A mathematical model for reliability management of agricultural machinery

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Abstract. The article examines the current and practically significant task of managing the reliability of agricultural machinery used in the production activities of Russian Federation penal system institutions. A model for optimal management of technical maintenance and repairs of technical equipment is proposed in order to increase their reliability. Additionally, the necessary optimality conditions for solving the optimization problem under consideration are formulated. An algorithm for the numerical solution of the formulated optimal control problem has been developed, and the results of testing the implemented algorithm are presented. The implementation of the proposed approach in the form of an automated decision support system serves to solve the problem of increasing the operational reliability of vehicles used in agricultural activities of correctional institutions.

1 Introduction

In the Penal System of the Russian Federation, agricultural activity plays an important role and serves to provide food for convicts and employees, as well as being a sphere of production of products intended for sale to third-party buyers. Effective management of food security in the penal system is aimed at improving the conditions of convicts and assisting the administration of correctional institutions in ensuring the maximum variety of food for the special contingent [1]. The importance of agricultural activities in correctional institutions of the Federal Penitentiary Service of Russia lies not only in the self-sufficiency of food for prisoners and employees, but also in reducing the costs of penal institutions for the purchase of food [2]. In this regard, a number of works by leading Russian scientists are devoted to solving issues of agricultural activity.

Since this activity is inextricably linked with the intensive use of agricultural machinery (tractors, combine harvesters, various trucks, etc.), the issues of increasing its operational reliability are being updated, including through high-quality management of technical maintenance and repairs of technical equipment. The use of mathematical models and information technologies provides the opportunity to create effective automated decision support systems to increase the reliability of technical equipment used in the production and economic sector of Russian Federation penal system institutions [3].
2 Materials and methods

The work examines a model of a non-redundant unit of agricultural machinery (vehicle) controlled during servicing, used in technological processes of crop production implemented in the activities of a correctional institution [4].

We believe that at any given time the technical device in question can be in a state of readiness for operation (“1”) or in a state of failure (“2”). Further, we assume that the technical system goes into a state of readiness with probability \( P_1 \), and into a state of failure with probability \( P_2 \) [5]. If a failure occurs (transition “1-2”), an immediate transition to restoring its readiness is carried out (transition “2-1”). The corresponding state graph for the technical device maintenance system is shown in Fig. 1.

![State graph of a non-redundant controlled unit of agricultural machinery](image)

Fig. 1. State graph of a non-redundant controlled unit of agricultural machinery

To describe the dynamics of the transition of the technical means under consideration into the indicated states, we will use Kolmogorov’s system of differential equations for systems with continuous time and a discrete set of states, while assuming that the failure flow is Poisson [6, 7]. For the state graph shown in Figure 1, the corresponding system of equations has the following form [3]:

\[
\begin{align*}
\frac{dP_1}{dt} &= -a_{12} P_1(t) + a_{21} P_2(t), \\
\frac{dP_2}{dt} &= -a_{12} P_2(t) + a_{12} P_1(t),
\end{align*}
\]

where \( t \in [0,T] \) – device operating time, \( a_{12} = w_{12} \), \( a_{21} = \frac{1}{T_{r_{12}}} = \mu \), \( w_{12} \) – parameter characterizing the Poisson flow of device failures, \( T_{r_{12}} \) – average time spent on troubleshooting during maintenance, \( \mu \) – intensity of restoration of the vehicle’s readiness for work. Graphs of the probability of readiness for operation and the probability of failure of a technical system as a function of time \( t \) are presented in Fig. 2.

The numerical solution of the system of differential equations is constructed using the initial data presented in Table 1.
Table 1 shows the values of probability $P_1$ of finding the technical system in a state of readiness for use.

**Table 1.** Values of probability $P_1$ of finding a technical device in a state of readiness for use.

<table>
<thead>
<tr>
<th>$T_{r}^{12}, h$</th>
<th>$w_{12}, h^{-1}$</th>
<th>10^{-3}</th>
<th>10^{-4}</th>
<th>10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0,9901</td>
<td>0,9990</td>
<td>0,9999</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0,9756</td>
<td>0,9975</td>
<td>0,9998</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0,9524</td>
<td>0,9950</td>
<td>0,9995</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0,9091</td>
<td>0,9901</td>
<td>0,9990</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3 shows graphs of the dependence of the probability of the operational state of the technical equipment on the parameter $w_{12}$.

**Fig. 3.** Graphs of the function $P_1(t)$ for different values of parameter $w_{12}$.

The presented graphs (Fig. 3) and tabular data show that at $w_{12} \leq 10^{-5} \, 1/h$, $w_{12}$ has an insignificant effect on the probability of the readiness state $P_1$, but with an increase in the value $w_{12}$ (at $w_{12} > 10^{-5} \, 1/h$), the situation changes: for example, an increase of $T_{r}^{12}$ by 90% at $w_{12} = 10^{-4} \, 1/h$ leads to an increase in the value of $P_1$ by 0.9%, and at $w_{12} = 10^{-3} \, 1/h$ – to an increase of 8%.

Considering the value $\frac{1}{T_{r}^{12}}$ as a control parameter (failure elimination) with the introduction of notations: $x_1(t) = P_1(t), \ x_2(t) = P_2(t), t \in [0, T]$, we come to the formulation...
of the Pareto-optimal problem of ensuring the maximum level of reliability of an agricultural machinery unit in the presence of a limitation on the amount of financial resources that can be spent on its maintenance [8, 9, 10].

The problem of optimal management of the technical system reliability will take the following form.

It is required to minimize functionality

$$J(u) = -\int_0^T x_1(t) \, dt + M \max \{a - x^*_1, 0\} \to \min$$  \hspace{1cm} (2)

taking into account dynamic restrictions:

$$\dot{x}_1(t) = -a_{12}x_1(t) + x_2(t)u(t),$$  \hspace{1cm} (3)

$$\dot{x}_2(t) = -x_2(t)u(t) + a_{12}(t)x_1(t),$$

restrictions on the control function:

$$0 \leq u(t) \leq u_{\max},$$  \hspace{1cm} (4)

under initial conditions:

$$x_1(0) = 1, \quad x_2(0) = 0, \quad u(0) = 0.$$  \hspace{1cm} (5)

For the optimal control problem in the above formulation, a wide range of analytical and numerical solution methods are available [11, 12]. Let us formulate the necessary optimality conditions in problem (2)–(5) in the form of Pontryagin’s maximum principle [13].

The Pontryagin function for the problem under consideration (2) – (5) will take the form:

$$H(\lambda_0, t, x_1, x_2, u, p_1(t), p_2(t)) = \lambda_0 x_1 + p_1(t)(-a_{12}x_1(t) + x_2(t)u(t)) + p_2(t)(-x_2(t)u(t) + a_{12}(t)x_1(t)) + \lambda_0 x_1 - p_1(t)a_{12}x_1(t) + p_2(t)a_{12}(t)x_1(t) + \left( p_1(t) - p_2(t) \right)x_2(t)u(t).$$  \hspace{1cm} (6)

Let us introduce a switching function of the following form:

$$\varphi(t) = (p_1(t) - p_2(t))x_2(t).$$  \hspace{1cm} (7)

Transforming expression (20), we present the Pontryagin function in the form:

$$H(\lambda_0, t, x_1, x_2, u, p_1(t), p_2(t)) = \lambda_0 x_1 - p_1(t)a_{12}x_1(t) + p_2(t)a_{12}(t)x_1(t) + \varphi(t)u$$  \hspace{1cm} (8)

According to Pontryagin’s maximum principle, optimal control satisfies the maximum condition [13]:

$$H(\lambda_0, t, x_1(t), x_2(t), u(t), p_1(t), p_2(t)) = \max_{u \in U}\frac{\partial H}{\partial u}(\lambda_0, t, x_1(t), x_2(t), u(t), p_1(t), p_2(t)).$$  \hspace{1cm} (9)

If $[\bar{x}, \bar{u}]$ – optimal process in problem (2)-(5), then for optimal control the following relations are satisfied:

$$\bar{u}(t) = \begin{cases} u_0, & \text{if } p_1(t) > p_2(t), \\ 0, & \text{if } p_1(t) < p_2(t), \\ \gamma \in [0, u_{\max}], & \text{if } p_1(t) = p_2(t), t \in [0, T], \end{cases}$$  \hspace{1cm} (10)

in this case, the following conditions hold for conjugate functions $p_1(t), p_2(t)$:

$$\dot{p}_i = \frac{\partial H}{\partial x_i} = \lambda_0 + p_1(t)a_{12} - p_2(t)a_{12}(t),$$  \hspace{1cm} (11)
\[
\dot{p}_2 = -\frac{\partial H}{\partial x_2} = (p_2(t) - p_1(t))u(t).
\]

Next, the transversality conditions are satisfied at the end point of the integration segment:

\[
p_1(T) = -2M \max\{a - x_1(T), 0\}, \quad p_2(T) = 0. \tag{12}
\]

Let us construct a discrete optimal control problem that approximates the original continuous optimal control problem (2) – (5), using the Euler scheme of the 1st order of accuracy to approximate the derivatives. To replace the target functional, the rule of left rectangles with sampling step \( \Delta t = \frac{T}{q} \) [13]. Here \( q \) – number of points of uniform partition of the segment \([0, T]\).

The objective function in the discrete optimal control problem will take the form:

\[
I([x],[u]) = -\sum_{i=0}^{q-1} x_i \Delta t + M \max^2\{a - x_i^q, 0\} \rightarrow \inf . \tag{13}
\]

The following ratios are used for recalculation of \( x_1^{i+1}, x_2^{i+1} \):

\[
x_1^{i+1} = x_i^i + \Delta t\left(-a_{12}x_i^i + x_2^i u^i\right),
\]

\[
x_2^{i+1} = x_i^i + \Delta t\left(-x_2^i u^i + a_{12}x_i^i\right). \tag{14}
\]

The initial conditions used are:

\[
x_1^0 = 1, x_2^0 = 0, \tag{15}
\]

boundary conditions for control are specified:

\[
0 \leq u^i \leq u_{\text{max}}, i = 0, q-1. \tag{16}
\]

Let us introduce the Lagrange function:

\[
L(\lambda_0, t, x_1, x_2, p, u) = -\lambda_0 \sum_{i=0}^{q-1} x_i \Delta t + M \max^2\{a - x_i^q, 0\} +
\]

\[
+ \sum_{i=0}^{q-1} p_1^{i+1} \left(x_i^{i+1} - x_i^i - \Delta t\left(-a_{12}x_i^i + x_2^i u^i\right)\right) +
\]

\[
+ \sum_{i=0}^{q-1} p_2^{i+1} \left(x_2^{i+1} - x_2^i - \Delta t\left(-x_2^i u^i + a_{12}x_i^i\right)\right). \tag{17}
\]

\section*{3 Results and discussion}

The numerical algorithm uses the method of fast automatic differentiation; control restrictions are taken into account using gradient projection [14, 15, 16]. Using the conditions for stationarity of the Lagrange function:

\[
\frac{\partial L(\lambda_0, t, x_1, x_2, p, u)}{\partial x_i^j} = 0, \quad \frac{\partial L(\lambda_0, t, x_1, x_2, p, u)}{\partial x_2^i} = 0, \tag{18}
\]

we calculate the conjugate variables using the relations:

\[
\frac{\partial L(\lambda_0, t, x_1, x_2, p, u)}{\partial x_i^j} = p_i^j - p_i^{j+1} (1 - a_{12} \Delta t) - p_2^{j+1} a_{12} \Delta t - \Delta t = 0, \tag{19}
\]
\[ \frac{\partial L(\lambda_0, t, x_1, x_2, p, u)}{\partial x_2^i} = -p_1^{i+1} \Delta t u^i + p_2^{i+1} (1-u^i \Delta t) = 0, \]

hence,

\[ p_1^i = p_1^{i+1} (1-a_{i2} \Delta t) + p_2^{i+1} a_{i2} \Delta t + \Delta t, \quad (20) \]

\[ p_2^i = p_2^{i+1} (1-u^i \Delta t) + p_1^{i+1} u^i \Delta t, \quad p_1^q = 2M \max \{a - x_1^q, 0\}, \quad p_2^q = 0. \]

We calculate:

\[ \frac{\partial L(\lambda_0, t, x_1, x_2, p, u)}{\partial u^i} = \left(p_2^{i+1} - p_1^{i+1}\right) x_2^i \Delta t. \quad (21) \]

The numerical algorithm is implemented in the form of a desktop application in the Lazarus IDE using the Free Pascal programming language. Figures 4 – 6 show graphs of optimal control and trajectories obtained during the operation of the method with the following parameter values: \( q=1000, \ T=3 \, \text{r}, \ w_{12} = 10^{-4} \, 1/h, \ x_1^0 = 1, \ x_2^0 = 0, \ u^0 = 0, \ a = 0.9997, \ \alpha^{(0)} = 0.1, \ \varepsilon_f = 10^{-8}, \ \varepsilon_x = 10^{-8}. \)

![Graph of optimal control](image1)

**Fig. 4.** Graph of optimal control \( \bar{u}(t) \)

![Graph](image2)

**Fig. 5.** Graph \( \bar{x}_i(t) \)
The proposed calculation formulas can be used to construct a computer model of the technical system reliability, which allows one to predict equipment failures, as well as develop optimal strategies when planning activities for the maintenance and repair of agricultural equipment. The implementation of the proposed approach in the form of an automated decision support system serves to solve the problem of increasing the operational reliability of vehicles used in agricultural activities of correctional institutions.

4 Conclusion

The constructed mathematical model of the ANN can be used to create an autopilot RS designed for emergency rescue operations in extreme conditions in the correctional facilities. Due to the flexibility, stability, and ability to adapt to external conditions, ANNs make it possible to solve control problems for complex technical systems in a wide range of parameters [15].

References
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