Pole placement with minimum gain and sparse static feedback

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Abstract. The work addresses the issue of developing polar positioning algorithms for linear control systems with despotic thinning constraints on a static feedback matrix. The design of sparse-limiting feedback control systems is considered to be one of the more complex issues, but the issue of defining the sparse feedback matrix, which allows the selection of eigenvalues or, in other words, the placement of poles, has not been fully studied. Algorithms for determining the eigenvalues, i.e., solving the problem of pole placement and finding the optimal solution of the technological process control system are considered in the work. The results of numerical analysis confirmed the effectiveness of these algorithms, which allows them to be used in solving practical problems of synthesizing adaptive control systems for technological objects.

1 Introduction

A significant number of works are devoted to the synthesis of static feedback in linear control systems. In them, the desired behavior of the system is determined, as a rule, by the requirement that the roots of its characteristic polynomial belong to a certain set of values or by the requirement to minimize the integral quadratic functional that evaluates the quality of transient processes. Accordingly, the problem of placing the poles of the transfer function of a closed-loop system (modal control) and the problem of synthesizing a linear-quadratic controller are considered. For them, there are effective methods that provide their exact solution, provided that all components of the state vector can be used in the controller and no obvious restrictions are imposed on the choice of values of feedback coefficients. However, these problems turn out to be difficult to solve if they take into account restrictions on the structure of the controller, in particular, consisting in the prohibition of using some state variables, which occurs, for example, when synthesizing output feedback [1-7].

In the development of time-linear invariant control systems using state-by-state feedback, polar positioning issues are still considered one of the problematic issues [1,5,8]. Pole positioning issues make it possible to ensure its quality in stable control of the system, as well as through the formation of a transition characteristic of the process. We define linear invariant control systems by time as:

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\[ D x(k) = A x(k) + B u(k), \]
\[ u(k) = F x(k), \]

where \( x \in R^n \) – time-invariant linear control system state vector; \( u \in R^m \) – control vector; Operators \( A \in R^{m \times n}, B \in R^{n \times m}, \) and \( D \) are time continuous differentiating or discrete displacement operators; \( F \in R^{m \times n} \) – closed system matrix eigenvalues; while \( A_c \triangleq A + BF \) belongs to the polynomial \( S = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). \( F - (A, B) \) is taken into account of the controllability property of the pair. Also, for systems with one input \((m = 1)\), the best solution is to calculate the feedback vector that determines the eigenvalues using the Ackerman equation. However, when the system has multiple inputs \((m > 1)\), the use of the inverse contact matrix is not considered to be very efficient in solving pole placement problems, and in this case there is flexibility in the placement of eigenvalues of the closed-loop control system. Using this flexibility, it is possible to select the feedback matrix when defining the system's eigenvalues. An example of this flexibility is increasing the reliability and quality of the control system by selecting feedback to reduce the sensitivity of the closed-loop control system to various disturbances [9-12].

Today, in the design of control systems, great importance is attached to the use of feedback, where the fixed element of the F matrix is equal to zero. The use of this type of feedback usually occurs in multidimensional systems where the adjusters are not able to take into account all the states of the system [3,13,14].

The design of feedback control systems that limit dilution is one of the more complex issues. To date, a lot of research has been done to solve these problems, but the problem of defining a sparse feedback matrix that allows choosing eigenvalues or, in other words, placing poles, has not been fully explored. In this paper, pole placement algorithms are proposed when the feedback matrix \( F \) has arbitrary sparsity constraints.

### 2 Setting private values or polar placement with a minimum reinforcement coefficient

Setting private values with a reduced amplification coefficient or placing poles in some of the necessary places indicated in set \( S = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is to find the private values of the closed contour (1) and (2) the inverse Matrix \( F \in R^{m \times n} \) with a minimum norm, which determines the equations and satisfies the given requirements.

\[ \hat{F} \in [0, 1]^{m \times n} - F \]

is a binary matrix indicating the required structure of the feedback matrix.

\[ F_{i,j} = \begin{cases} 0 & \text{if } \hat{F}_{i,j} = 0, \\ * & \text{if } \hat{F}_{i,j} = 1, \end{cases} \]

where \* – real number; \( F^c = 1_{m \times n} - \hat{F} \) – additional matrix of sparse structure; \( X = [x_1, x_2, \ldots, x_n] \in C^{n \times n}, x_i \neq 0_n \) is a closed contour matrix \( A_c(F) = A + BF \) is a nonsingular eigenvector matrix.

Pole placement issues can be summarized as follows:

\[ \min_{F^c} \frac{1}{2} \| F \|_F^2 \]  
\[ (A + BF)X = X \Lambda, \]  
\[ x_i^T x_i = 1 \forall i = 1, 2, \ldots, n \]  
\[ F^c \cap F = 0_{m \times n}. \]

where \( \Lambda = \text{diag}(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}^T) \) – Sa diagonal matrix composed of elements of a set of eigenvalues; Equation (4) is a constraint equation for determining the eigenvalue; (6) equation - the equation representing the rarefaction constraints; Equation (5) is \( X \) the equation defining the eigenvector matrix.

\( \hat{F} \) – with respect to sparse constraints \((A, B)\). The formed modes of are invariant using the state feedback matrices of the linear system \( A \) has a private value:
The following can be formulated for solving the optimization problems presented in equation (3):

1) $S$ -set of eigenvalues $(A,B,F)$ formed roots (modes) of:

$$\Gamma_f(A,B,F) \subseteq S.$$ 

2) $S$ -for the set of eigenvalues, equations (4) – (6) have at least one that satisfies the constraints $F$– feedback matrix will be available.

Equation (3) is necessary to implement the optimization problem formulated 1).

Formulated 2) is restrictive, since a static inverse matrix with a certain sparsity template may not be able to give arbitrary locations (i.e., the prediction $a$ for an arbitrary set $S$ that satisfies) the specific values of the closed contour [15-19]. For this reason, let's consider a way to calculate the optimal inverse matrix using at least one such inverse matrix.

3 Defining specific values i.e. solving the problem of polar placement

This section proposes a solution to optimization problems. In this case, Lagrange multipliers are used to move to the optimization condition.

If there is a set of private values $S$ if it contains complex elements, $X$ the corresponding eigenvectors in are also complex. Thus, the existing constraints on the eigenvalues in equation (4) are complex constraints, giving rise to the following combined constraint:

$$(A + BF)X^* = X^*A^*.$$

In optimization problems $X$ and $X^*$ we consider it to be an independent variable.

According to him, two united $(x_i,x_j)$ right eigenvectors with eigenvalues $(\lambda_i,\lambda_j)$ is also considered united. So, if $\lambda_i = \lambda_j^*$ if so, then $x_i = x_j^*$. The same sequence applies to the left eigenvectors.

The optimization theory monad first-order optimization condition is that the corresponding optimal point satisfies the generalization condition, i.e. $J_b$ – The Jacobian constraint must have full color. Before presenting the main result, we formulate a regularization condition for the optimization problem as in Equation (3).

$J_b$ calculate the $x \equiv vec(X) \in C^{n^2}$, $f \equiv vec(F) \in R^{mn}$ requires vectorization of the matrix constraints Equations (4) - (6). In this case, $z \equiv [x^T, x^{H}, f^T]^T$ is equal, and it covers all the independent variables that make up the optimization issue in this case. Where $n_s$ is the total number of thinned feedback constraints equal to $F^c$ unit numbers:

$$n_s = |\{i,j: F^c = [\tilde{f}^c_{ij}, \tilde{f}^c_{ij} = 1]\}|.$$ 

In this case, equation (6) can be written as follows:

$$Qf = 0_{n_s}. (8)$$

In this equation $Q$ for some binary matrices $Q \in \{0,1\}^{n_s \times mn}$ satisfies

The Jacobian constraint equations expressed in the system of equations (3)-(6) are defined as follows:

$$J_b(z) = \begin{bmatrix} A_c(F \otimes (-A^T)) & 0_{n^2 \times n^2} & X^T \otimes B \\ 0_{n^2 \times n^2} & A_c(F \otimes (-A^H)) & X^{H} \otimes B \\ 0_{n^2 \times n^2} & \hat{x}^* & 0_{n \times mn} \end{bmatrix}, \quad (9)$$

where $\hat{x} \equiv (1_n \otimes x^H) \circ (I_n \otimes 1_n^T)$.

To do this, we construct $J_b$ and, in the system of equations (3) – (6), we shift the constraints to the vector representation in the form of Equation (3) and calculate the
derivative of its founders with respect to \( z \). Taking into account \( \vec{A}(B) = (I \otimes A)vec(B) = (B^T \otimes I)vec(A), \) \( \vec{A}(ABC) = (C^T \otimes A)vec(B) \) we write the equation (4) as follows:

\[
[(A + BF) \oplus (-A^T)]x = 0_{n^2}, \tag{10}
\]
\[
[A \oplus -(A^T)]x + (X^T \otimes B)f = 0_{n^2}. \tag{11}
\]

\( x \) with respect to Equation (10) and \( f \) Differentiating equation (11) with respect to the first (block) \( J_b \) gives the string. A similar vectorization of the constraints on the expression (7) in the combined eigenvalues and \( z \) differentiation with respect to the second (blocked) \( J_b \) given in line. (5) based on the expression \( i = 1,2,\ldots, n \) for \( x \) and \( x^* \) the relative differentiability constraint gives:

\[
\frac{d}{dx} \begin{bmatrix}
x_1^H x_1 - 1 \\
n_2^H x_2 - 1 \\
\vdots \\
n_n^H x_n - 1
\end{bmatrix} = \begin{bmatrix}
x_1^H 0^n_T & \cdots & 0^n_T \\
0^n_T & x_2^H & \cdots & 0^n_T \\
\vdots & \vdots & \ddots & \vdots \\
0^n_T & 0^n_T & \cdots & x_n^H
\end{bmatrix} = \hat{x},
\]

where \( \hat{x}^* \)– eigenvalue. Also \( Q = [e_{q_1}, e_{q_2} \ldots e_{q_n}]^T \) and \( \{q_1, \ldots, q_n\} = \text{supp} \ (\text{vec}(\hat{F})) \)– \( \text{vec}(\hat{F}) \) is a complex of indices indicating the indices of the vector, where \( n_s \) equation (6) with rarefaction constraints can be written as equation (8). (8) equation \( z \) fourth (blocked) by differentiating by \( J_b \) string can be obtained.

Also, in the course of the research work, along with solving the problems of pole placement, methods were developed to determine the optimality conditions (equation (3)) in solving the optimization problem.

\[
\hat{z} = [\hat{x}^T, \hat{x}^H, \hat{f}^T]^T \text{ equivalent } \left( \hat{X}, \hat{F} \right) \text{ if the values satisfy the constraints in (4)–(6). } (\hat{X}, \hat{F}) \text{ can be considered as the local minimum value of equation (3), where:}
\]

\[
\hat{F} = -\hat{F} \circ (B^T \hat{L} \hat{X}^T), \tag{12}
\]

where \( \hat{X} \) and \( \hat{L} = A_c(\hat{F}) \)are the left and right matrices of the eigenvector, which must satisfy the following:

\[
(A + B\hat{F})\hat{X} = \hat{X}\Lambda, \tag{13}
\]
\[
(A + B\hat{F})^T \hat{L} = \hat{L}\Lambda^T, \tag{14}
\]
\[
J_b(\hat{z}) \text{– full color,} \tag{15}
\]
\[
P(\hat{z})D\hat{P}(\hat{z}) > 0, \tag{16}
\]

\[
\hat{D} \triangleq \begin{bmatrix}
0_{n^2 \times n^2} & 0_{n^2 \times n^2} & \hat{L}^H \\
0_{n^2 \times n^2} & 0_{n^2 \times n^2} & \hat{L}^T \\
\hat{L} & \hat{L}^* & 2I_{mn}
\end{bmatrix}, \tag{17}
\]

\[
\hat{L} \triangleq T_{n,m}(B^T \hat{L} \otimes I_n) \text{ and } P(\hat{z}) – J_b(\hat{z}). \text{ When calculating the projection matrix of, it can be defined as:}
\]

\[
P(\hat{z}) = I_{2n^2 + mn} – J_b^+(\hat{z})J_b(\hat{z}). \tag{18}
\]

### 4 Conclusion

Thus, the work proposed polar positioning algorithms for linear control systems with arbitrary thinning constraints on a static feedback matrix. Algorithms for determining eigenvalues, that is, solving the problem of pole placement and finding the optimal solution of the technological process control system through it, are considered in the article.
References

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