Coupled Problem on Thermo-Elasticity in Strains for an Isotropic Parallelepiped

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Abstract. This work is devoted to mathematical and numerical modeling of the coupled dynamic problem of thermoelasticity in deformations. A numerically related boundary value problem of thermoelasticity in deformations for a parallelepiped with the corresponding initial and boundary conditions is formulated and solved. Grid equations are constructed using the finite-difference method in the form of explicit and implicit schemes. In this case, the solution of the explicit scheme is reduced to recurrent relations with respect to deformations and temperature. In the case of implicit difference schemes, the equations are solved by sequential application of the sweep method. The validity of the formulated boundary value problems is justified by comparing the numerical results obtained with different methods based on two different models.

1. Introduction

In scientific and technical applications, determining the safety margins and elemental reliability of structures while accounting for thermomechanical dynamic variables is critical. Duhamel and Neumann were the first to examine the mathematical models that explain the process of thermoelastic deformation. In these models, it was assumed that elastic deformation and thermal expansion comprised the entire deformation. Temperature fields are typically considered as a solution to the heat flow equation for tackling thermoelastic problems.

The amount of scientific literature devoted to examining how mechanical and thermal variables interact to cause thermoelastic deformation has increased significantly in recent years. Combining the heat influx equation with the equations of motion of deformable solids allows one to account for the mutual influence of the thermomechanical forces. The study of the thermomechanical state of elastic bodies usually involves coupled and uncoupled boundary value problems of thermoelasticity with respect to displacements and temperature. The coupled thermoelasticity problem consists of an equation of motion and an equation of heat influx regarding the displacement and temperature with corresponding initial and boundary conditions. In the case of stationary problems, if the inertial terms are neglected, the equations of motion and heat influx can be considered independently of each other, and the thermoelastic problem becomes uncoupled.

The formulation of the boundary value problems of thermoelasticity with respect to stresses and strains is an urgent problem in solid mechanics. The boundary value problems of thermoelasticity with respect to stresses and strains can be formulated within the framework of the Saint-Venant deformation compatibility condition [1].

With the support of the equation of motion and the Duhamel-Neumann relation, it is known that the requirements for compatibility of deformations can be expressed in terms of the stress tensor as the Beltrami-Michell equations [4,5]. Nine equations and three boundary conditions constitute a boundary value problem using the Beltrami-Michell equations and three equations of motion [11].

Borodachev's publications [7,8] demonstrate how the second set of Beltrami-Michell equations depends on the first set of three equations. The study carried out by Pobedry [2,3,4] shows that the equilibrium equations and compatibility criteria are simplified to form a boundary value problem with six equations for stresses. The Beltrami-Michell equations were derived from the Pobedry equations in a specific scenario. The consideration of boundary value problems in displacements and stresses and their equivalency are discussed in [7]. The existence and uniqueness of solutions to boundary-value problems have been discussed in these publications [8, 9]. Nowatsky's work [10] examined the Beltrami-Michell equations with temperature considered. Thermoelasticity-related issues were examined in [15]. Konovalov's [22] works focus on the analysis of dynamic boundary value issues in stresses.

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There are few boundary value issues with stresses that have been numerically solved, despite the availability of efficient techniques for tackling applicable problems, such as finite-difference methods, BEM, and FEM. Let us take note of Filonenko-Borodich's classic work [17]. The variational-difference technique was used to examine the equilibrium problem of a parallelepiped in stresses in [4,5,15].
This study is devoted to the formulation and numerical solution of the related problems of thermoelasticity in deformations. The finite-difference approach was used to create explicit and implicit schemes for discrete equations. A parallelepiped thermo-elasticity dynamic problem with respect to deformation was addressed numerically. Explicit schemes can be solved for recurring relations between the strain tensor components and temperature. The variable direction method was applied sequentially in the relevant directions to eliminate implicit schemes.

2. Statement of the coupled problem of thermoelasticity for a parallelepiped

It is known [4,5,6] that the associated boundary value problem of thermoelasticity for isotropic ones consists of the equation of motion

$$\sigma_{ij,j} + \rho X_i = \rho \ddot{u}_i, \tag{1}$$

Duhamel-Neumann relations

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij}, \tag{2}$$

Cauchy relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{3}$$

heat gain equation

$$\lambda_0\theta_{,ii} - C_\varepsilon\dot{\theta} - T_0\gamma\dot{\varepsilon}_{ii} = -w,\ \gamma = (3\lambda + 2\mu)\alpha, \tag{4}$$

and, initial and boundary conditions

$$u_i\,|_{t=t_0} = \varphi_i,\ \frac{\partial u_i}{\partial t}\,|_{t=t_0} = \psi_i,\ T\,|_{t=t_0} = \tilde{T}, \tag{5}$$

$$u_i\,|_{\Sigma_1} = u_i^0,\ \sigma_{ij}n_j\,|_{\Sigma_2} = S_i,\ T\,|_{\Sigma} = T^0. \tag{6}$$

where σ_{ij} – stress tensor, ε_{ij} – strain tensor, u_i – displacement, T – temperature, $\lambda,\ \mu$ – elastic Lame constants, θ – spherical part of the strain tensor, S_i – surface load, n_i – components of the outer normal to the surface Σ, X_i – body forces, δ_{ij} – Kronecker symbol.
Using relations (2) and (3), the equation of motion can be written relative to displacements [10, 17,18] i.e.

$$\mu\nabla^2 u_i + (\lambda + \mu)\theta_{,i} - (3\lambda + 2\mu)\alpha\frac{\partial T}{\partial x_i} + \rho X_i = \rho\ddot{u}_i, \tag{7}$$

where ∇^2 – Laplace operator, $\theta = \varepsilon_{kk}$.
Differentiating equation (7) with respect to x_j i.e.

$$\mu\nabla^2 u_{i,j} + (\lambda + \mu)\theta_{,ij} - (3\lambda + 2\mu)\alpha T_{,ij} + \rho X_{i,j} = \rho\ddot{u}_{i,j}, \tag{8}$$

and, swapping the indices *i* and *j* in (8)

$$\mu\nabla^2 u_{j,i} + (\lambda + \mu)\theta_{,ji} - (3\lambda + 2\mu)\alpha T_{,ij} + \rho X_{j,i} = \rho\ddot{u}_{j,i}, \tag{9}$$

and by adding equations (8) and (9) we can find [2,14,24] that

$$\mu\nabla^2\varepsilon_{ij} + (\lambda + \mu)\theta_{,ij} - (3\lambda + 2\mu)\alpha T_{,ij} + \frac{1}{2}\rho(X_{i,j} + X_{j,i}) = \rho\ddot{\varepsilon}_{ij}. \tag{10}$$

The last relation is an analogue of the deformation compatibility equation for thermomechanical dynamic processes, and consists of six equations. Equation (10) can also be found from the Saint-Venant condition using the equation of motion and the Duhamel-Neumann relation. In the absence of body forces, equation (10) has the form

$$\mu\nabla^2\varepsilon_{ij} + (\lambda + \mu)\theta_{,ij} - (3\lambda + 2\mu)\alpha T_{,ij} = \rho\ddot{\varepsilon}_{ij}. \tag{11}$$

Note that the last equation is obtained from the compatibility condition taking into account Hooke's law and the equation of motion, and in fact represents another form of the deformation compatibility condition. Therefore, following work [10], we can call them differential equations of compatibility of deformations.

To formulate [4, 9, 12, 14] the associated boundary value problem of elasticity theory, it is necessary to add the equations of motion (1) and the equations of heat influx to equation (10) [23, 24]

$$\lambda_0\theta_{,ii} - C_\varepsilon\dot{\theta} - T_0\gamma\dot{\varepsilon}_{ii} = -w. \tag{12}$$

Equations (11-12), in the three-dimensional case take the form i.e.

$$\begin{gathered}
(\lambda+2\mu)\frac{\partial^2\varepsilon_{11}}{\partial x^2} + (\lambda+\mu)(\frac{\partial^2\varepsilon_{22}}{\partial x^2} + \frac{\partial^2\varepsilon_{33}}{\partial x^2}) + \mu(\frac{\partial^2\varepsilon_{11}}{\partial y^2} + \frac{\partial^2\varepsilon_{11}}{\partial z^2}) - \gamma\frac{\partial^2 T}{\partial x^2} = \rho\frac{\partial^2\varepsilon_{11}}{\partial t^2},\\
(\lambda+2\mu)\frac{\partial^2\varepsilon_{22}}{\partial y^2} + (\lambda+\mu)(\frac{\partial^2\varepsilon_{11}}{\partial y^2} + \frac{\partial^2\varepsilon_{33}}{\partial y^2}) + \mu(\frac{\partial^2\varepsilon_{22}}{\partial x^2} + \frac{\partial^2\varepsilon_{22}}{\partial z^2}) - \gamma\frac{\partial^2 T}{\partial y^2} = \rho\frac{\partial^2\varepsilon_{22}}{\partial t^2},\\
(\lambda+2\mu)\frac{\partial^2\varepsilon_{33}}{\partial z^2} + (\lambda+\mu)(\frac{\partial^2\varepsilon_{11}}{\partial z^2} + \frac{\partial^2\varepsilon_{22}}{\partial z^2}) + \mu(\frac{\partial^2\varepsilon_{33}}{\partial x^2} + \frac{\partial^2\varepsilon_{22}}{\partial y^2}) - \gamma\frac{\partial^2 T}{\partial z^2} = \rho\frac{\partial^2\varepsilon_{33}}{\partial t^2},\\
\mu(\frac{\partial^2\varepsilon_{12}}{\partial x^2} + \frac{\partial^2\varepsilon_{12}}{\partial y^2} + \frac{\partial^2\varepsilon_{12}}{\partial z^2}) + (\lambda+\mu)(\frac{\partial^2\varepsilon_{11}}{\partial x\partial y} + \frac{\partial^2\varepsilon_{22}}{\partial x\partial y} + \frac{\partial^2\varepsilon_{33}}{\partial x\partial y}) - \gamma\frac{\partial^2 T}{\partial x\partial y} = \rho\frac{\partial^2\varepsilon_{12}}{\partial t^2},\\
\mu(\frac{\partial^2\varepsilon_{13}}{\partial x^2} + \frac{\partial^2\varepsilon_{13}}{\partial y^2} + \frac{\partial^2\varepsilon_{13}}{\partial z^2}) + (\lambda+\mu)(\frac{\partial^2\varepsilon_{11}}{\partial x\partial z} + \frac{\partial^2\varepsilon_{22}}{\partial x\partial z} + \frac{\partial^2\varepsilon_{33}}{\partial x\partial z}) - \gamma\frac{\partial^2 T}{\partial x\partial z} = \rho\frac{\partial^2\varepsilon_{13}}{\partial t^2},\\
\mu(\frac{\partial^2\varepsilon_{23}}{\partial x^2} + \frac{\partial^2\varepsilon_{23}}{\partial y^2} + \frac{\partial^2\varepsilon_{23}}{\partial z^2}) + (\lambda+\mu)(\frac{\partial^2\varepsilon_{11}}{\partial y\partial z} + \frac{\partial^2\varepsilon_{22}}{\partial y\partial z} + \frac{\partial^2\varepsilon_{33}}{\partial y\partial z}) - \gamma\frac{\partial^2 T}{\partial y\partial z} = \rho\frac{\partial^2\varepsilon_{23}}{\partial t^2},\\
\lambda_0(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) - C_\varepsilon\frac{\partial T}{\partial t} - \gamma T_0\frac{\partial(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})}{\partial t} = 0,
\end{gathered} \tag{13}$$

For a system of differential equations (13) can be considered with the following boundary conditions consisting of: initial conditions

$$\begin{gathered}
T(x,y,z,t)|_{t=0} = \tilde{T}_i,\ \varepsilon_{11}(x,y,z,t)|_{t=0} = \xi_i,\ \varepsilon_{22}(x,y,z,t)|_{t=0} = \psi_i,\ \varepsilon_{33}(x,y,z,t)|_{t=0} = \zeta_i,\\
\varepsilon_{12}(x,y,z,t)|_{t=0} = \vartheta_i,\ \varepsilon_{13}(x,y,z,t)|_{t=0} = \omega_i,\ \varepsilon_{23}(x,y,z,t)|_{t=0} = \varsigma_i,\\
\frac{\partial}{\partial t}\varepsilon_{11}(x,y,z,t)|_{t=0} = \xi^1_i,\ \frac{\partial}{\partial t}\varepsilon_{22}(x,y,z,t)|_{t=0} = \psi^1_i,\ \frac{\partial}{\partial t}\varepsilon_{33}(x,y,z,t)|_{t=0} = \zeta^1_i,\\
\frac{\partial}{\partial t}\varepsilon_{12}(x,y,z,t)|_{t=0} = \upsilon^1_i,\ \frac{\partial}{\partial t}\varepsilon_{13}(x,y,z,t)|_{t=0} = \omega^1_i,\ \frac{\partial}{\partial t}\varepsilon_{23}(x,y,z,t)|_{t=0} = \varsigma^1_i,
\end{gathered} \tag{14}$$

boundary conditions for temperature fields

$$\begin{gathered}
T(x,y,z,t)|_{x=0} = T_1,\ T(x,y,z,t)|_{x=l_1} = T_2,\ T(x,y,z,t)|_{y=0} = T_3,\ T(x,y,z,t)|_{y=l_2} = T_4,\\
T(x,y,z,t)|_{z=0} = T_5,\ T(x,y,z,t)|_{z=l_3} = T_6.
\end{gathered} \tag{15}$$

and, trivial deformations corresponding, according to Hooke's law, to the load-free surface of the parallelepiped:

$$\begin{gathered}
\varepsilon_{11}(x,y,z,t)|_{x=0,l_1} = 0,\ \varepsilon_{11}(x,y,z,t)|_{y=0,l_2} = 0,\ \varepsilon_{11}(x,y,z,t)|_{z=0,l_3} = 0,\\
\varepsilon_{22}(x,y,z,t)|_{x=0,l_1} = 0,\ \varepsilon_{22}(x,y,z,t)|_{y=0,l_2} = 0,\ \varepsilon_{22}(x,y,z,t)|_{z=0,l_3} = 0,\\
\varepsilon_{33}(x,y,z,t)|_{x=0,l_1} = 0,\ \varepsilon_{33}(x,y,z,t)|_{y=0,l_2} = 0,\ \varepsilon_{33}(x,y,z,t)|_{z=0,l_3} = 0,\\
\varepsilon_{12}(x,y,z,t)|_{x=0,l_1} = 0,\ \varepsilon_{12}(x,y,z,t)|_{y=0,l_2} = 0,\ \varepsilon_{12}(x,y,z,t)|_{z=0,l_3} = 0,\\
\varepsilon_{23}(x,y,z,t)|_{x=0,l_1} = 0,\ \varepsilon_{23}(x,y,z,t)|_{y=0,l_2} = 0,\ \varepsilon_{23}(x,y,z,t)|_{z=0,l_3} = 0,\\
\varepsilon_{13}(x,y,z,t)|_{x=0,l_1} = 0,\ \varepsilon_{13}(x,y,z,t)|_{y=0,l_2} = 0,\ \varepsilon_{13}(x,y,z,t)|_{z=0,l_3} = 0.
\end{gathered} \tag{16}$$

3. Solution methods

This paragraph is devoted to the construction of a numerical model of the three-dimensional problems considered in paragraph 2.

By constructing three families of parallel lines $x_1 = ih_1$, $x_2 = jh_2$, $x_3 = kh_3$ $(i, j, k = \overline{0, n_m}, m = 1,2,3)$, $t_l = l\tau$, $(l = 0,1,2,...)$ in parallelepiped $t \geq 0$, $0 \leq x_i \leq l_i$ and following the works [6, 13, 22] replacing the derivatives in equation (13) with difference relations, we can find the following grid equations

$$(\lambda+2\mu)\frac{\varepsilon^{11}_{i+1,j,k,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i-1,j,k,l}}{h_1^2}+(\lambda+\mu)(\frac{\varepsilon^{22}_{i+1,j,k,l}-2\varepsilon^{22}_{i,j,k,l}+\varepsilon^{22}_{i-1,j,k,l}}{h_1^2}+$$
$$+\frac{\varepsilon^{33}_{i+1,j,k,l}-2\varepsilon^{33}_{i,j,k,l}+\varepsilon^{33}_{i-1,j,k,l}}{h_1^2})+\mu(\frac{\varepsilon^{11}_{i,j+1,k,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j-1,k,l}}{h_2^2}+\frac{\varepsilon^{11}_{i,j,k+1,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j,k-1,l}}{h_3^2}) \quad (17)$$
$$-\gamma\frac{T_{i+1,j,k,l}-T_{i,j,k,l}+T_{i-1,j,k,l}}{h_1^2}=\rho\frac{\varepsilon^{11}_{i,j,k,l+1}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j,k,l-1}}{\tau^2}.$$

In a similar way we find the finite difference equation for ε_{22}, ε_{33}, ε_{12}, ε_{13}, ε_{23}, T.
These difference equations within the domain have a second order of approximation $O(h^2, \tau^2)$, and are explicit. Therefore, having resolved these difference equations for ε_i^{l+1} and T_i^{l+1}, respectively, we obtain the following recurrence relations i.e.

$$\varepsilon^{11}_{i,j,k,l+1}=\frac{\tau^2}{\rho}((\lambda+2\mu)\frac{\varepsilon^{11}_{i+1,j,k,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i-1,j,k,l}}{h_1^2}+(\lambda+\mu)(\frac{\varepsilon^{22}_{i+1,j,k,l}-2\varepsilon^{22}_{i,j,k,l}+\varepsilon^{22}_{i-1,j,k,l}}{h_1^2}+$$
$$+\frac{\varepsilon^{33}_{i+1,j,k,l}-2\varepsilon^{33}_{i,j,k,l}+\varepsilon^{33}_{i-1,j,k,l}}{h_1^2})+\mu(\frac{\varepsilon^{11}_{i,j+1,k,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j-1,k,l}}{h_2^2}+\frac{\varepsilon^{11}_{i,j,k+1,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j,k-1,l}}{h_3^2}) \quad (18)$$
$$-\gamma\frac{T_{i+1,j,k,l}-T_{i,j,k,l}+T_{i-1,j,k,l}}{h_1^2})+2\varepsilon^{11}_{i,j,k,l}-\varepsilon^{11}_{i,j,k,l-1}.$$

A In a similar way, one can find equations for the remaining components of the strain and temperature tensor. Relations (18) make it possible to find the values of the desired functions $\varepsilon_{11}(x, y, t)$ on layer t^{l+1} if the values of these functions on the two previous layers are known. The values of function $\varepsilon_i(x, y, t)$ on two initial layers $k = 0$ and $k = 1$ can be found from the initial conditions.
It is known that in explicit schemes the time step τ t is significant compared to *h*. Typically, the following convergence condition $\frac{\tau^2}{h} < 1$ is required [10]. It is possible to construct difference schemes without restrictive conditions for grid steps in *x* and *t*. Why replace the index *l* in the first terms of equations (17) with *l+1*, then the difference scheme becomes implicit

$$(\lambda+2\mu)\frac{\varepsilon^{11}_{i+1,j,k,l+1}-2\varepsilon^{11}_{i,j,k,l+1}+\varepsilon^{11}_{i-1,j,k,l+1}}{h_1^2}+(\lambda+\mu)(\frac{\varepsilon^{22}_{i+1,j,k,l}-2\varepsilon^{22}_{i,j,k,l}+\varepsilon^{22}_{i-1,j,k,l}}{h_1^2}+$$
$$+\frac{\varepsilon^{33}_{i+1,j,k,l}-2\varepsilon^{33}_{i,j,k,l}+\varepsilon^{33}_{i-1,j,k,l}}{h_1^2})+\mu(\frac{\varepsilon^{11}_{i,j+1,k,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j-1,k,l}}{h_2^2}+\frac{\varepsilon^{11}_{i,j,k+1,l}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j,k-1,l}}{h_3^2}) \quad (19)$$
$$-\gamma\frac{T_{i+1,j,k,l}-T_{i,j,k,l}+T_{i-1,j,k,l}}{h_1^2}=\rho\frac{\varepsilon^{11}_{i,j,k,l+1}-2\varepsilon^{11}_{i,j,k,l}+\varepsilon^{11}_{i,j,k,l-1}}{\tau^2}.$$

and, it can be reduced to the following tridiagonal form, solved by the variable direction method

$$a\varepsilon^{11}_{i+1,j,k,l+1}+b\varepsilon^{11}_{i,j,k,l+1}+c\varepsilon^{11}_{i-1,j,k,l+1}=f^{11}_{ijkl} \quad (20)$$

where a, b, c, $f_i^k -$ coefficients.
In a similar way, for the remaining equations (13), they can be reduced to a three-diagonal form, with different coefficients, i.e.

$$
\begin{aligned}
&\ddot{a}\varepsilon^{22}_{i+1,j,k,l+1} + \ddot{b}\varepsilon^{22}_{i,j,k,l+1} + \ddot{c}\varepsilon^{22}_{i-1,j,k,l+1} = f^{22}_{ijkl},\\
&\dot{a}\varepsilon^{33}_{i+1,j,k,l+1} + \dot{b}\varepsilon^{33}_{i,j,k,l+1} + \dot{c}\varepsilon^{33}_{i-1,j,k,l+1} = f^{33}_{ijkl},\\
&\tilde{a}\varepsilon^{12}_{i+1,j,k,l+1} + \tilde{b}\varepsilon^{12}_{i,j,k,l+1} + \tilde{c}\varepsilon^{12}_{i-1,j,k,l+1} = f^{12}_{ijkl},\\
&\underset{\cdot}{a}\varepsilon^{13}_{i+1,j,k,l+1} + \underset{\cdot}{b}\varepsilon^{13}_{i,j,k,l+1} + \underset{\cdot}{c}\varepsilon^{13}_{i-1,j,k,l+1} = f^{13}_{ijkl},\\
&\underset{\cdot\cdot}{a}\varepsilon^{23}_{i+1,j,k,l+1} + \underset{\cdot\cdot}{b}\varepsilon^{23}_{i,j,k,l+1} + \underset{\cdot\cdot}{c}\varepsilon^{23}_{i-1,j,k,l+1} = f^{23}_{ijkl},\\
&AT_{i+1,j,k,l+1} + BT_{i,j,k,l+1} + CT_{i-1,j,k,l+1} = F^{T}_{ijkl}.
\end{aligned}
\tag{21}
$$

From equations (21) it follows that the solution of difference equations is a sequential four-fold application of the variable direction method along the corresponding coordinate axes.

4. Numerical Results

Explicit and implicit difference schemes of the coupled problem of thermoelasticity in deformations were solved by recurrence relations and the variable direction method, respectively, under the following initial and boundary conditions:

$$
\begin{aligned}
&T(x,y,z,t)|_{t=0} = T_0 \sin(\frac{\pi x_i}{l_1})\sin(\frac{\pi y_j}{l_2})\sin(\frac{\pi k_z}{l_3}),\\
&\varepsilon_{11}(x,y,z,t)|_{t=0} = 0,\ \varepsilon_{22}(x,y,z,t)|_{t=0} = 0,\ \varepsilon_{33}(x,y,z,t)|_{t=0} = 0,\\
&\varepsilon_{12}(x,y,z,t)|_{t=0} = 0,\ \varepsilon_{13}(x,y,z,t)|_{t=0} = 0,\ \varepsilon_{23}(x,y,z,t)|_{t=0} = 0,\\
&\frac{\partial}{\partial t}\varepsilon_{11}(x,y,z,t)|_{t=0} = 0,\ \frac{\partial}{\partial t}\varepsilon_{22}(x,y,z,t)|_{t=0} = 0,\ \frac{\partial}{\partial t}\varepsilon_{33}(x,y,z,t)|_{t=0} = 0,\\
&\frac{\partial}{\partial t}\varepsilon_{12}(x,y,z,t)|_{t=0} = 0,\ \frac{\partial}{\partial t}\varepsilon_{13}(x,y,z,t)|_{t=0} = 0,\ \frac{\partial}{\partial t}\varepsilon_{23}(x,y,z,t)|_{t=0} = 0,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&\varepsilon_{11}(x,y,z,t)|_{x=0} = 0,\ \varepsilon_{11}(x,y,z,t)|_{x=l_1} = 0,\ \varepsilon_{11}(x,y,z,t)|_{y=0} = 0,\ \varepsilon_{11}(x,y,z,t)|_{y=l_2} = 0,\\
&\varepsilon_{22}(x,y,z,t)|_{x=0} = 0,\ \varepsilon_{22}(x,y,z,t)|_{x=l_1} = 0,\ \varepsilon_{22}(x,y,z,t)|_{y=0} = 0,\ \varepsilon_{22}(x,y,z,t)|_{y=l_2} = 0,\\
&\varepsilon_{12}(x,y,z,t)|_{x=0} = 0,\ \varepsilon_{12}(x,y,z,t)|_{x=l_1} = 0,\ \varepsilon_{12}(x,y,z,t)|_{y=0} = 0,\ \varepsilon_{12}(x,y,z,t)|_{y=l_2} = 0,\\
&T(x,y,z,t)|_{x=0} = 0,\ T(x,y,z,t)|_{x=l_1} = 0,\ T(x,y,z,t)|_{y=0} = 0,\ T(x,y,z,t)|_{y=l_2} = 0.
\end{aligned}
\tag{23}
$$

and, source data

$$
\begin{aligned}
&T_0 = 20,\ \lambda = 0.78,\ \lambda_0 = 0.06,\ \alpha = 0.05,\ \mu = 0.5,\\
&\rho = 0.86,\ c_\varepsilon = 3.5,\ h_1 = h_2 = 0.1,\ l_1 = l_2 = 1.
\end{aligned}
$$

Table 1. Values of function $T(x,y,z,t)$ at *t=0.05*, *z=0.4* (explicit scheme)

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y=0.1	0.000	1.770	3.365	4.630	5.441	5.720	5.441	4.630	3.365	1.770	0.000
y=0.2	0.000	3.365	6.395	8.795	10.333	10.863	10.333	8.795	6.395	3.365	0.000
y=0.3	0.000	4.630	8.795	12.092	14.205	14.933	14.205	12.092	8.795	4.630	0.000
y=0.4	0.000	5.441	10.333	14.205	16.686	17.539	16.686	14.205	10.333	5.441	0.000
y=0.5	0.000	5.720	10.863	14.933	17.539	18.437	17.539	14.933	10.863	5.720	0.000
y=0.6	0.000	5.441	10.333	14.205	16.686	17.539	16.686	14.205	10.333	5.441	0.000
y=0.7	0.000	4.630	8.795	12.092	14.205	14.933	14.205	12.092	8.795	4.630	0.000
y=0.8	0.000	3.365	6.395	8.795	10.333	10.863	10.333	8.795	6.395	3.365	0.000
y=0.9	0.000	1.770	3.365	4.630	5.441	5.720	5.441	4.630	3.365	1.770	0.000
y=1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 1 shows the temperature distribution over coordinate and time obtained using an explicit scheme. Comparison of the results from tables (Tables 1-4) show that the numerical results found by recurrence relations and by the sweep method are quite close, which ensures the validity of the formulated boundary value problems and the reliability of the obtained numerical results.

Table 2. Values of function $T(x,y,z,t)$ at *t=0.05*, *z=0.4* (implicit schema)

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y=0.1	0.000	1.770	3.365	4.630	5.441	5.721	5.441	4.630	3.365	1.770	0.000
y=0.2	0.000	3.365	6.395	8.795	10.334	10.864	10.334	8.795	6.395	3.365	0.000
y=0.3	0.000	4.630	8.795	12.092	14.206	14.933	14.206	12.092	8.795	4.630	0.000
y=0.4	0.000	5.441	10.334	14.206	16.687	17.541	16.687	14.206	10.334	5.441	0.000
y=0.5	0.000	5.720	10.864	14.933	17.541	18.438	17.541	14.933	10.864	5.720	0.000
y=0.6	0.000	5.441	10.334	14.206	16.687	17.541	16.687	14.206	10.334	5.441	0.000
y=0.7	0.000	4.630	8.795	12.092	14.206	14.933	14.206	12.092	8.795	4.630	0.000
y=0.8	0.000	3.365	6.395	8.795	10.334	10.864	10.334	8.795	6.395	3.365	0.000
y=0.9	0.000	1.770	3.365	4.630	5.441	5.721	5.441	4.630	3.365	1.770	0.000
y=1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 3. Values of function $\varepsilon_{11}(x,y,z,t)$ at *t=0.05*, *z=0.5* (explicit scheme)

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y=0.1	0.000	0.004	0.008	0.012	0.014	0.014	0.014	0.012	0.008	0.004	0.000
y=0.2	0.000	0.008	0.016	0.022	0.026	0.027	0.026	0.022	0.016	0.008	0.000
y=0.3	0.000	0.012	0.022	0.030	0.036	0.038	0.036	0.030	0.022	0.012	0.000
y=0.4	0.000	0.014	0.026	0.036	0.042	0.044	0.042	0.036	0.026	0.014	0.000
y=0.5	0.000	0.014	0.027	0.038	0.044	0.046	0.044	0.038	0.027	0.014	0.000
y=0.6	0.000	0.014	0.026	0.036	0.042	0.044	0.042	0.036	0.026	0.014	0.000
y=0.7	0.000	0.012	0.022	0.030	0.036	0.038	0.036	0.030	0.022	0.012	0.000
y=0.8	0.000	0.008	0.016	0.022	0.026	0.027	0.026	0.022	0.016	0.008	0.000
y=0.9	0.000	0.004	0.008	0.012	0.014	0.014	0.014	0.012	0.008	0.004	0.000
y=1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 4. Values of function $\varepsilon_{11}(x,y,z,t)$ at *t=0.05*, *z=0.5* (implicit schema)

	x=0	*x=0.1*	*x=0.2*	*x=0.3*	*x=0.4*	*x=0.5*	*x=0.6*	*x=0.7*	*x=0.8*	*x=0.9*	*x=1*
y=0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y=0.1	0.000	0.004	0.008	0.012	0.014	0.014	0.014	0.012	0.008	0.004	0.000
y=0.2	0.000	0.008	0.016	0.022	0.026	0.027	0.026	0.022	0.016	0.008	0.000
y=0.3	0.000	0.012	0.022	0.030	0.036	0.037	0.036	0.030	0.022	0.012	0.000
y=0.4	0.000	0.014	0.026	0.036	0.042	0.044	0.042	0.036	0.026	0.014	0.000
y=0.5	0.000	0.014	0.027	0.037	0.044	0.046	0.044	0.037	0.027	0.014	0.000
y=0.6	0.000	0.014	0.026	0.036	0.042	0.044	0.042	0.036	0.026	0.014	0.000
y=0.7	0.000	0.012	0.022	0.030	0.036	0.037	0.036	0.030	0.022	0.012	0.000
y=0.8	0.000	0.008	0.016	0.022	0.026	0.027	0.026	0.022	0.016	0.008	0.000
y=0.9	0.000	0.004	0.008	0.012	0.014	0.014	0.014	0.012	0.008	0.004	0.000
y=1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

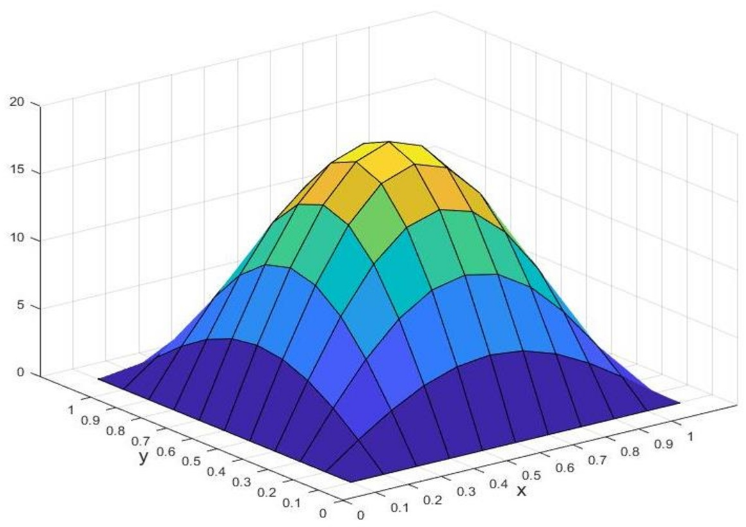

Fig. 1. Temperature distribution graph (explicit scheme) at *t=0.05, z=0.5*

5. Conclusion

Within the framework of the Saint-Venant compatibility conditions, differential equations of dynamic thermoelasticity with respect to deformations were constructed using the Lamé equations. A coupled boundary value problem of thermoelasticity in deformations was formulated for a parallelepiped with a free surface located in a dome-shaped temperature field. The finite-difference approach is used to create explicit and implicit schemes for discrete equations. Explicit scheme solutions can be simplified to recurring relations concerning deformation and temperature components. Implicit schemes are solved using alternating directions according to the method of variable directions in the corresponding directions.

Numerical algorithms and corresponding software have been developed to solve three-dimensional thermoelastic boundary value problems. The influence of the temperature field on the distribution of the strains and stresses was studied.

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