# Uneven time series forecasting using a modified exponential smoothing method

*A.N.* Mironov[1*], *S.U.* Pirogov[1], *A.I.* Petuhov[1], *V.K.* Sinilov[1], and *O.L.* Shestopalova[2]

[1]Military Space Academy named after A.F. Mozhaisky, 197198 St. Petersburg, Russia
[2]Moscow Aviation Institute (National Research University), 125993 Moscow, Russia

**Abstract.** The article is devoted to the problem of forecasting time series with an uneven distribution of observations over time. The exponential smoothing model is used as the basic forecasting model, in which the variable weights of observations decrease exponentially. The exponential smoothing model allows us to take into account the attenuation of the correlation of cross sections of a random process of time series change over time. However, this does not take into account the factors of temporal unevenness of the results of observations and the finiteness of the sample of observations. The article describes a method for predicting an uneven time series based on a modified exponential smoothing model, in which the transition from exponential smoothing to decreasing non-exponential smoothing is carried out. The modified sequence of the weights of the observations is determined by adjusting the classically calculated exponential weights, taking into account the actual irregularity of the observations.

## 1 Introduction

The results of periodic measurements of parameters of various processes in economics, medicine, engineering and other applied fields are usually presented in the form of time series (TS) [1-5]. A time series is understood as time-ordered measurement results of values of a certain controlled parameter at discrete time intervals $\Delta t = h$ [6]. Let's imagine TS as: $c(t_i) = f(t_i) + \varepsilon(t_i)$, where $f(t_i)$ is the trend or the deterministic (non–random) component of the random TS; $\varepsilon(t_i)$ - the random component of TS with a mathematical expectation equal to zero; $t_i$, $i = \overline{1,n}$ – the moments of measurement of TS values; $n$ - the number of measurements (observations) of the parameter.

In the traditional formulation [6-9], the task of predicting TS consists in extrapolating TS $\{c(t_1), c(t_2), ..., c(t_n)\}$ ,formed by parameter values $c$ , obtained at regular intervals using a segment of the Taylor series, i.e. a polynomial of the degree of $p$ decomposition of the function $f(t)$ in the neighborhood $t = t_n$ – the moment of the last observation. In this

* Corresponding author: mironov-anik@yandex.ru

case, the task is reduced to estimating the first $p$ derivatives of the function $f(t)$ at a point $t = t_n$.

Suppose that the function $f(t)$ has the form:

$$f(t) = \sum_{j=0}^{p} a_j \frac{(t - t_n)^j}{j!}, \tag{1}$$

where $a_j = f^{(j)}(t_n)$, $j = 1,2,..., p$.

Equality (1) is valid in the vicinity of a point $t = t_n$. If this rule were true over the entire observation interval, estimates of the derivatives $a_j (j = \overline{0, p})$ could be obtained using the least squares method (LSM) [10, 11].

For TS with a significant random component $\varepsilon(t_i)$, the use of LSM to determine the parameters of the forecast model can lead to significant errors. To avoid these errors, in a number of papers [12-14] it is proposed to consider observations as unequal. Weights are attributed to them, decreasing depending on the prehistory of observations of TS values according to the law of geometric progression. Such methods are called TS forecasting methods with discounting of observational results [6].

Among the methods with discounted observations, the exponential smoothing (ES) method or the Brown method is the most well-known [7-9, 10-14]. Formulas for estimating derivatives $a_j = f^{(j)}(t_n)$ determined by the ES method are derived on the assumption that there is not a finite, but an infinite series of observations

$$..., c(t_{-m}), c(t_{-m+1}),..., c(t_{-1}), c(t_0), c(t_1),..., c(t_n), \text{ where } \lim_{m\to\infty} t_{-m} = -\infty. \tag{2}$$

In the infinite series (2), an observation made at a time $t_{n-k}$ is assigned a weight equal to $\alpha(1-\alpha)^k$, where $\alpha$ is a constant number, called the smoothing parameter and equal to the weight of the last observation $c(t_n)$, $0 < \alpha < 1$.

It can be shown that the estimates of the derivatives $\tilde{a}_j = \tilde{f}^{(j)}(t_n)$ of the function $f(t)$ at the last observation point, obtained using the ES method, coincide with the estimates of the coefficients $\tilde{a}_{<p+1>} = < a_0, a_1,..., a_p >$ when solving the following optimization problem

$$\tilde{a}_{<p+1>} = \arg\min \alpha \sum_{k=0}^{\infty} (1-\alpha)^k \left[ c(t_{n-k}) - \sum_{j=0}^{p} \frac{a_j}{j!} (t_{n-k} - t_n)^j \right]^2. \tag{3}$$

Therefore, the ES method is a generalization of the LSM for the case when the observations are not equivalent. In addition to the fact that the ES method does not take into account the finite sample size of observations, it requires uniformity in time of time series counts, i.e. compliance with the condition $h = t_{i+1} - t_i = const$, for $\forall i = \overline{0, n-1}$.

In reality, the samples of TS observations are finite and uneven in time. The reasons for the uneven TS are, firstly, technological limitations on the possibility of uniform measurements. For example, when measurements can be carried out only when the controlled equipment is functioning, and the nature of its involvement in the technological process is fundamentally uneven in time. Secondly, part of the measurements may be lost when transmitting the results of parameter monitoring over lines and communication channels due to interference, which transforms uniform TS to an uneven appearance [15-17].

In order to avoid forecasting errors due to the finiteness of the observations, as well as in order to generalize the ES method to the case of unequal observations, we give a brief conclusion of the formulas used in the ES method.

## 2 Materials and methods

The ES method introduces quantities $S_{t_m}^{[k]}$ called exponential averages of the k order for TS $\{c(t_m)\}$, $-\infty < m \le n$.

Next, a system of linear equations with unknown values of derivative $\widetilde{a}_j$ estimates is compiled. The right - hand sides of the equations contain the values of exponential averages $S_{t_m}^{[k]}$. By solving these equations relatively $\widetilde{a}_j$, an expression for the estimate $\widetilde{a}_j$ is obtained in the form of a linear combination of exponential averages. The exponential averages themselves are found recursively, based on the values of TS $\{c(t_m)\}$.

Knowing the estimates $a_j$ ( $j = \overline{0, p}$ ) of the derivatives of the function $f(t)$ at the last observation point $t_n$, the following expression is taken as the parameter value predicted at the time $t_n + \tau$

$$c^*(t_n + \tau) = \widetilde{a}_0 + \widetilde{a}_1\tau + \frac{\widetilde{a}_2}{2!}\tau + \ldots + \frac{\widetilde{a}_p}{p!}\tau^p. \tag{4}$$

Consider a series of observations (2). The exponential averages of the first order for the series are determined by the formula

$$S_{t_m}^{[1]} = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i c(t_{m-i}). \tag{5}$$

Exponential averages of a higher order for a series are determined recursively by the formula

$$S_{t_m}^{[k+1]} = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i S_{t_{m-i}}^{[k]}. \tag{6}$$

where $k$ = 1,2,3,... .

In order to combine formulas (5) and (6) and ensure the unity of terminology, it is convenient to call the values of series (2) exponential means of the zero-left order. Thus, by definition $S_{t_m}^{[0]} \equiv c(t_m)$. Formula (6), which is now valid for k = 0, can be written as follows

$$S_{t_m}^{[k+1]} = \alpha S_{t_m}^{[k]} + (1-\alpha) S_{t_{m-i}}^{[k+1]}, \tag{7}$$

where $k$ = 0,1,2,...; $m$ = ...–$k$,...,–1,0,1,...,$n$.

The recurrent formula (7) gives an algorithm for generating values, which is conveniently presented in the form of Table 1.

**Table 1.** An algorithm for generating values $S_{t_m}^{[k+1]}$ based on a recurrent formula (7)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | $S_0^{[0]}$ | $S_1^{[0]}=c_1$ | … | $S_{m-1}^{[0]}=c_{m-1}$ | $S_m^{[0]}=c_m$ | … | $S_n^{[0]}=c_n$ |
| ⋮ | | | | | | | ⋮ |
| $k$ | $S_0^{[k]}$ | | … | | $S_{t_m}^{[k]}$ ↓α | … | $S_{t_n}^{[k]}$ |
| $k$+1 | $S_0^{[k+1]}$ | | … | $S_{t_{m-1}}^{[k+1]} \xrightarrow{1-\alpha}$ | $S_{t_m}^{[k+1]}$ | … | $S_{t_n}^{[k+1]}$ |
| ⋮ | | | | | | … | ⋮ |
| $p$+1 | $S_0^{[p+1]}$ | | … | | | .. | $S_{t_n}^{[p+1]}$ |

A specially selected value $S_0^{[k]}$ is taken instead of the entire sum of terms corresponding to observations $c(t_m)$ with m ≤ 0.in the case when TS $\{c_1, c_2,...,c_n\}$ is not an infinite.

The choice of these initial values is, in our opinion, the most difficult and at the same time the most vulnerable place in the ES method. It seems advisable to use an algorithm to find the optimal value $S_0^{[k]^*}$ according to the criterion of the minimum error of the forecast. This procedure can be implemented by dividing some training data set $\{c(t_1), c(t_2),..., c(t_n)\}$ into two parts: basic and verification. In the first part of the scale, the parameters of the forecast model are estimated, in the second - the forecast error. By varying the value $S_0^{[k]}$, its value $S_0^{[k]^*}$ is determined such that the forecast error is minimal.

In accordance with the main theorem of the ES method proved by Brown, any k-th derivative ($k$ =0,1,2,...,$p$) $a_k$ in equation (1) can be expressed in terms of linear combinations of exponential averages up to (p+1) th order.

The system of (p+1) linear equations expressing $S_{t_n}^{[k]}$ ($k=\overline{1, p+1}$) with the help $f^{(j)}(t_n)$, ($j$ = 0,1,...,$p$) has the form [9]:

$$S_{t_n}^{[k]} = \sum_{j=0}^{p} (-1)^i \frac{f^{(j)}(t_n)}{j!} \frac{\alpha^k}{(k-1)!} \sum_{j=0}^{\infty} j^k (1-\alpha)^j \frac{(k+j-1)!}{j!}, \; k=\overline{1, p+1} \quad (8)$$

The linear system (8) can be written in matrix form. To do this, we denote

$$\vec{S} = \begin{Vmatrix} S_{t_n}^{[1]} \\ S_{t_n}^{[2]} \\ \vdots \\ S_{t_n}^{[p+1]} \end{Vmatrix}; \ \vec{\alpha} = \begin{Vmatrix} f^{(0)}(t_n) \\ \frac{1}{1!} f^{(1)}(t_n) \\ \vdots \\ \frac{1}{p!} f^{(p)}(t_n) \end{Vmatrix};$$

$$M = \left\| (-1)^k m_{ik} \right\|^{p+1} \text{ – matrix with dimension } (p+1)\times(p+1), \quad (9)$$

where $m_{ik} = \varphi_{ik}(\alpha) = \dfrac{\alpha^i}{(i-1)!} \sum_{j=0}^{\infty} j^k (1-\alpha)^j \dfrac{(i+j-1)!}{j!}$.

The elements of the matrix M are functions α and can be calculated for matrix values $i$ and $k$ with $1 \le \alpha, k \le p+1$.

The system (8) will be written as $\vec{S} = M\vec{\alpha}$. The solution of the system (8) or (9) can be written as $\vec{\alpha} = M^{-1}\vec{S}$. Let's find a vector $\vec{\alpha}$ for $p$ equal to 1 and 2. First of all, let's calculate the elements of the matrix

$$m_{11} = \alpha \sum_{j=0}^{\infty} j(1-\alpha)^j = \alpha(1-\alpha)\sum_{j=0}^{\infty} j(1-\alpha)^{j-1} = \alpha(1-\alpha)\frac{1}{\alpha^2} = \frac{1-\alpha}{\alpha};$$

$$m_{21} = \frac{\alpha^2}{1!}\sum_{j=0}^{\infty} j(1-\alpha)^j (j+1) = \alpha^2(1-\alpha)\sum_{j=0}^{\infty}(j+1)(1-\alpha)^{j-1} = \frac{2\alpha^2(1-\alpha)}{\alpha^3} = \frac{2(1-\alpha)}{\alpha};$$

$$m_{12} = \frac{\alpha}{0!}\sum_{j=0}^{\infty} j^2(1-\alpha)^j = \alpha^2\left\{(1-\alpha)^2\sum_{j=0}^{\infty} j(j-1)(1-\alpha)^{j-2} + \sum_{j=0}^{\infty} j^2(1-\alpha)^j\right\} =$$

$$= \frac{2(1-\alpha)^2 + (1-\alpha)\alpha}{\alpha^2} = \frac{(1-\alpha)(2-\alpha)}{\alpha^2};$$

$$m_{22} = \alpha^2\sum_{j=0}^{\infty} j^2(1-\alpha)^j(j+1) = \alpha^2\sum_{j=0}^{\infty} j^2(j+1)(j-1)(1-\alpha)^j + \alpha^2\sum_{j=0}^{\infty}(j+1)(1-\alpha)^j =$$

$$= \alpha^2\left\{(1-\alpha)^2\frac{\sigma}{\alpha^4} + (1-\alpha)\frac{2}{\alpha^3}\right\} = \frac{1-\alpha}{\alpha^3}\left[\frac{6(1-\alpha)\alpha}{1} + 2\alpha^2\right] = \frac{1-\alpha}{\alpha^3}(6\alpha - 6\alpha^2 + 2\alpha^2) =$$

$$= \frac{1-\alpha}{\alpha^3}(6\alpha - 4\alpha^2) = \frac{1}{\alpha^2}2(1-\alpha)(3-2\alpha);$$

$$m_{32} = \frac{\alpha^3}{2!}\sum_{j=0}^{\infty} j^2(1-\alpha)^j(j+1)(j+2) = \frac{\alpha^3}{2!}\sum_{j=0}^{\infty}(j+2)(j+1)j[(j-1)+1)](1-\alpha)^j =$$

$$= \frac{\alpha^3}{2!}\sum_{j=0}^{\infty}(j+2)(j+1)j(j-1)(1-\alpha)^j + \frac{\alpha^3}{2!}\sum_{j=0}^{\infty}(j+2)(j+1)j(1-\alpha)^j =$$

$$= \frac{1}{2!\alpha^5}[\alpha^3(1-\alpha)^2 4!] + \frac{\alpha^3 3!}{\alpha^4 2!}(1-\alpha) = \frac{1}{2!\alpha^2}\{(1-\alpha)3![4(1-\alpha)+\alpha]\} =$$

$$= \frac{1}{\alpha^2}[3(1-\alpha)(4-3\alpha)];$$

$$m_{i0} = \frac{\alpha^i}{(i-1)!}\sum_{j=0}^{\infty}\frac{1}{j!}[(1-\alpha)^j(i-1+j)!] = \frac{\alpha^2}{(i-1)!}\sum_{j=0}^{\infty}(1-\alpha)^{j+i-2}(j+1)\times$$

$$\times(j+2)\dots(j+i-1) = \frac{\alpha^i}{(i-1)!}\sum_{j=0}^{\infty}(j+i-1)(j+i-2)\dots(j+1)(1-\alpha)^j =$$

$$= \frac{\alpha^i}{(i-1)!}[(i-1)(i-2)\dots1 + i(i-1)(i-2)\dots2(1-\alpha)+\dots] =$$

$$= \frac{\alpha^i}{(i-1)!}\left[\frac{1}{1-x}\right]^{i-1}\Bigg|_{x=1-\alpha} = \frac{\alpha^i}{(i-1)!}\frac{(i-1)!}{\alpha^i} = 1;$$

$$m_{i1} = \frac{\alpha^i}{(i-1)!}\sum_{j=0}^{\infty}j(1-\alpha)^j\frac{(i-1+j)!}{(i-1)!} = \frac{\alpha^i(1-\alpha)}{(i-1)!}\sum_{j=0}^{\infty}(1-\alpha)^{j-1}\frac{(i-1+j)}{(i-1)!} =$$

$$= \frac{\alpha^i(1-\alpha)}{(i-1)!}\sum_{j=0}^{\infty}(1-\alpha)^k\frac{1}{k!}(i+k)! = \frac{\alpha^i(1-\alpha)i!}{(i-1)!\alpha^{i+1}} = \frac{i(1-\alpha)}{\alpha}.$$

Similarly, the elements of the matrix $||m_{ik}||$ can be calculated for higher values (for $p$ =3,4,5...).

Let us check for $p$ = 1 the equivalence of the solutions obtained from system (8) and from condition (3). If $p$ = 1, then from formula (2) we obtain $f(t) = a_0 + a_1(t-t_n)$.

Let 's denote $\Phi(a_0,a_1) = \alpha\sum_{k=0}^{\infty}(1-\alpha)^k[c(t_{n-k})+a_0-a_1(t_{n-k}-t_n)]^2$.

From condition (3), the following system of equations can be obtained for determining the coefficients $a_0$ and $a_1$:

$$\begin{aligned}\frac{\partial\Phi}{\partial a_0} &= -\alpha\sum_{k=0}^{\infty}(1-\alpha)^k c(t_{n-k}) + a_0\sum_{k=0}^{\infty}\alpha(1-\alpha)^k + a_1\sum_{k=0}^{\infty}\alpha(1-\alpha)^k(t_{n-k}-t_n);\\ \frac{\partial\Phi}{\partial a_1} &= -\alpha\sum_{k=0}^{\infty}(1-\alpha)^k[c(t_{n-k})+a_0+a_1(t_{n-k}-t_n)](t_{n-k}-t_0) = 0.\end{aligned} \quad (10)$$

Transform (10) so as to express exponential averages in the form of linear combinations of coefficients $a_0$ and $a_1$. The first equation of the system (10) can be written as

$$S_{t_n}^{[1]} = a_0 - a_1 h\sum_{k=0}^{\infty}k\alpha(1-\alpha)^k = a_0 - a_1 h\frac{1-\alpha}{\alpha}, \quad (11)$$

where $h = t_n - t_{n-1}$ = const is the time step between adjacent observation points.

The second equation is written as

$$\sum_{k=0}^{\infty}k\alpha(1-\alpha)^k c(t_{n-k}) = a_0\sum_{k=0}^{\infty}k\alpha(1-\alpha)^k - a_1 h\sum_{k=0}^{\infty}k^2\alpha(1-\alpha)^k. \quad (12)$$

It follows from expression (12) for $m_{12}$ that

$$\sum_{k=0}^{\infty}k^2\alpha(1-\alpha)^k = \frac{1}{\alpha^2}(1-\alpha)(2-\alpha). \quad (13)$$

 https://doi.org/10.1051/e3sconf/202453103020

In order to express the left side of equation (13) in terms of $S_{t_n}^{[1]}$ and $S_{t_n}^{[2]}$, we write down the expanded expression for $S_{t_n}^{[2]}$:

$$S_{t_n}^{[2]} = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i S_{t_n}^{[1]} = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i \sum_{i=0}^{\infty} \alpha(1-\alpha)^i c(t_{n-i-j}) =$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha^2 (1-\alpha)^{i+i} c(t_{n-i-j}) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \alpha^2 (1-\alpha)^k c(t_{n-k}) = \sum_{k=0}^{\infty} (k+1)\alpha^2 (1-\alpha)^k c(t_{n-k}) =$$

$$= \sum_{k=0}^{\infty} k\alpha^2 (1-\alpha)^k c(t_{n-k}) + \sum_{k=0}^{\infty} \alpha^2 (1-\alpha)^k c(t_{n-k}) = \alpha\left[\sum_{k=0}^{\infty} k\alpha(1-\alpha)^k c(t_{n-k})\right] + \alpha S_{t_n}^{[1]}.$$

Therefore, the left side of equation (13) can be transformed to the form:

$$\left(S_{t_n}^{[2]} - \alpha S_{t_n}^{[1]}\right)/\alpha .$$

Taking into account the obtained ratio, equation (13) is written as

$$\left(S_{t_n}^{[2]} - \alpha S_{t_n}^{[1]}\right)\frac{1}{\alpha} = \frac{a_0}{\alpha}(1-\alpha) - \frac{a_1 h}{\alpha^2}(1-\alpha)(2-\alpha) . \tag{14}$$

To represent the exponential average $S_{t_n}^{[2]}$ as a linear combination of $a_0$ and $a_1$, use the ratio (11). Substitute the value $S_{t_n}^{[1]}$ into equation (14). As a result, we get

$$S_{t_n}^{[2]} = \alpha a_0 - (a_1 h)(1-\alpha) + a_0(1-\alpha) - a_1 h \frac{1}{\alpha}(1-\alpha)(2-\alpha) =$$

$$= \alpha a_0 + a_0 - \alpha a_0 - a_1 h(1-\alpha)\left[1 + \frac{2-\alpha}{\alpha}\right] = a_0 - a_1 h \frac{(1-\alpha)^2}{\alpha}.$$

So, after the transformations, the system (10) obtained as a consequence of condition (3) can be written as

$$\begin{cases} S_{t_n}^{[1]} = a_0 - a_1 h \dfrac{1}{\alpha}(1-\alpha); \\ S_{t_n}^{[2]} = a_0 - a_1 h \dfrac{2}{\alpha}(1-\alpha). \end{cases} \tag{15}$$

If we compare the value $x = -ih$ ($i$ = 0, 1,...,$n$–1) to each observation $c_{n-i}$, then the weights of the observations can be calculated using the formula

$$\alpha_{n-i} = \int_{-(i+1)h}^{-ih} e^x dx = e^{-ih} - e^{-(i+1)h} = (1-e^{-h})e^{-ih} . \tag{16}$$

Let's choose $h$ in such a way that equality $1 - e^{-h} = \alpha$ is fulfilled, where α is the smoothing parameter selected empirically by the formula α = 2/($m$+1); $m$ is the number of observations that must be taken into account as much as possible in the forecast/

Then expression (16) will take the form $\alpha_{n-i} = \alpha(1-\alpha)^i$, $(i = 0,1,...,n-1)$,, i.e. the weights used in the exponential smoothing method are actually obtained.

Let's take this approach to determine the weights of observations in the case of unequal observations. To do this, consider the auxiliary variable

$$x = a\frac{t - t_n}{t_n - t_1}, \tag{17}$$

where $a$ is some positive constant.

If $t = t_j$,, then $x = x_j = a(t_j - t_n)\,(t_n - t_1)^{-1}$, $j = \overline{1, n}$. Note that $x_1 = -a$; $x_n = 0$.

Thus, formula (17) defines the linear transformation of the observation interval ($t_1 \div t_n$) in the interval ($-a \div 0$). Let's determine the weight of the $j$-th observation using the formula

$$\alpha_j = \int_{x_{j-1}}^{x_j} e^x dx = \alpha e^{x_j} = \alpha e^{a\frac{t_j - t_n}{t_n - t_1}}. \tag{18}$$

The expressions (17) and (18) obtained make it possible to calculate the weights of unequally spaced observations. It is easy to verify that all the properties of the weights listed above are fulfilled, i.e. the proposed method is indeed a generalization (modification) of the ES method. After the weights of the observations are found, the optimization problem is solved:

$$\widetilde{a}_{<p+1>} = <\widetilde{a}_0, \widetilde{a}_1, ..., \widetilde{a}_p> = \underset{\{\widetilde{a}_{<p+1>}\}}{\arg\min} \sum_{i=0}^{n} \alpha_i \left( c_i - \sum_{k=0}^{p} \frac{\widetilde{a}_k}{k!} x_i^k \right)^2 . \tag{19}$$

It can be shown that the coefficient estimates obtained from condition (19) satisfy the following system of linear algebraic equations

$$\sum_{k=0}^{p} \widetilde{a}_k \left( \frac{1}{k!} \sum_{j=0}^{n} \alpha_j (t_j - t_n)^{k+l} \right) = \sum_{j=0}^{n} \alpha_j (t_j + t_n)^l c_j ,$$

where $l = 0,1,...,p$.

The system of equations (19) can be solved by numerical methods on a computer. The value of the determining parameter predicted at the time $t_n + \tau$ is determined by the formula

$$\widetilde{c}^{*}_{t_n+\tau} = \widetilde{a}_0 + \widetilde{a}_1 \tau + ... + \frac{\widetilde{a}_p}{p!} \tau^p .$$

## 3 Results

The prediction of the quantitative values of the parameters by the developed modified ES method consists in the sequential completion of the following stages (see Figure 1).

Step 1. Formation of the initial data.

Step 2. For the known moments of observations $t_1$, $t_2$,...,$t_n$ and $p$ we determine the weights of the observations $\alpha_1$, $\alpha_2$,..., $\alpha_n$ according to formulas (17) and (18).

Step 3. Determine the matrices $\boldsymbol{V}$, $\boldsymbol{V}^{-1}$, $\boldsymbol{K}$, $\boldsymbol{K}^{-1}$ and $\boldsymbol{X}$ by formulas (20)-(24).

If the weights of the observations are $\alpha_1$, $\alpha_2$,..., $\alpha_n$, then the matrix $\boldsymbol{V}$ is diagonal and has the form

**Step 1**
Formation of the initial data

$$t_1, t_2, \dots, t_{n;;} \quad \vec{c}^{\,\mathrm{T}} = (c_1, c_2, \dots, c_n)\,;\ \alpha$$

**Step 2**
Determining the weights of observations $\alpha_1, \alpha_2, \dots, \alpha_n$ for known moments of observation $t_1, t_2, \dots, t_n$ and α

$$x = \alpha \frac{t - t_n}{t_n - t_1},$$

$$\alpha_j = \int_{x_{j-1}}^{x_j} e^x dx = \alpha e^{x_j} = \alpha e^{a\frac{t_j - t_n}{t_n - t_1}}$$

**Step 3**
Formation of matrix
$\boldsymbol{V}$, $\boldsymbol{V}^{-1}$, $\boldsymbol{K}$, $\boldsymbol{K}^{-1}$ и $\boldsymbol{X}$

$$\mathbf{V} = \begin{Vmatrix} \frac{1}{\alpha_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\alpha_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\alpha_n} \end{Vmatrix} = \left\| \frac{\delta_{ij}}{\alpha_i} \right\|_1^n, \quad \delta_{ij} = \begin{cases} 1 & \text{при } i = j, \\ 0 & \text{при } i \neq j. \end{cases} \qquad \mathbf{V}^{-1} = \left\| \alpha_i \delta_{ij} \right\|_1^n;$$

$$\mathbf{K} = \left\| \frac{\delta_{ij}}{\sqrt{\alpha_i}} \right\|_1^n; \quad \mathbf{K}^{-1} = \left\| \sqrt{\alpha_i}\, \delta_{ij} \right\|_1^n;$$

$$\mathbf{X} = \begin{Vmatrix} 1 & t_1 & \frac{t_1^2}{2!} & \cdots & \frac{t_1^p}{p!} \\ 1 & t_2 & \frac{t_2^2}{2!} & \cdots & \frac{t_2^p}{p!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & t_n & \frac{t_n^2}{2!} & \cdots & \frac{t_n^p}{p!} \end{Vmatrix}$$

**Step 4**
Determination of coefficient estimates $a_0, \dots, a_p$ a segment of the Taylor series for given (known) parameter values measured at time points $t_1, t_2, \dots, t_n$ and forming a vector $\vec{c}^{\,\mathrm{T}}$

$$\hat{\vec{a}} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{U} = (\mathbf{X}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{V}^{-1}\vec{c}$$

**Step 5**
Determination of the residual sum of squares and initial estimates of the variance of the deviation of the values of parameters $c$ from those calculated using the approximate formula

$$\vec{f}^{\,\mathrm{T}}\vec{f} = (\vec{U} - \mathbf{B}\hat{\vec{a}})^{\mathrm{T}}(\vec{U} - \mathbf{B}\hat{\vec{a}}) = (\vec{c} - \mathbf{X}\hat{\vec{a}})^{\mathrm{T}}\mathbf{V}^{-1}(\vec{c} - \mathbf{X}\hat{\vec{a}})$$

$$S^2 = (\vec{f}^{\,\mathrm{T}}\vec{f})(n - p - 1)^{-1};$$

$$c_i \approx a_0 + a_1 t_i + \frac{a^2}{2!} t_i^2 + \dots + \frac{a^p}{p!} t_i^p$$

**Step 6**
Calculation of the covariance matrix estimation $\hat{\vec{a}}$ of the vector $\vec{a} = (a_0, \dots, a_p)$

$$D[\hat{\vec{a}}] = M[\hat{\vec{a}}\ \hat{\vec{a}}^{\mathrm{T}}] = \sigma^2(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B})^{-1} = \sigma^2(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}$$

**Step 7**
Calculation of the predicted parameter values c

$$c^*(t_n + \tau) = \tilde{a}_0 + \tilde{a}_1\tau + \frac{\tilde{a}_2}{2!}\tau + \dots + \frac{\tilde{a}_p}{p!}\tau^p.$$

**Fig. 1.** The structure of the method for predicting changes in uneven TS of limited duration

$$\mathbf{V} = \begin{Vmatrix} \frac{1}{\alpha_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\alpha_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\alpha_n} \end{Vmatrix} = \left\| \frac{\delta_{ij}}{\alpha_i} \right\|_1^n, \quad \delta_{ij} = \begin{cases} 1 \text{ при } i = j, \\ 0 \text{ при } i \neq j. \end{cases} \tag{20}$$

2. The matrices $V^{-1}$, $K$, $K^{-1}$ are determined by the formulas (the notation is the same)

$$\mathbf{V}^{-1} = \left\| \alpha_i \delta_{ij} \right\|_1^n; \tag{21}$$

$$\mathbf{K} = \left\| \frac{\delta_{ij}}{\sqrt{\alpha_i}} \right\|_1^n; \tag{22}$$

$$\mathbf{K}^{-1} = \left\| \sqrt{\alpha_i} \delta_{ij} \right\|_1^n. \tag{23}$$

3. The matrix X has the form:

$$\mathbf{X} = \begin{Vmatrix} 1 & t_1 & \frac{t_1^2}{2!} & \cdots & \frac{t_1^p}{p!} \\ 1 & t_2 & \frac{t_2^2}{2!} & \cdots & \frac{t_2^p}{p!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & t_n & \frac{t_n^2}{2!} & \cdots & \frac{t_n^p}{p!} \end{Vmatrix}. \tag{24}$$

Step 4. For the given (known) values of the parameters measured at time $t_1$, $t_2,\ldots,t_n$ and forming a vector $\vec{c}^{\,\mathrm{T}} = (c_1, c_2, \ldots, c_n)$, we determine the estimates of the coefficients $a_0,\ldots,a_p$ of the segment of the Taylor series according to the formula (25).

$$\hat{\vec{a}} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{U} = (\mathbf{X}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{V}^{-1}\vec{c}.$$

In the case of an arbitrary positive definite matrix $V$, the estimate (25) is called a generalized MLS, and if the matrix $V$ is diagonal, then the estimate $\hat{\vec{a}}$ obtained by formula (25) is called a weighted MLS estimate.

Step 5. Using formula (26), we determine the residual sum of squares and using formula (28), the initial estimates of the variance of the deviation of the values of parameters c from those calculated using the approximate formula (26)

$$c_i \approx a_0 + a_1 t_i + \frac{a^2}{2!} t_i^2 + \ldots + \frac{a^p}{p!} t_i^p, \tag{26}$$

The remaining sum of squares is determined by the formula

$$\vec{f}^{\,\mathrm{T}} \vec{f} = (\vec{U} - \mathbf{B}\hat{\vec{a}})^{\mathrm{T}} (\vec{U} - \mathbf{B}\hat{\vec{a}}) = (\vec{c} - \mathbf{X}\hat{\vec{a}})^{\mathrm{T}} \mathbf{V}^{-1} (\vec{c} - \mathbf{X}\hat{\vec{a}}). \tag{27}$$

The residual sum of squares has *(n–p–1)* degrees of freedom and can be used to estimate the $S^2$ value of $\sigma^2$

$$S^2 = (\vec{f}^{\mathrm{T}} \vec{f})(n - p - 1)^{-1}. \quad (28)$$

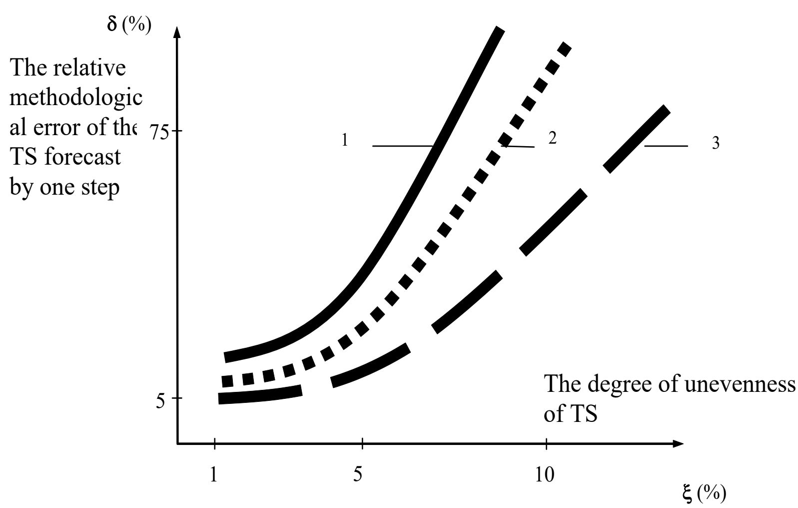
Step 6. Calculate the covariance matrix of estimate $\hat{\vec{a}}$ of the vector $\vec{a}=(a_0,...,a_p)$ using the formula (29).

$$D[\hat{\vec{a}}]=M[\hat{\vec{a}}\,\hat{\vec{a}}^{\mathrm{T}}] = \sigma^2(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B})^{-1} = \sigma^2(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{V}^{-1}\boldsymbol{X})^{-1} \quad (29)$$

Step 7. Using the formula (4), we calculate the predicted values of the parameters.

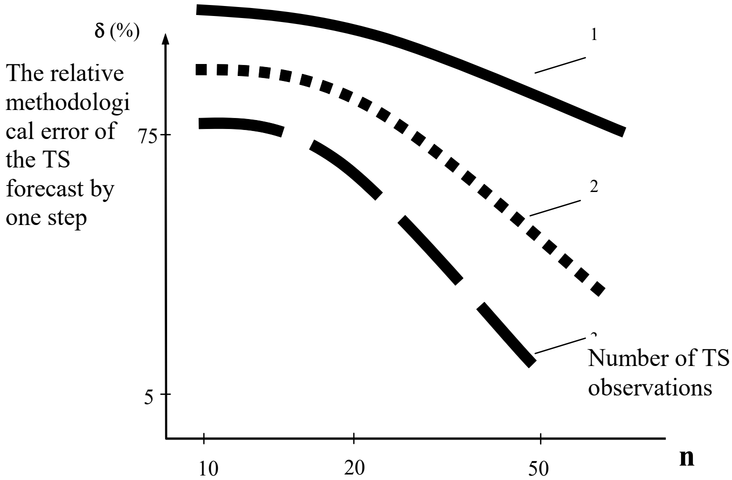
## 4 Discussions

In Figure 2 and 3 show experimental curves of the dependence of the relative methodological error of the TS forecast results by one step on the degree of unevenness and the sample size of the initial time series for three forecasting models: MLS with a linear trend without differentiation; exponential smoothing and modified exponential smoothing based on the developed method for predicting parameters from uneven time series of observations of limited duration.




1 – extrapolation of MLS with a linear trend;
2 – exponential Brown smoothing;
3 – modified exponential smoothing.

**Fig. 2**. Experimental dependences of the relative methodological error of the TS forecast results by one step on the degree of unevenness of the initial time series




1 – extrapolation of MLS with a linear trend;
2 – exponential Brown smoothing;
3 – modified exponential smoothing.

**Fig. 3**. Experimental dependences of the relative methodological error of the TS forecast results by one step on the value of the choice of the initial time series

The analysis of the curves shows a decrease in the relative methodological error of the TS forecast in the range of approximately 10 to 25 percent, depending on the previously used models, as well as the actual unevenness and sample size of observations of the predicted TS.

## 5 Conclusion

Thus, the article describes the developed method for predicting changes in controlled parameters based on uneven time series of observations of limited duration. The described approach to forecasting is characterized by the use of a modified exponential smoothing method with weight coefficients that take into account the influence of factors of limited volume and irregularity of the initial data collection process. In the proposed modified method, in order to take into account the influence of factors of unevenness and finiteness of the sample of observations, a transition from exponential smoothing to decreasing non-exponential with a linear transformation of the observation interval was carried out. The actual unevenness of TS is taken into account by introducing the unevenness of individual observations.

In this case, the modified sequence of observation weights is determined by adjusting the classically calculated exponential weights taking into account the actual unevenness of the TS and then optimized according to the criterion of the minimum interpolation error on the final sample of observations. The use of the modified method makes it possible to reduce the methodological error of forecasting and increase the accuracy and reliability of forecasts.

## References

1. A.M. Sunchalin, A.L. Sunchalin, "Chronoeconomics", **1 (22),** 26-30 (2020)
2. A. A. Akhmetshina, A young scientist, **50 (288)**, 161-163 (2019)
3. 3. D.B. Egorov, S.D. Zakharov Artificial Intelligence in Healthcare, **1,** 21-26 (2020)
4. D.N.Savinskaya, et.al ,, Modern Economy: problems and solutions, **11 (143),** 56-64 (2021).
5. M.I.Gorlov, et.al , Microelectronics, **5 (35),** 392-400 (2006)
6. Y.P. Lukashin, Adaptive methods of short-term time series forecasting (Moscow, Finance and Statistics, 2003).
7. S.K. Prajakta, Kanwal Rekhi School of Information Technology Journal, **1(33)**, 1-13 (2004)
8. S. Makridakis, et.al, Forecasting Method and Applications; **3 ed**. (Wiley, 2003)
9. S Everette, Jr.Gardner, The state of the art, **Part II. June 3** (2005).
10. Rob J Hyndman. Computational technologies **11(2),** 171-188 (2002)
11. .Rob J Hyndman and Muhammad Akram, Computational technologies **13(5),** 73-87 (2006).
12. E Ostertagova, Proceedings of the 4th International Conference on Modelling of Mechanical and Mechatronic Systems, Technical University of Kosice, Slovak Republic, 380–384, (2011).
13. Z.P Li,, et.al, Acta automatica sinica,. **11 (34),** 1404– 1409 (2008).
14. D. A. Turko, Research and development in the field of mechanical engineering, energy and management : proceedings of the VIII International Conference. inter-university. scientific and technical Conf. of students, undergraduates and postgraduates, Gomel, 412-415 (2008)
15. O.V. Russkov, et.al , Modern science: actual problems of theory and practice. Series: Natural and Technical Sciences, **04**, 142-147 (2021)
16. O.V. Russkov, et.al , National Association of Scientists, **4,** 55-59 (2014)
17. O.V. Russkov, et.al , Eurasian Union of Scientists, **9,** 28-33 (2014)