ mapping in $B \subset C(\bar{D})$ to $C(\bar{D})$:

$$\mathbf{F}u(x) = q(x)\Re_1 u(x) + \frac{(1-q(x))}{c} \sum_{n=0}^{\infty} \Re_2\,(a_n(x) u^n(x)),$$

which has a Fréchet derivative at every point of $B_0 \subset B$ a convex closed set. Recall that for the nonlinear operator **F:**
$\mathbf{F}v(x) = \int_B K\,(x,y) g(v(y),y) dy$ the Fréchet derivative at a point is determined by the equality [8] , $\boldsymbol{F}'(v)u = \int_B K\,(x,y) g_v'(v(y),y) u(y) dy$.

In the next theorem, we define the conditions imposed on the coefficients that ensure the convergence of the iterative method for the integral equation $u(x) = \boldsymbol{F}u(x)$ corresponding to the probabilistic representation (8).

Theorem 1. Let $B_0 = \{v: \|v\| < 1\}$ and $M = \sum_{n=1}^{\infty} n\,\bar{a}_n < c$. Then the operator $\boldsymbol{F}$ is a compression operator, i.e. , $\|\boldsymbol{F}u - \boldsymbol{F}v\| \le q_0 \|u - v\|$, $(u, v \in B_0)$, $q_0 < 1$, and iteration method for any initial element $u_0 \in B_0$converge to a unique solution $u(x)$ to the equation $u(x) = \boldsymbol{F}u(x)$.

Proof. Let it be$B \subset C(\bar{D})$ and $q_0 = \sup_{v \in B_0} \|\boldsymbol{F}'(v)\|$. As is known from [6], the operator is a compression operator and therefore, there is a fixed point, that is, which is a solution to the integral equation (7). In this case, the value u(x) can be obtained as the limit of the iteration sequence $\{u_n(x)\}$, here $u_{n+1}(x) = \boldsymbol{F}u_n(x)$, where $u_0(x)$ is an arbitrary element of B_0. Let's define a subset $B_0 \in B$, where $q_0 < 1$. The Fréchet derivative of a nonlinear operator $\boldsymbol{F}$ at a point is defined as follows:

$$\mathbf{F}'(v)u = q(x)\Re_1'(v)u + \frac{(1-q(x))}{c} \sum_{n=0}^{\infty} \Re_{2n}'(v)u = q(x) \int_{\bar{D}} \frac{u(y)}{4\pi R^2} I_{S_R(x)}(y) d\mu(y) +$$

$$+ \frac{(1-q(x))}{c} \sum_{m=0}^{\infty} n \int_{\bar{D}} p\,(x,y) a_n(y) v^{n-1} u(y)\; I_{K_R(x)}(y) d\mu(y).$$

Here

$$\Re_{2n}(u) = \Re_2(a_n(x,t)u^n(x,t)) = \iint_{\bar{D}} p\,(x,y)a_n(y,\tau)u^n(y,\tau)d\mu(y).$$

Let us estimate the norm of the derivative. From the properties of the norms it follows:

$$\|\boldsymbol{F}'(v)\| \le q(x)\|\Re_1'(v)\| + \frac{(1-q(x))}{c}\sum_{n=1}^{\infty}\|\Re_{2n}'(v)\| \le$$

$$\le q(x)\|\Re_1\| + \frac{(1-q(x))}{c}\sum_{n=0}^{\infty} n\,\bar{a}_n\|\Re_2\|\|v\|^{n-1}$$

Due to conditions $B_0 = \{v: \|v\| < 1\}$ and $M = \sum_{n=1}^{\infty} n\,\bar{a}_n < c$, we have $q_0 = \sup_{v\in B_0}\|\boldsymbol{F}'(v)\| < 1$, and the theorem has been proven.

If the conditions of the theorem are satisfied, for solution to the nonlinear integral equation, which we will further evaluate, we could apply the method of statistical modeling.

Let's get started to construct a random branching process consistent with a probabilistic representation. For these purposes in accordance with representation (8), we construct a walk on sphere process with branching in the phase space D, in which particles of the same type participate.

Let $M = \sum_{n=1}^{\infty} n\,\bar{a}_n$, and $0 < \alpha < 1$. At the initial moment there is one particle at point $x_0 = x$. Let it be known $n > 0$ and x_n. For a unit of time, the particle with probability $q = \frac{R\sqrt{c}}{sh(R\sqrt{c})}$ goes to a point x_{n+1} uniformly distributed on the sphere $S_R(x)$ and with probability $1-q$ goes to a point x_{n+1}, distributed in the ball $K_R(x)$ with density $p(x,y)$. In the second case with the probability $\pi_n = \frac{\alpha}{M}\bar{a}_n,\ (n = 1,2,\cdots)$ a particle is divided into n new particles or with the probability $\pi_0 = 1 - \sum_{n=1}^{\infty}\pi_n$ absorbed. The new particles behave independently of each other in a similar way to the particle under consideration. The process ends if all particles died out in $\bar{D}$, or if all particles fall into the Γ. The parameter αallows you to adjust the number of branches.

Now we will construct Monte Carlo estimators for the problem (1), when the process of walk on spheres with branching from the point $x_0 = x$ is implemented, i.e. at the beginning there is one particle at the point x. Per unit time, the particle with probability $q = \frac{R\sqrt{c}}{\mathrm{sh}\,(R\sqrt{c})}$ passes to the point $x_1 = y_1$, which is chosen uniformly over the sphere $S_R(x)$ and with probability $1-q$ to the point $x_1 = y_2$, which is chosen inside the ball $K_R(x)$ according to the density $p(x,y)$. In the second case, branching will occur with probability π_n $(n \neq 0)$, i.e. the particle is divided into n particles and the estimator is multiplied by $W = \frac{Ma_n(x_1)}{c\bar{a}_n\alpha}$. In the case of $n = 0$, the particle is absorbed at the point x_1 and the estimator is multiplied by $\frac{a_0(x_1)}{c\pi_0}$. Further an independent walk of trajectories is constructed from the point x_1. The new particles behave independently of each other in a similar way to the considered particle. The process terminates either if all particles in $\bar{D}$ die, or if all particles reach the ε neighborhood of the boundary.

Let us define an estimator in a recursive way. Let $\zeta_0(x) = u(x), \zeta_k(x) = \Psi(\zeta_{k-1}(x))$, where

$$\Psi(\zeta(x)) = \begin{cases} \zeta(y_1), & \text{with probability } q(x) \\ W \cdot \prod_{i=1}^{n} \zeta^{(i)}(y_2), & \text{with probability } (1-q(x))\pi_n,\ n \neq 0 \\ W_0, & \text{with probability } (1-q(x))\pi_0 \end{cases}$$

Here $\zeta^{(i)}(y)$ are independent realizations of the random variable $\zeta(y)$. Let $\Re_k$ be the σ-algebra generated by the sequences $\{\omega_i\}_{i=0}^{k-1}, \{\alpha_i^0\}_{i=0}^{k-1}, \{\alpha_i^1\}_{i=0}^{k-1}, \{\alpha_i^2\}_{i=0}^{k-1}$. The following assertion is true.

Theorem 2. The sequence $\{\zeta_k(x)\}_{k=0}^{\infty}$ forms a martingale with respect to $\{\Re_k\}_{k=0}^{\infty}$. If $M < c$, then $\zeta_k(x,t)$ is a uniformly integrable martingale.

Proof. It is obvious from the definition of $\Re_k$ that $\zeta_k(x)$ is $\Re_k$-measurable. From the properties of conditional mathematical expectations, it follows that:

$$\mathbf{E}(\zeta_{n+1}(x)/\Re_n) = \mathbf{E}(\Psi(\zeta_n(x))/\Re_n) = q(x)\mathbf{E}(\zeta_n(y_1)/\Re_n) +$$
$$+(1-q(x))\sum_{i=1}^{\infty} \pi_n \mathbf{E}\left(W \cdot \prod_{j=1}^{i} \zeta_n^{(j)}(y_2)/\Re_n \right) + (1-q(x))\pi_0 \mathbf{E}(W_0/\Re_n)$$

In view of the validity of the probabilistic representation (8) we get:

$$\mathbf{E}(\zeta_{n+1}(x)/\Re_n) = q(x)\int_{S_R(x)} \zeta_n(y)d\omega +$$
$$+(1-q(x))\int_{K_R(x)} p(x,y)\left[\sum_{i=1}^{\infty} \pi_n \frac{Ma_n(y)}{c\bar{a}_n\alpha}\zeta_n^i(y) + \pi_0 \frac{g}{c\pi_0}\right] = \zeta_n(x)$$

Thus, the sequence $\{\zeta_k(x)\}_{k=0}^{\infty}$ forms a martingale with respect to $\{\Re_k\}_{k=0}^{\infty}$. To prove the uniform integrability of $\zeta_k(x)$, it suffices to show that $|\zeta_k(x)| \le C, (C = \text{const})$. Since the parameter α is chosen from the condition $\frac{M}{c} \le \alpha < 1$. Since $u(x) \in C(\bar{D}) \cap C^2(\bar{D})$ and $\bar{D}$ is a bounded domain, then $|u(x)| \le$ const for $(x) \in \bar{D}$. Further, under the conditions of the theorem, $|W_n(y)| = \left|\frac{Ma_n(y)}{c\bar{a}_n\alpha}\right| \le 1$, and hence $|\zeta_n(x)| \le C < \infty, (C =$ const $)$. Hence it follows that the sequence $\{\zeta_n(x,t)\}$ is uniformly integrable. The theorem has been proven.

In order to reduce the time of obtaining one implementation of the estimator for the solution of problem (1), we consider a slightly modified process with a smaller number of branches and estimates on it with an arbitrarily small bias. Let us take ε-small enough and consider the inner ε - neighborhood of the boundary: Γ_ε. Let N_1 be the moment of termination of the process inside the region, and N_ε be the moment of the first hit of all particles in Γ_ε. $N = \min\{N_1, N_\varepsilon\}$ process stop time. Then the probability of breaking the trajectory at the point x_n will be equal to:

$$g(x_n) = \begin{cases} 1, & x_n \in \Gamma_\varepsilon \\ \big(1-q(x_{n-1})\big)\pi_0, & x_n \in \bar{D} \setminus \Gamma_\varepsilon \end{cases}$$

From the above conditions it is easy to show, that $N < \infty$.

Theorem 3. If $M < c$, then $\zeta_N(x)$ is an unbiased estimator for $u(x)$ with finite variance.

Proof. Since $\zeta_n(x)$ is a uniformly integrable martingale and N is a Markov moment, then according to Doob's free choice transformation theorem [11] for the martingale $\{\zeta_k(x)\}_{k=0}^{\infty}$, we get $\mathbf{E}\zeta_N(x) = \mathbf{E}\zeta_1(x)$. From the definition of $\zeta_1(x)$ and the probabilistic representation (8) it follows that

$$\mathrm{E}(\zeta_1(\mathrm{x})) = \mathrm{q}(\mathrm{x})\mathrm{Eu}(\mathrm{y}_1) + (1 - \mathrm{q}(\mathrm{x}))\mathrm{Ef}\big(\mathrm{y}_2, \mathrm{u}(\mathrm{y}_2)\big)/\mathrm{c} = \mathrm{u}(\mathrm{x})$$

By consequence of the conditions of Theorem 2, $\mathbf{E}(\zeta_N(x))^2 < \infty$ and hence its variance is finite. The theorem has been proven. It should be noted that all obtained results could be extended to the Semi-Linear Neumann Boundary Value Problems [see:12,13].

Next, from $\zeta_N(x)$ a biased but practically realizable estimator $\zeta_N^*(x)$ is constructed in the standard way. Let x^* be the boundary point closest to $x\Gamma$. $\zeta_N^*(x)$ is obtained by replacing $u(x_N)$ in $\zeta_N(x)$ with $\varphi(x_N^*)$. Let us estimate the displacement $\zeta_N^*(x)$. It's clear that $|\mathbf{E}\zeta_N^*(x) - u(x)| \le \mathbf{E}|\zeta_N^*(x) - \zeta_N(x)|$. If $N = N_1$, then $T_N = \{\theta\}$ and the process terminates without hitting Γ_ε. In this case $\zeta_N^*(x) = \zeta_N(x)$ and the offset is zero. If $N = N_\varepsilon$, then $Z_N = (x_N^1, n_N^1; x_N^2, n_N^2; \dots; x_N^k, n_N^k)$, where $x_N^i \in \underline{\Gamma}, i = 1,2, \dots, k$ and the number k does not depend on N. Given that $|W_n(y)| \le 1$ for arbitrary n and $y \in \bar{D}$, we have

$$\zeta_{\mathrm{N}}^*(\mathrm{x}) - \zeta_{\mathrm{N}}(\mathrm{x}) \le \left(\prod_{\mathrm{i}=1}^{\mathrm{k}} \left[\varphi\big(\mathrm{x}_{\mathrm{N}}^{\mathrm{i}} *\big)\right]^{\mathrm{n}_{\mathrm{N}}^{\mathrm{i}}} - \prod_{\mathrm{i}=1}^{\mathrm{k}} \left[\mathrm{u}\big(\mathrm{x}_{\mathrm{N}}^{\mathrm{i}} *\big)\right]^{\mathrm{n}_{\mathrm{N}}^{\mathrm{i}}} \right)$$

Let $L(\varepsilon)$ be the modulus of continuity of the function $u(x)$; then

$$|\mathrm{E}\zeta_{\mathrm{N}}^*(\mathrm{x}) - \mathrm{u}(\mathrm{x})| \le \mathrm{L}(\varepsilon)\mathrm{E}\big(\mathrm{n}_{\mathrm{N}}^1 + \mathrm{n}_{\mathrm{N}}^2 + \cdots + \mathrm{n}_{\mathrm{N}}^{\mathrm{k}}\big).$$

Considering that the average number of particles in the N th generation

$$\mathrm{E}\big(\mathrm{n}_{\mathrm{N}}^1 + \mathrm{n}_{\mathrm{N}}^2 + \cdots + \mathrm{n}_{\mathrm{N}}^{\mathrm{k}}\big) \le \mathrm{K}^{\mathrm{N}} < 1$$

we find that the bias does not exceed $L(\varepsilon)$. The finiteness of the variance $\zeta_N^*(x)$ follows from $|W_n(y)| \le 1$.

4 The numerical experiment

The results of numerical experiment are given in this section. The following problems (Pr.1 and Pr.2) were considered:

(Pr.1)
$$\begin{cases} -\Delta \mathrm{u}(\mathrm{x}) + 4\mathrm{u}(\mathrm{x}) = \sin(\mathrm{u}) + \mathrm{f}_0, \ \mathrm{x} \in \mathrm{D} = \{(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) : \mathrm{x}_1^2 + \mathrm{x}_2^2 + \mathrm{x}_3^2 \le 1\} \\ \mathrm{u}(\mathrm{x}) = 0.4\exp(-\mathrm{x}_1 - \mathrm{x}_2 - \mathrm{x}_3), \ \mathrm{x} \in \partial\mathrm{D} \end{cases}$$

where $f_0(x) = 0.4\exp(-x_1 - x_2 - x_3) - \sin\big(0.4\exp(-x_1 - x_2 - x_3)\big)$,

(Pr.2)
$$\begin{cases} -\Delta u(x) + 4u(x) = \cos(u) + f_0, \ x \in D = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \le 1\}, \\ u(x) = 0.4\exp(-x_1 - x_2 - x_3), \ x \in \partial D. \end{cases}$$

where $f_0(x) = 0.4\exp(-x_1 - x_2 - x_3) - \cos\big(0.4\exp(-x_1 - x_2 - x_3)\big)$.

Denote (x_0) as the point where the problem is solved. α is a parameter, which is used in π_n and regulating the quantity of the branches. Let's designate the average quantity of the particles of the tree as AQP and the average number of branches as AQB. U_{ex} is the exact solution in point x_0, that is $U_{ex} = u(x_0)$. MS is the estimator of $MS = \bar{\xi}$ and σ is the

statistical estimator of the value $\sqrt{D\xi/N}$, where $D\xi$ is the variance of the estimator. 3σ is the confidence interval for the estimator. Let's designate err as the difference between the exact solution and estimator, err $= |u(x_0) - MS|$.

For the computational experiment, the values $N = 5000$ (the number of simulated trajectories) and $\varepsilon = 0.005$ (the neighborhood of the boundary) were used. The results of the computational experiment are given in the following table 1.

Table 1. Results of computational experiments

Pr	α	x_0	U_{ex}	S	DS	err	3σ	AQB	AQP
1	0,9	(-0,4; 0,2; -0,6)	0,89022	0,89204	0,20223	0,00182	0,01908	1,05	12,32
1	0,9	(-0,7; 0,2; -0,1)	-0,56464	-0,56740	0,30100	0,00276	0,02328	1,06	12,43
2	0,8	(0,9; 0,2; 0,1)	0,00032	-0,00049	0,00371	0,00082	0,00259	1,08	13,18
2	0,8	(0,9; 0,2; 0,1)	0,00032	0,00198	0,00281	0,00166	0,00225	1,07	13,22

The results of computational experiment show that with the discussed algorithms we can obtain practically efficient estimators. The parameter α, which is used in calculating the probabilities of the branching, gives us a chance to control the average quantity of the branches and average number of the particles in the tree. In our case, with the probability 0.997, the exact solution of the problem will be in the confidence interval $(\bar{\xi} - 3\sigma, \bar{\xi} + 3\sigma)$. Since in the given examples, the exact solutions are known, we can make sure that all the estimators are in the confidence interval (see table 1).

5 Conclusion

In this work we proposed Monte Carlo algorithms for the solution of the interior Dirichlet boundary value problem for the Helmholtz operator with a polynomial nonlinearity on the right-hand side.

Partial differential equations and Monte Carlo method, which have been used to model many phenomena in the natural sciences and engineering. In some cases, our method could be applied in the area reaction diffusion equations, homogenization of PDEs, stochastic dynamics, interacting particle systems, non-linear Poisson-Boltzmann equations, electrostatics in bio-molecular structure and dynamics and etc. Some last applications this method to the heat conductivity problems with quadratic nonlinearity was in paper [14].

The preliminary results of on computational experiment show, that with the discussed algorithms we can obtain practically efficient estimators. The parameter, which is used in calculating the probabilities of the branching, gives us the opportunity to control the average quantity of branches and the average number of the particles in the constructed tree. Thus, the estimators were constructed on trees with a minimal number of branches. In our approach we have used an estimator via absorption. In the future we could built an estimator via collision and hope will get a new result. The results of numerical experiments and applications to concert problems we are going to give in our next publications.

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