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Characterization of Directed Graphs Representing C*-Algebra of Complex Matrices

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Abstract. Quantum mechanics is a study that plays a major role in determining the biological intelligence of Artificial Intelligence (AI). Point particle systems in quantum mechanics can be explained using C*-Algebra which is called CAR-algebra. There is a special case in the CAR-algebra which is isomorphic to the C*-algebra of complex matrices. On the other hand, C*-algebras of direct sum of complex matrix spaces is isomorphic to C*-algebra constructed by orthogonal projection and partial isometries operators via the Cuntz-Krieger relations of a directed graph. This article will provide a basis for the relationship between quantum mechanics and graphs through a discussion of the characterization of graphs that can represent C*-algebra of complex matrices. It is found that C*-algebra complex matrices $n \times n$ is a directed graph without cycles with $n-1$ arrows, a single source, and has n path from the source.

1 Introduction

The era of Artificial Intelligence (AI) is inevitable. As AI becomes increasingly sophisticated, various fields are starting to use it to simplify almost all required processes, even as if it will replace every human job. AI is created in such a way as to resemble human nature by matching biological intelligence. According to Kak [1], this biological matching requires the concept of quantum mechanics. Ajagekar [2] in his article discusses how quantum AI systems are also very useful for sustainable and renewable energy. Understanding quantum mechanical systems is of course inseparable from mathematical concepts in their formulation. One concept that is quite popular in modeling physical observations from quantum mechanics is the C*-algebra [3]. In the paper Bru [4], one of the C*-algebras for quantum mechanics is called the CAR (Canonical Anticommutation Relation)-algebra. In the special case when the configuration of point particles is finite, the CAR-algebra formed will be isomorphic to the C*-algebra of complex matrices of size $2^n \times 2^n$

C*-algebra is an abstract structure that is quite complicated to understand. To make it easier, it can be explained through operators in Hilbert space, $B(H)$, Raeburn [5] makes it even more interesting by representing it using a directed graph so as to characterize the C*-

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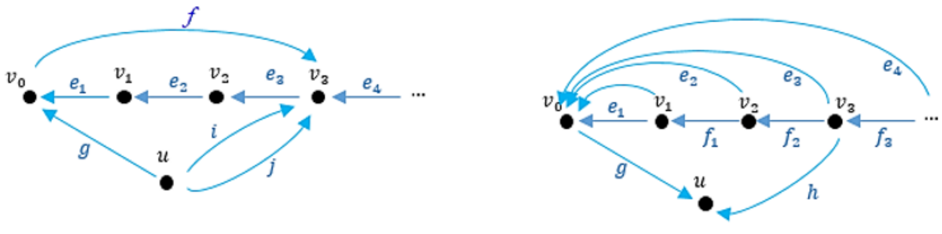
algebra through illustrations. He used orthogonal projection and partial isometry operators combined with vertices and directed edges or arrows on the graph. As the CAR-algebra that is isomorphic to the C*-algebra of complex matrices, special case of the C-algebra of Raeburn's graph is also a C*-algebra of complex matrices. However, the characterization of graphs that represent C*-algebra complex matrices has not yet been carried out because a particular graph is given first, while on the other hand it has not been shown whether that graph as the only representation or whether there will be other forms. Therefore, to connect CAR-algebra and C*-algebra of graphs through the same complex matrix space, this article discussed the characterization of graphs that represent C*-algebra of complex matrices. Thus this discussion can make quantum mechanics represented more interestingly through mathematical concepts, namely graphs.

In this article, the terminology and notation used in [5] and [6] will be used. Here, the graphs used are all directed graphs, so the word graph will refer to directed graphs.

A directed graph E is a system (E^0, E^1, s, r) consisting of a set of *vertices* E^0, a set of *directed edges* or *arrows* E^1, a *source function* $s: E^1 \to E^0$ with $s(e)$ is the *source vertex* of the arrow $e \in E^1$, and a *range function* $r: E^1 \to E^0$ with $r(e)$ is the *final vertex* of $e \in E^1$. A directed graph is *row-finite* if each final vertex receives only a finite number of arrows.

Consider the following graphs.



(a) (b)

Fig. 1. (a) A row-finite graph
(b) Not a row-finite graph

In Figure 1. (a), each final vertex receives at most 3 arrows, so the graph above is a row-finite graph. Meanwhile, in Figure 1. (b), there is a final vertex that receives an infinite number of arrows, namely v_0. Then, the graph in the figure is not a row-finite graph.

Apart from the term source vertex, there is a term called *source* which is defined as a vertex that does not receive any single arrow. There is also a term called *sink* which is defined as a vertex that does not emit any single arrow. For example, in Figure 1. (a), there is one vertex which is the source, namely u, while in Figure 1. (b), the vertex u is the sink. It is very important to pay attention to the use of the word “source” in these two different terms. For the source vertex is always accompanied by the sentence "from the arrow …". For an example, the vertex u in Figure 1. (a) is the source, on the other hand the vertex u is also the source vertex of the arrow g, i, and j.

C*-algebra according to Strung [7] is a complete normed space A over a field □ equipped with associative multiplication operation satisfying $\|ab\| \leq \|a\|\|b\|$, and equipped with involution and satisfy identity $\|a^*a\| = \|a\|^2$.

In this article, the C*-algebra discussed is specifically the C*-algebra of operators, that is every closed subspace of $B(\mathcal{H})$, for a Hilbert space $\mathcal{H}$, with

$$B(\mathcal{H}) = \{ T: \mathcal{H} \to \mathcal{H} \mid \text{bounded and linear operator}\} \quad (1)$$

and the norm defined by $\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\}$ [8].

Closed subspaces as C*-algebras will be constructed by two types of operators in (1), namely orthogonal projections and partial isometries. *Orthogonal projection* is a linear operator $P: \mathcal{H} \to \mathcal{H}$ such that for a subspace M of the Hilbert space $\mathcal{H}$ holds $Px \in M$ and $(x - Px) \perp M, \forall x \in \mathcal{H}$ [9].

Furthermore, because each projection that used in this article is an orthogonal projection, the term will be simplified to just a projection.

Theorem 1.1 [10]
An operator P on a Hilbert space $\mathcal{H}$ is a projection on M subspaces of $\mathcal{H}$ if and only if $P^2 = P = P^*$.

A *partial isometry* in a Hilbert space $\mathcal{H}$ is a linear operator $S: \mathcal{H} \to \mathcal{H}$ so that its restriction on a subspace $(\ker S)^\perp$ is an isometry, that is, it satisfies $\|Sx\| = \|x\|$, $\forall x \in (\ker S)^\perp$, [11].

Theorem 1.2 [11]
An operator S on a Hilbert space $\mathcal{H}$ is a partial isometry if and only if S^*S is a projection.

Based on [12], every operator in C*-algebra has a complex matrix representation through its basis. With this representation, the vector space of complex matrices $n \times n$, $M_n(\mathbb{C})$, equipped with the matrix multiplication operation, adjoint matrices as an involution, and the matrix norm form a C*-algebra. The basis of $M_n(\mathbb{C})$ is $\{E_{ij} = \{\delta_{ij}\} \mid i,j \in \{1,2,\dots,n\}\}$.

2 Cuntz-Krieger graph algebra

Given a row-finite graph, the C*-algebra formed is constructed by orthogonal projection operators and partial isometries that satisfy the Cuntz-Krieger relation. These operators were grouped into a family known as the Cuntz-Krieger family. Formally by [13], *Cuntz-Krieger family* is a family $E\{S,P\}$ of a row-finite directed graph E in a Hilbert space $\mathcal{H}$ consisting of a set of projections $\{P_v \in B(\mathcal{H}) \mid v \in E^0\}$ and a set of partial isometries $\{S_e \in B(\mathcal{H}) \mid e \in E^1\}$ that satisfy two Cuntz-Krieger relations:

(CK 1) $S_e^*S_e = P_{s(e)}, \forall e \in E^1$. (2)
(CK 2) $P_v = \sum_{e\in I} S_e S_e^*, \forall v \in E^1$ with v is not a source and $I = \{e \in E^1 \mid r(e) = v\}$. (3)

The C*-algebra constructed by orthogonal projection operators and partial isometry in the Cuntz-Krieger family is denoted by $C^*(S,P)$.

Theorem 2.1 [5]
Let S_e, for some $e \in E^1$, be a partial isometry on the Hilbert space $\mathcal{H}$ of a Cuntz-Krieger family $E\{S,P\}$ then $S_e = P_{r(e)}S_e = S_e P_{s(e)}$.

Theorem 2.2 [5]
Let $E\{S,P\}$ be a Cuntz-Krieger family on the Hilbert space $\mathcal{H}$ then:
1) Every projection of a set $\{S_e S_e^* \mid e \in E^1\}$ is mutually orthogonal;
2) $S_e^* S_f \neq 0 \Rightarrow e = f$;
3) $S_e S_f \neq 0 \Rightarrow s(e) = r(f)$;
4) $S_e S_f^* \neq 0 \Rightarrow s(e) = s(f)$.

Theorem 2.2, especially part 3) interprets that if $S_e S_f$ define a nonzero partial isometry, then $s(e) = r(f)$, which means that arrows e and f are connected, in graph theory it is called a path. In general, a *path* of length n is a sequence of arrows $\mu = e_1 e_2 \dots e_n$ with $s(e_i) = r(e_{i+1})$, for $i = 1,2,\dots,n-1$. The *length of path* μ is denoted by $|\mu|$. A path $\mu = e_1 e_2 \dots e_n$ can be denoted more simply as $\mu = \{e_i\}_{|\mu|}$. Define a vertex $v \in E^0$ as a path of length 0 and arrow $e \in E^1$ as a path of length 1. There is also a *cycles*, namely a path that has the same source and final vertex.

Fig. 2. A path $\mu = e_1 e_2 \dots e_n$ of length n [6].

Define $E^n = \{ \mu \mid |\mu| = n$, for n non negative integer$\}$ as the set of all paths. By this definition, E^0 and E^1 respectively remain consistent with the definition of a set of vertices and a set of edges, only they are viewed as a set of paths. Also define the set of all paths regardless of length, namely $E^* = \bigcup_{n \geq 0} E^n$.

As with the arrows, source and range function are also defined for a path, that is, if as $\mu = \{e_i\}_{|\mu|} \in E^*$ then $r(\mu) = r(e_1)$ dan $s(\mu) = s(e_{|\mu|})$.

Returning again to Theorem 2.2 part 3), for the path $\mu = ef$ we can define a non-zero partial isometry $S_{ef} \coloneqq S_e S_f$. More generally, for a path $\mu = e_1 e_2 \dots e_n$ we can define the partial isometry $S_\mu \coloneqq S_{e_1} S_{e_2} \dots S_{e_n}$. Specifically for vertex $v \in E^0$ as a path with length 0, the partial isometry is an orthogonal projection, that is, $S_v \coloneqq P_v$.

Theorem 2.3 [5]
Let S_μ, for a path $\mu \in E^*$, be a partial isometry in the Hilbert space $\mathcal{H}$ with initial projection $P_{s(\mu)}$ and final space $S_\mu(\mathcal{H}) \subset P_{r(\mu)}(\mathcal{H})$, then $S_\mu^* S_\mu = P_{s(\mu)}$ and $P_{r(\mu)} S_\mu S_\mu^* = S_\mu S_\mu^*$.

Corollary 2.4 [5]
Let $E\{S, P\}$ be a Cuntz-Krieger family on a Hilbert space $\mathcal{H}$. If $\mu, \alpha \in E^*$, then:

1) if $|\mu| = |\alpha|$ and $\mu \neq \alpha$, then $(S_\mu S_\mu^*)(S_\alpha S_\alpha^*) = 0$;
2) $S_\mu^* S_\alpha = \begin{cases} S_{\mu'}^* & \text{if } \mu = \alpha\mu', \exists \mu' \in E^* \\ S_{\mu'}^* & \text{if } \alpha = \mu\alpha', \exists \alpha' \in E^* \\ 0 & \text{otherwise;} \end{cases}$
3) If $S_\mu S_\alpha \neq 0$, then $\mu\alpha \in E^*$ and $S_\mu S_\alpha = S_{\mu\alpha}$;
4) If $S_\mu S_\alpha^* \neq 0$, then $s(\mu) = s(\alpha)$.

Corollary 2.5 [5]
Let $E\{S, P\}$ be a Cuntz-Krieger family on a Hilbert space $\mathcal{H}$. If $\mu, \alpha, \beta \in E^*$, then:

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'} S_\beta^* & \text{if } \alpha = \nu\alpha', \exists \alpha' \in E^* \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu', \exists \nu' \in E^* \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Corollary 2.5 shows that the product of two similar forms, namely $S_\mu S_\nu^*$ and $S_\alpha S_\beta^*$, produced a similar form, namely $S_{\mu\alpha'} S_\beta^*$ or $S_\mu S_{\beta\nu'}^*$ as in (4). Corollary 2.4 part 4) provided a guarantee that if the result is a non-zero operator then the two related paths have the same source vertex. Consequently, span $\{S_\mu S_\nu^* \mid \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}$ being a subalgebra of a

C*-algebra and its cover being C*-subalgebra. Through this C*-subalgebra, a C*-algebra as a representation of a row-finite graph can be formed, as follows.

Corollary 2.6 [5]
Let $E\{S,P\}$ be a Cuntz-Krieger family on a Hilbert space $\mathcal{H}$. Then:

$$C^*(S,P) = \overline{\operatorname{span}}\left\{S_\mu S_\nu^* \,\middle|\, \mu,\nu \in E^*,\ s(\mu) = s(\nu)\right\}. \tag{5}$$

3 Graph representation for C*-algebra of complex matrices

There are two discussions regarding graph representation for C*-algebra of complex matrices. First, a graph representation for the C*-algebra of direct sums of complex matrix spaces as proven by Raeburn [5]. Second, graph representation for C*-algebra of complex matrix spaces. From the second discussion, we discussed further about characterization of each graph that represent C*-algebra of complex matrix spaces.

3.1 Graph representation for C*-algebra of direct sum of complex matrix spaces

Raeburn [5] has proved that a graphs without cycles is isomorphic to a C*-algebra of direct sums of complex matrix spaces. The theorem is as follows.

Theorem 3.1.1 [5]
Let E be a row-finite graph with finite vertices with no cycles and $w_1,\ w_2,\ \ldots,\ w_n$ are the sources in E. For every Cuntz-Krieger family $E\{S,P\}$ in which each projection is nonzero, then

$$C^*(S,P) \cong \bigoplus_{i=1}^{n} M_{|s^{-1}(w_i)|}(\mathbb{C}) \tag{6}$$

where $s^{-1}(w_i) = \{\mu \in E^* |\ s(\mu) = w_i\}$.

Proof.
For any $i \in \{1,2,\ldots,n\}$, let $A_i = \operatorname{span}\left\{S_\mu S_\nu^* \,\middle|\, \mu,\nu \in E^*,\ s(\mu) = s(\nu) = w_i\right\}$.
The results of the products $\left(S_\mu S_\nu^*\right)\left(S_\alpha S_\beta^*\right)$ on A_i according to (4) on Corollary 2.5 are classified based on $\alpha = \nu\alpha'$ or $\nu = \alpha\nu'$ for α' or ν' having a source vertex w_i. This form requires that there is a cycle in E and w_i is not a source, meanwhile E has no cycle and w_i is a source. Thus, such α' or ν' does not exist so it must be classified based on $\alpha = \nu$. Then

$$\left(S_\mu S_\nu^*\right)\left(S_\alpha S_\beta^*\right) = \begin{cases} S_\mu S_\beta^* & \text{if } \alpha = \nu \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

where $s^{-1}(w_i) = \{\mu \in E^* |\ s(\mu) = w_i\}$.
Consider a basis $\left\{E_{\mu\nu} \middle|\, \mu,\nu \in s^{-1}(w_i)\right\}$ of $M_{n_i}(\mathbb{C})$, then $E_{\mu\nu}$ is a matrix of size $|s^{-1}(w_i)| \times |s^{-1}(w_i)|$ so that $M_{n_i}(\mathbb{C}) = M_{|s^{-1}(w_i)|}(\mathbb{C})$.
Define $S_\mu S_\nu^* \mapsto E_{\mu\nu}$, then $A_i \cong M_{|s^{-1}(w_i)|}(\mathbb{C})$.
For $i \neq j$, $s(\mu) = w_i$ and $s(\nu) = w_j$, which means $s(\mu) \neq s(\nu)$, then based on Corollary 2.4 part 4), $S_\mu S_\nu^* = 0$. Consequently, $A_i \cap A_j = \{0\}$ for every $i \neq j$. Thus,

$$C^*(S,P) \cong \operatorname{span}\left\{\bigcup_{i=1}^{n} A_i\right\} \cong \bigoplus_{i=1}^{n} A_i \cong \bigoplus_{i=1}^{n} M_{|s^{-1}(w_i)|}(\mathbb{C}) \tag{8}$$

■

By the Theorem 3.1.1, the C*-algebra of a direct sum of n complex matrix spaces $M_{n_i}(\mathbb{C})$ can be characterized by a row-finite graph without cycles with n number of sources and for each source, the number of paths with the source vertex being that source is n_i.

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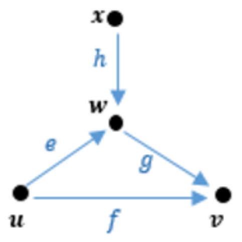
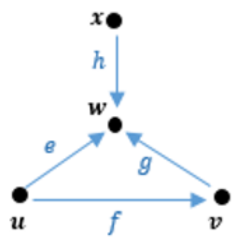
Example 3.1.2



(a)



(b)

Fig. 3. (a) A graph with no cycles, two sources, and some vertex receives two arrows

(b) A graph with no cycles, two sources, and some vertex receives three arrows

For Figure 3. (a), because it does not have a cycle and there are two sources, namely u and x, then Theorem 3.1.1 can be applied. Note that $s^{-1}(u) = \{u, e, f, ge\}$ has 4 paths and $s^{-1}(x) = \{x, h, gh\}$ has 3 paths, then the C*-algebra represented by the graph as in (6) is

$$C^*(S,P) \cong M_{|s^{-1}(u)|}(\mathbb{C}) \oplus M_{|s^{-1}(x)|}(\mathbb{C}) \cong M_4(\mathbb{C}) \oplus M_3(\mathbb{C}). \quad (9)$$

Theorem 3.1.1 can also be applied to the graph in Figure 3. (b) because it has no cycles. There are two sources, namely u and x with $s^{-1}(u) = \{u, e, f, gf\}$ and $s^{-1}(x) = \{x, h\}$, then the C*-algebra represented by the graph is

$$C^*(S,P) \cong M_4(\mathbb{C}) \oplus M_2(\mathbb{C}) \quad (10)$$

The C*-algebra represented by the graph in Theorem 3.1.1 is not the C*-algebra of complex matrices but the direct sum of several complex matrix spaces. The next subsection discussed the C*-algebra of complex matrices.

3.2 Graph representation for C*-algebra of complex matrix spaces

Taking the special case of Theorem 3.1.1, graph representation for C*-algebra of complex matrices of size n is contained in the following theorem.

Corollary 3.2.1
Let E be a row-finite graph with finite vertices with no cycles and there is only one source. Let w be the sources in E. If $|s^{-1}(w)| = n$, then for every Cuntz-Krieger family $E\{S, P\}$ in which each projection is nonzero

$$C^*(S,P) \cong M_n(\mathbb{C}) \quad (11)$$

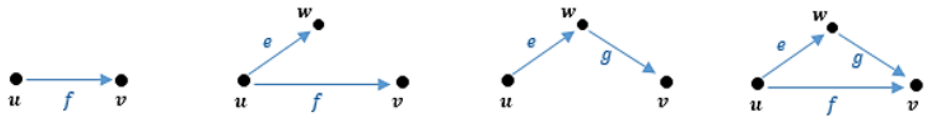
Proof.
There is only one source, so $C^*(S,P) = \text{span}\left\{S_\mu S_\nu^* \,\middle|\, \mu, \nu \in E^*,\ s(\mu) = s(\nu) = w\right\}$ which is the single form of A_i in the proof of Theorem 3.1.1. As a result

$$C^*(S,P) \cong M_{|s^{-1}(w)|}(\mathbb{C}). \quad (12)$$

Since $|s^{-1}(w)| = n$, then $C^*(S,P) \cong M_n(\mathbb{C})$. ∎

From Corollary 3.2.1, we can see that one of the characteristics of graph representing a C*-algebra of a complex matrix space $M_n(\mathbb{C})$ is a row-finite graph without cycles with only 1 source and the number of paths from that source is n.

Example 3.2.2
Consider the following two graphs.



(a) (b) (c) (d)

Fig. 4. (a) A graph with two vertices and one source
(b) A graph with three vertices, one source, and two sinks
(c) A graph with three vertices, one source, and one sinks receiving one arrow
(d) A graph with three vertices, one source, and one sinks receiving two arrows

All graphs in the Figure 4 above do not have cycles and only have one source, namely u. Thus, Corollary 3.2.1 can be applied. For Figure 4. (a), since $s^{-1}(u) = \{u, f\}$, there are two paths, then the C*-algebra represented by the graph is

$$C^*(S,P) \cong M_2(\mathbb{C}). \tag{13}$$

For Figure 4. (b), $s^{-1}(u) = \{u, e, f\}$, there are three paths, then

$$C^*(S,P) \cong M_3(\mathbb{C}). \tag{14}$$

As in Figure 4. (b), the graph in Figure 3. (c) has three paths, namely $s^{-1}(u) = \{u, e, ge\}$, then

$$C^*(S,P) \cong M_3(\mathbb{C}). \tag{15}$$

For Figure 4. (d), with $s^{-1}(u) = \{u, e, f, ge\}$, then

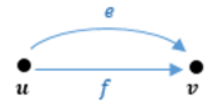
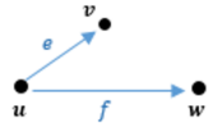
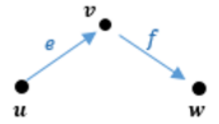
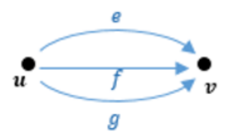
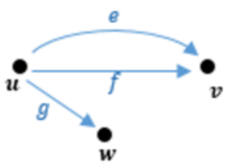
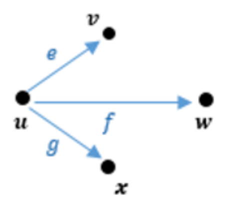
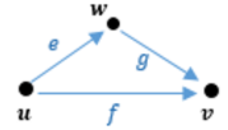
$$C^*(S,P) \cong M_4(\mathbb{C}). \tag{16}$$

The results in Example 3.2.2 show that even though they both have 3 vertices and single source, the graph in Figure 3. (d) provides a C*-algebra $M_4(\mathbb{C})$ representation, different from Figure 3. (b) and (c) which both are graph representation of C*-algebra $M_3(\mathbb{C})$.

Corollary 3.2.1 and Example 3.2.2 provide an illustration of how to find the C*-algebra of complex matrices to be represented by a given graph. Conversely, it is clear that if a C*-algebra of complex matrices is given, it can be represented by a graph without cycles containing only 1 source and 5 paths originating from that source. However, the graphs in Figure 3. (b) and (c) show that the C*-algebra of complex matrices can have a variety of graphs representing them. Thus, it is necessary to review the further characteristics of each graph that represents a C*-algebra of complex matrices. The following table provides examples of the characteristics of each graph that represents several C*-algebras of complex matrices.

Table 1. Configuration of graph representations of some C*-algebra of complex matrices

C*-Algebra	Graph Characteristics	Possible Path	Graph Illustration
$M_2(\mathbb{C})$	- No cycles - 1 source (say u) - 2 paths from u	$\{u, e\}$	

$M_3(\mathbb{C})$	- No cycles - 1 source (say u) - 3 paths from u	$\{u, e, f\}$	 or 
		$\{u, e, fe\}$	
$M_4(\mathbb{C})$	- No cycles - 1 source (say u) - 4 paths from u	$\{u, e, f, g\}$	 or  or 
		$\{u, e, f, ge\}$	 or

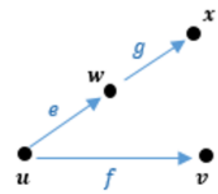
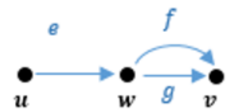
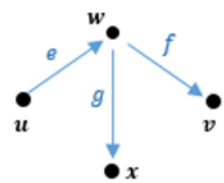
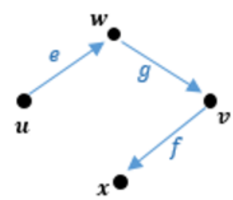
			
		$\{u, e, fe, ge\}$	 or 
		$\{u, e, ge, fge\}$	

Table 1 shows that each $M_n(\mathbb{C})$ for $n = 2,3,4$ can be represented by various types of graphs. For $M_2(\mathbb{C})$, it is only represented by a graph with 2 vertices and 1 arrow. For $M_3(\mathbb{C})$, if it is represented by a graph with 2 vertices then there will be a vertex that receives 2 arrows and if it is represented by a graph with 3 vertices then each vertex receives 1 arrow. For $M_4(\mathbb{C})$, if it is represented by a graph with 2 vertices then there will be a vertex that receives 3 arrows, if it is represented by a graph with 3 vertices then there will be a vertex that receives 2 arrows, and if it is represented by a graph with 4 vertices then each vertex receives 1 arrow.

From the case above, it can be seen that the number of arrows for each $M_n(\mathbb{C})$ is $n - 1$. Furthermore, for $M_n(\mathbb{C})$ to be represented by a graph with n vertices, it is required that every vertex except the source receives exactly 1 arrow. The following example of C*-algebra of complex matrices $M_5(\mathbb{C})$ will illustrate its generalization.

Example 3.2.3

Consider C*-algebra of complex matrices $M_5(\mathbb{C})$. Since it has 5 basis, from the proof of Corollary 3.1.1, for graph E, define its Cuntz-Krieger C*-algebra by

$$C^*(S, P) = \overline{\text{span}}\left\{S_\mu S_\nu^* \;\middle|\; \mu, \nu \in E^*,\ s(\mu) = s(\nu) = w\right\} \tag{17}$$

https://doi.org/10.1051/e3sconf/202448303004

where u is the only source of graph with 5 paths originating from that source. Then the possible paths are $\{u, e, f, g, h\}$, $\{u, e, fe, g, h\}$, $\{u, e, fe, gfe, h\}$, $\{u, e, fe, gfe, hgfe\}$, $\{u, e, fe, ge, h\}$, $\{u, e, fe, ge, hge\}$, $\{u, e, fe, ge, he\}$, and $\{u, e, fe, g, hg\}$.

The following is the illustration of a graph with 5 vertices for each path.

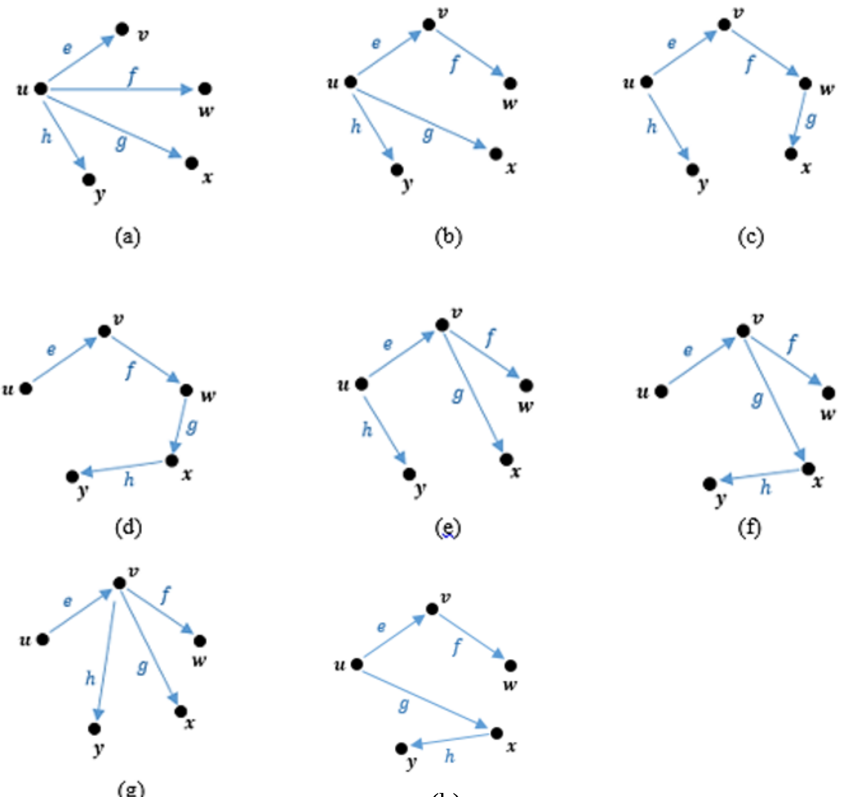


Fig. 5. (a) A 5 vertices graph with path $\{u, e, f, g, h\}$
(b) A 5 vertices graph with path $\{u, e, fe, g, h\}$
(c) A 5 vertices graph with path $\{u, e, fe, gfe, h\}$
(d) A 5 vertices graph with path $\{u, e, fe, gfe, hgfe\}$
(e) A 5 vertices graph with path $\{u, e, fe, ge, h\}$
(f) A 5 vertices graph with path $\{u, e, fe, ge, hge\}$
(g) A 5 vertices graph with path $\{u, e, fe, ge, he\}$
(h) A 5 vertices graph with path $\{u, e, fe, g, hg\}$

Even though they are different, there is a similar characteristics of each graph in the Figure 5 above, they have 4 arrows and each non-source vertex receives exactly 1 arrow.

As in Table 1, the graph representation for $M_5(\mathbb{C})$ is not only for graphs that have 5 vertices. The following is an illustration of a graph with less than 5 vertices for paths $\{u, e, f, g, h\}$ and $\{u, e, fe, g, h\}$.

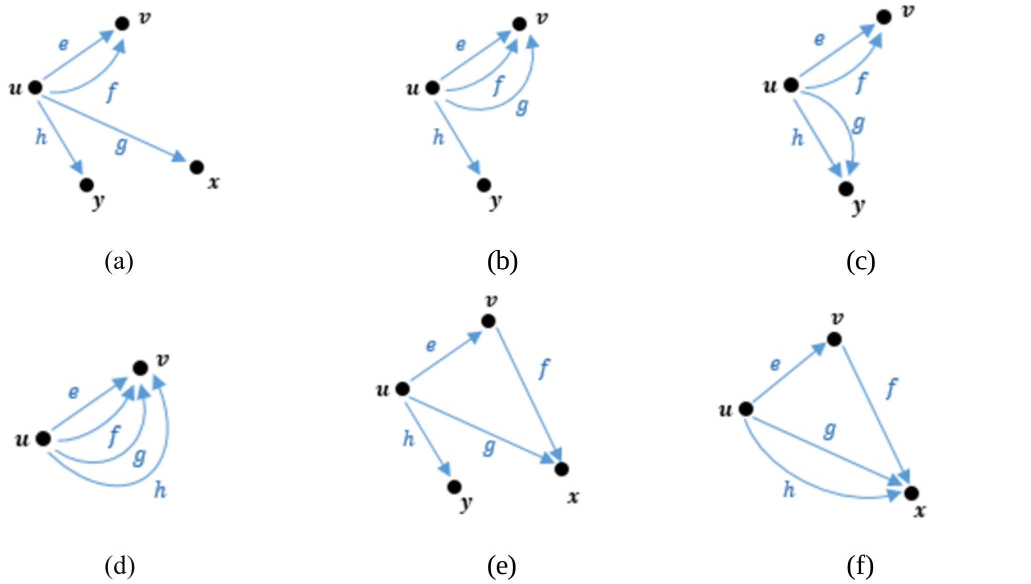


Fig. 6. (a) A 4 vertices graph with path $\{u, e, f, g, h\}$ and 1 vertex receives 2 arrows
(b) A 3 vertices graph with path $\{u, e, f, g, h\}$ and 1 vertex receives 3 arrows
(c) A 3 vertices graph with path $\{u, e, f, g, h\}$ and 2 vertices receives 2 arrows
(d) A 2 vertices graph with path $\{u, e, f, g, h\}$ and 1 vertex receives 3 arrows
(e) A 4 vertices graph with path $\{u, e, fe, g, h\}$ and 1 vertex receives 2 arrows
(f) A 3 vertices graph with path $\{u, e, fe, g, h\}$ and 1 vertex receives 3 arrows

As in the 5 vertices case, even though the graph in Figure 6 has less than 5 vertices, the number of arrows remains consistent at 4. However, this case is more difficult to characterize compared to the 5 vertices case. It can be seen in Figure 5. (b) and (c), these graphs both have 3 vertices but differ in terms of the most arrows received by the vertices. This shows that for a number of vertices less than the number of paths cannot be used as a simple single characterization. The simplest characterization is that if the number of vertices is less than and equal to the number of paths then there exists vertex that receive more than 1 arrow.

The results in Example 3.2.3 show that the graph representation of the C*-algebra $M_5(\mathbb{C})$ has similar characteristics to the complex matrix C*-algebra in Table 1. However, for more general C*-algebra of complex matrices still needs to be checked.

4 Conclusion

The characteristic of graphs that represents the C*-algebra of complex matrices $n \times n$ for $n = 1,2,3,4,5$ is a graph that has $n - 1$ arrows, exactly one source, and there are n paths originating from that source. The number of vertices in the graph representation varies between 2 to n, where each choice will affect the number of arrows received from each non-source vertex. This characterization still needs to be generalized for any C*-algebra of complex matrices. From the results of this characterization, C*-algebra, especially complex matrix spaces, can be viewed more realistically through graphs. This graph representation can also be linked to C* algebra in quantum mechanical systems so that it is hoped that it can explain quantum mechanical systems more realistically and be used as a basis for AI development.

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