Study on coupled problems of thermoelasticity in Strains

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Abstract. In the work, within the framework of the strain compatibility conditions of Saint-Venant, two equivalent dynamic boundary value problems of thermoelasticity with respect to strains are formulated. In the case of the first boundary value problem, the dynamic equations of thermoelasticity are obtained from the compatibility conditions, in the second case, instead of the first three equations of thermoelasticity, the equations of motion expressed with respect to deformations are considered. Discrete analogues of boundary value problems are constructed using the finite-difference method in the form of explicit and implicit schemes. The solution of explicit schemes is reduced to recurrent relations with respect to deformations and temperature. Implicit schemes are solved by sequential application of the elimination method. The validity of the formulated thermoelastic boundary value problems is substantiated by comparing the numerical results of the problem of a thermoelastic parallelepiped obtained by different methods, as well as solving a similar problem in displacements.

1. Introduction

The formulation of boundary value problems of thermoelasticity with respect to stresses and strains is an urgent problem in solid mechanics. Boundary-value problems of thermoelasticity with respect to stresses and strains can be formulated within the framework of the Saint-Venant’s compatibility conditions [1]. It is known that the strain compatibility conditions, using the Duhamel–Neumann relation and the equation of motion, can be written relative to the stress tensor in the form of the Beltrami-Michell equations [2]. The Beltrami-Michell equations in combination with three equations of motion constitute a boundary value problem with nine equations and three boundary conditions [3]. In the works of Pobedry [4], the compatibility conditions and equilibrium equations are reduced to a boundary value problem consisting of six equations for stresses. In a particular case, the Beltrami-Michell equations follow from the Pobedry equations [5]. The works of Borodachev [6,7] show that the first group, consisting of three equations, is dependent on the second group of three Beltrami-Michell equations. To the study of dynamic boundary value problems in stresses, questions of the equivalence of setting boundary value problems in displacements and stresses are considered in [8]. Questions of the existence and uniqueness of solutions to boundary value problems are considered in [9]. In [10], a more detailed review of the formulation of boundary value problems regarding stresses was carried out. Despite the existing effective methods for solving applied problems, such as FEM, BEM, finite-difference and other methods, there are few numerically solved boundary value problems regarding stresses. Let us note the classic works of Filonenko-Borodich [11]. The problem of equilibrium of a parallelepiped under stress using the variational-difference method was considered in [12].

This work is devoted to the formulation of coupled problems of thermoelasticity in strains and their numerical solution. Discrete equations are compiled using the finite-difference method in the form of explicit and implicit schemes. A numerically coupled dynamic problem of thermoelasticity with respect to strains for a parallelepiped is solved. The solution of explicit schemes is reduced to recurrent relations with respect to the components of the strain tensor and temperature. Solving implicit schemes comes down to sequential application of the elimination method in the appropriate directions.

2. Formulation of the coupled problem of thermoelasticity in strains

It is known [2] that the coupled boundary value problem of thermoelasticity for isotropic ones consists of the equation of motion

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\[ \sigma_{ij,j} + \rho X_i = \rho \ddot{u}_i, \]  (1)

Duhamel-Neumann relations

\[ \sigma_{ij} = \lambda \delta_{ij,j} + 2 \mu \epsilon_{ij} - (3\lambda + 2\mu)\alpha (T - T_0) \delta_{ij}, \]  (2)

Cauchy relations

\[ \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \]  (3)

heat equation

\[ \lambda \theta_{,i,i} - C_e \theta - T_0 \gamma \theta_{,i} = -w, \quad \gamma = (3\lambda + 2\mu)\alpha, \]  (4)

and, initial and boundary conditions

\[ u_i |_{t=0} = \varphi_i, \quad \frac{\partial u_i}{\partial t} |_{t=0} = \psi_i, \quad T |_{t=0} = \bar{T}, \]  (5)

\[ u_i |_{\Sigma} = u_i^0, \quad \sigma_{ij,n} |_{\Sigma} = S_i, \quad T |_{\Sigma} = T^0. \]  (6)

where \( \sigma \) — stress tensor, \( \epsilon \) — strain tensor, \( u \) — displacement, \( T \) — temperature, \( \lambda, \mu \) — elastic Lame constants, \( \theta \) — spherical part of the strain tensor, \( S_i \) — surface load, \( n_i \) — components of the outer normal to the surface \( \Sigma \), \( X_i \) — body forces, \( \delta_{ij} \) — Kronecker symbol, \( T_0 \) — initial temperature, \( \alpha \) — coefficient of thermal expansion.

Using relations (2) and (3), the equation of motion can be written relative to displacements \([3,8]\) i.e.

\[ \mu \nabla^2 u_i + (\lambda + \mu) \theta_{,i} - (3\lambda + 2\mu)\alpha \frac{\partial T}{\partial X_i} + \rho X_i = \rho \ddot{u}_i. \]  (7)

where \( \nabla^2 \) — Laplace operator, \( \theta = \epsilon_{ik} \).

Differentiating equation (7) by \( x_j \) i.e.

\[ \mu \nabla^2 u_{i,j} + (\lambda + \mu) \theta_{,i,j} - (3\lambda + 2\mu)\alpha \frac{\partial T}{\partial X_i} + \rho X_{i,j} = \rho \ddot{u}_{i,j}, \]  (8)

and, swapping the indices \( i \) and \( j \) in (8)

\[ \mu \nabla^2 u_{j,i} + (\lambda + \mu) \theta_{,j,i} - (3\lambda + 2\mu)\alpha \frac{\partial T}{\partial X_i} + \rho X_{j,i} = \rho \ddot{u}_{j,i}, \]  (9)

and by adding equations (8) and (9) we can find \([14,15]\) that

\[ \mu \nabla^2 \epsilon_{ij} + (\lambda + \mu) \theta_{,ij} - (3\lambda + 2\mu)\alpha \frac{\partial T}{\partial X_{ij}} + \frac{1}{2} \rho (X_{i,j} + X_{j,i}) = \rho \ddot{\epsilon}_{ij}. \]  (10)

Equation (10) can also be found from the Saint-Venant’s compatibility condition using the motion equation and the Duhamel-Neumann relation. The last relation consists of six equations and represents the dynamic equations of thermoelasticity with respect to strains. Equation (10) in the absence of body forces has the form:

\[ \mu \nabla^2 \epsilon_{ij} + (\lambda + \mu) \theta_{,ij} - (3\lambda + 2\mu)\alpha \frac{\partial T}{\partial X_{ij}} = \rho \ddot{\epsilon}_{ij}. \]  (11)

To formulate \([16,17]\) the coupled problem of thermoelasticity, it is necessary to add the heat influx equations to equation (10)

\[ \lambda \theta_{,i} - C_e \theta - T_0 \gamma \theta_{,i} = -w, \]  (12)

and initial and boundary conditions

\[ T |_{t=0} = \bar{T}, \quad \epsilon_{ij} |_{t=0} = \xi_i, \quad \epsilon_{ij} |_{\Sigma} = \xi_i, \quad T |_{\Sigma} = \psi_i. \]  (13)

Equations (11-13) represent the coupled boundary thermoelasticity in strains (problem A).

Considering, as the first three equations (11), in problem A, the three equations of motion (1) expressed with respect to strains, we can find another formulation of the coupled thermoelastic problem in strains (problem B). Note that in order to increase the order of approximation of difference equations, in boundary value problem B, the equations of motion can be used in a differentiated form.

3. Finite-difference equations
Let us consider boundary value problems A and B in parallelepipeds \( 0 \leq x_i \leq l_i, \ i = 1, 2, 3 \). By replacing the derivatives in equation (11) with difference relations relative to grid \( x_i = i h_i, \ x_i = j h_i, \ x_i = k h_i \) \((i, j, k = 0, 1, 2, 3), \ t_i = l \tau, \) the following grid equations for problem A can be found:

\[
(\lambda + 2\mu) \epsilon_{i,j,k,l}^{11} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{11}}{h_i^2} + (\lambda + 2\mu) \epsilon_{i,j,k,l}^{12} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{12}}{h_i^2} + \\
(\lambda + 2\mu) \epsilon_{i,j,k,l}^{23} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{23}}{h_i^2} + (\lambda + 2\mu) \epsilon_{i,j,k,l}^{13} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{13}}{h_i^2} \\
+ (\lambda + 2\mu) \epsilon_{i,j,k,l}^{11} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{11}}{h_i^2} + (\lambda + 2\mu) \epsilon_{i,j,k,l}^{12} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{12}}{h_i^2} + \\
(\lambda + 2\mu) \epsilon_{i,j,k,l}^{23} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{23}}{h_i^2} + (\lambda + 2\mu) \epsilon_{i,j,k,l}^{13} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{13}}{h_i^2} \\
- \gamma \epsilon_{i,j,k,l}^{11} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} = \rho \epsilon_{i,j,k,l}^{11} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} \\
(14)
\]

In a similar way, we find explicit finite-difference equations with the order of approximation \( O(h^2, \tau^2) \) for \( \epsilon_{i=22}, \epsilon_{i=33}, \epsilon_{i=12}, \epsilon_{i=13}, \epsilon_{i=23}, T. \) Having resolved these difference equations for \( \epsilon_{i=11}, T_{i=11} \) respectively, we obtain the following recurrence relations i.e.

\[
\epsilon_{i,j,k,l+1}^{11} = \frac{\tau^2}{\rho} ((\lambda + 2\mu) \epsilon_{i,j,k,l}^{11} - 2\epsilon_{i,j,k,l}^{11} + \epsilon_{i,j,k,l}^{11}) \frac{\epsilon_{i,j,k,l}^{11}}{h_i^2} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{11}}{h_i^2} + \\
(\lambda + 2\mu) \epsilon_{i,j,k,l}^{12} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{12}}{h_i^2} + (\lambda + 2\mu) \epsilon_{i,j,k,l}^{23} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{23}}{h_i^2} + \\
(\lambda + 2\mu) \epsilon_{i,j,k,l}^{13} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{13}}{h_i^2} + (\lambda + 2\mu) \epsilon_{i,j,k,l}^{11} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{11}}{h_i^2} + \\
(\lambda + 2\mu) \epsilon_{i,j,k,l}^{12} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{12}}{h_i^2} + (\lambda + 2\mu) \epsilon_{i,j,k,l}^{23} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{23}}{h_i^2} + \\
(\lambda + 2\mu) \epsilon_{i,j,k,l}^{13} + (\lambda + \mu)(\epsilon_{i,j,k,l}^{22} + \epsilon_{i,j,k,l}^{33}) \frac{\epsilon_{i,j,k,l}^{13}}{h_i^2} \\
- \gamma \epsilon_{i,j,k,l}^{11} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} = \rho \epsilon_{i,j,k,l}^{11} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} + \epsilon_{i,j,k,l}^{11} \frac{T_{i,j,k,l}}{h_i^2} \\
(15)
\]

In a similar way, we can find equations for the remaining components of the strain and temperature tensor. Relations (15) make it possible to find the values of the desired functions \( \epsilon_i(x, y, t) \) on layer \( t = 1 \) if the values of these functions on the two previous layers are known. The values of function \( \epsilon_i(x, y, t) \) on two initial layers \( k = 0 \) and \( k = 1 \) can be found from the initial conditions.

It is known that in explicit schemes the time step \( \tau \) is significant compared to \( h. \) Typically, the following convergence condition \( \frac{\tau^2}{h} < 1 \) is required [5,18]. Difference schemes can be constructed without restrictive conditions for grid steps in \( x_i \) and \( t. \) Why in the first terms of equations (15) we replace the index \( l \) with \( l+1, \) then the difference scheme becomes implicit and it can be reduced to the following tridiagonal form, solved by the variable direction method

\[
a \epsilon_{i,j,k,l}^{11} + b \epsilon_{i,j,k,l}^{12} + c \epsilon_{i,j,k,l}^{13} = f_{i,j,l}^{11},
\]

where \( a, b, c, f^{11} \) – coefficients. In a similar way, equations can be found for the components of deformation \( \epsilon_{i,j,k,l}^{22}, \epsilon_{i,j,k,l}^{33}, \epsilon_{i,j,k,l}^{12}, \epsilon_{i,j,k,l}^{13}, \epsilon_{i,j,k,l}^{23} \) and temperature

\[
AT_{i,j,k,l}^{11} + BT_{i,j,k,l}^{11} + CT_{i,j,k,l}^{11} = F_{i,j,l}^{11},
\]

From equations (16-17) it follows that difference equations are solved by the alternating directions method along the corresponding coordinate axes at a specific value of \( t. \) Thus, the solution to Problem B was reduced, respectively, to recurrence relations for explicit schemes, and the application of the method of alternating directions in the case of implicit schemes.

4. Numerical results

We will consider coupled boundary value problems A and B in a load-free thermoelastic parallelepiped under the following initial and boundary conditions regarding the components of deformation and temperature, i.e.

\[
T_{|t=0} = T_0 \sin(\frac{\pi x_1}{l_1}) \sin(\frac{\pi x_2}{l_2}) \sin(\frac{\pi x_3}{l_3}),
\]

\[
\epsilon_{ij} \left|_{t=0} = 0, \quad \frac{\partial \epsilon_{ij}}{\partial t} \left|_{t=0} = 0, \quad T \left|_{t=0} = 0.\right.
\]

(18)
initial data

\[ T_0 = 20, \ \lambda = 0.78, \ \lambda_0 = 0.06, \ \alpha = 0.05, \ \mu = 0.5, \]
\[ \rho = 0.86, \ c_v = 3.5, \ h = h_0 = 0.1, \ l_i = l_i = 1. \]

Table 1 shows the values of temperature \( T \) and deformation \( \varepsilon \), obtained by solving boundary value problems A and B, using the method of alternating directions-ADM (implicit scheme) and recurrence relations (explicit scheme), and compared with the solution of a similar problem in displacements.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Function values ( T(x, y, z, t) ) at ( t=0.05, z=0.5, y=0.5 )</th>
</tr>
</thead>
</table>
| Problem A (explicit scheme) | \begin{tabular}{c|cccccc}
  \( x=0 \) & 6.014 & 11.420 & 15.697 & 18.437 & 19.380 \\
  \( x=0.1 \) & 6.016 & 11.421 & 15.698 & 18.438 & 19.381 \\
  \( x=0.2 \) & 6.017 & 11.423 & 15.706 & 18.447 & 19.389 \\
  \( x=0.3 \) & 6.018 & 11.425 & 15.714 & 18.455 & 19.383 \\
  \( x=0.4 \) & 6.019 & 11.427 & 15.722 & 18.463 & 19.380 \\
  \( x=0.5 \) & 6.020 & 11.429 & 15.730 & 18.471 & 19.379 \\
\end{tabular} |
| Problem A (ADM)  | \begin{tabular}{c|cccccc}
  \( x=0 \) & 6.014 & 11.420 & 15.697 & 18.437 & 19.380 \\
  \( x=0.1 \) & 6.016 & 11.421 & 15.698 & 18.438 & 19.381 \\
  \( x=0.2 \) & 6.017 & 11.423 & 15.706 & 18.447 & 19.389 \\
  \( x=0.3 \) & 6.018 & 11.425 & 15.714 & 18.455 & 19.383 \\
  \( x=0.4 \) & 6.019 & 11.427 & 15.722 & 18.463 & 19.380 \\
  \( x=0.5 \) & 6.020 & 11.429 & 15.730 & 18.471 & 19.379 \\
\end{tabular} |
| Problem B (explicit scheme) | \begin{tabular}{c|cccccc}
  \( x=0 \) & 6.017 & 11.426 & 15.706 & 18.447 & 19.389 \\
  \( x=0.1 \) & 6.019 & 11.428 & 15.715 & 18.456 & 19.385 \\
  \( x=0.2 \) & 6.021 & 11.430 & 15.724 & 18.464 & 19.381 \\
  \( x=0.3 \) & 6.023 & 11.432 & 15.733 & 18.473 & 19.377 \\
  \( x=0.4 \) & 6.025 & 11.434 & 15.742 & 18.482 & 19.373 \\
  \( x=0.5 \) & 6.027 & 11.436 & 15.751 & 18.491 & 19.369 \\
\end{tabular} |
| Problem in displacements | \begin{tabular}{c|cccccc}
  \( x=0 \) & 6.017 & 11.399 & 15.608 & 18.431 & 19.379 \\
  \( x=0.1 \) & 6.019 & 11.401 & 15.617 & 18.440 & 19.375 \\
  \( x=0.2 \) & 6.021 & 11.403 & 15.625 & 18.449 & 19.371 \\
  \( x=0.3 \) & 6.023 & 11.405 & 15.634 & 18.458 & 19.367 \\
  \( x=0.4 \) & 6.025 & 11.407 & 15.643 & 18.467 & 19.363 \\
  \( x=0.5 \) & 6.027 & 11.409 & 15.652 & 18.475 & 19.359 \\
\end{tabular} |

In Figure 1 and 2 compare curves \( \varepsilon_x \), constructed from the results of problems A and B at the midpoint of the parallelepiped, obtained using recurrence relations (explicit scheme) and the alternating directions method (implicit scheme). A comparison of deformations in tables and figures shows that the numerical results obtained by the ADM and recurrence relations are quite close, which ensures the validity of the formulated boundary value problems and the reliability of the obtained numerical results.

![Fig. 1. Comparison of deformations \( \varepsilon_x \) obtained using explicit and implicit schemes at \( t=0.05, y=0.5, z=0.5 \) for problem A](image-url)
5. Conclusions
Within the framework of the strain compatibility conditions of Saint-Venant, differential equations of dynamic thermoelasticity with respect to strains are formulated. Two coupled boundary value problems of thermoelasticity with respect to strains and temperature are formulated. By comparing the numerical results of these boundary value problems in strains, as well as a similar problem in displacements, for solving the problem of a thermoelastic parallelepiped, the validity of the formulated thermoelastic problems is shown. Discrete equations are compiled by the finite-difference method in the form of explicit and implicit schemes, and solved by the method of alternating directions and by recurrence relations with respect to the components of the strain tensor and temperature. Numerical algorithms and corresponding software have been developed for solving three-dimensional thermoelastic boundary value problems. The influence of the temperature field on the distribution of strains and stresses was studied.

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